Communications in Mathematics and Applications
Vol. 16, No. 2, pp. 669–677, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i2.2906

http://www.rgnpublications.com

Research Article

An Investigation of the Bifurcation of Traveling Wave Solutions in Time-Fractional Nonlinear Differential Equations of the Symmetric Case

Mustafa T. Yaseen*[1] and Mudhir A. Abdul Hussain[2]

[1] *Department of Management and Marketing for Oil & Gas, College of Industrial Management, Basra University for Oil & Gas, Basrah, Iraq*
[2] *Department of Mathematics, Education College for Pure Sciences, University of Basrah, Basrah, Iraq*
***Corresponding author:** mustafa.taha@buog.edu.iq

Received: October 11, 2024 **Revised:** January 25, 2025 **Accepted:** February 19, 2025

Abstract. This study considers examining and bifurcation of traveling wave solutions in time-fractional nonlinear differential equations of the symmetric case. We blend Li and He's derivative of fractional order techniques with Lyapunov-Schmidt reduction in our approach. To simplify the analysis, the initial fractional equation that is differential is transformed into a partial differential equation through the utilization of the fractional complex transform. This conversion results in a condensed equation, presented as a pair of nonlinear algebraic equations, tackling the core issue. Furthermore, our investigation involves examining linear approximation solutions for a *nonlinear fractional equation* (NFE) that is differential.

Keywords. Differential equations, Bifurcation analysis, Complex transform

Mathematics Subject Classification (2020). 35R11, 37G10, 35C07

1. Introduction

A wide range of research and engineering fields are increasingly adopting *fractional differential equations* (FDEs). Due to their wide range of applications in several fields of applied sciences, their adaptability has attracted the interest of many academics in recent years. Finite difference equations offer a robust basis for developing models of various phenomena such as electromagnetics, solid mechanics, fluid mechanics, viscoelasticity, bio population dynamics, electrochemistry, and signal processing (Berger [5], Ghanim *et al*. [7], Vainberg and Trenogin [21]).

Recent study introduced a new numerical method for addressing nonlinear time-fractional differential equations, particularly in the Caputo framework. The approach integrates the Laplace transform with an adapted Adomian decomposition method, termed LTAADM. This method was contrasted with the conventional Laplace transform integrated with the standard Adomian technique (LADM). To demonstrate the benefits of the proposed method, LMADM and LTSAT were utilized on several nonlinear time-fractional differential equations (AL-Humedi and Hasan [1]).

The propagation of long-wave nonlinear waves in various environments is an important physical phenomenon, widely studied in fields such as ocean dynamics, laboratory experiments on stratification, and atmospheric research. To analyze the propagation of these long-wave nonlinear waves from a purely physical point of view, many mathematical models have been developed, many of which are based on the well-known *Korteweg de Vries* (KdV) equation, which serves as a basic model for examining weak long-wave nonlinearity.

Research suggests that the KdV equation arises from applying a multiscale asymptotic approach to the fundamental Euler equations governing incompressible and nonviscous fluids. It typically describes small-amplitude, long-wavelength surface waves in shallow water, as well as internal waves in shallow, densely layered fluids. In the first-order turbulent expansion, only first-order dispersion and nonlinearity are considered, leading to the KdV equation. However, a deeper understanding of physical processes often requires the inclusion of higher-order nonlinear and dispersive effects. In such cases, removing the second-order components from the perturbation expansions and applying the perturbation technique to the leading Euler equations results in a fifth-order KdV-like equation.

Several nonlinear equations encountered in fields such as mathematics, engineering and physics can be expressed as operator equations:

$$H(x,\lambda)=b, \quad x\in O\subset X,\ b\in Y,\ \lambda\in\mathbb{R}^n. \tag{1.1}$$

Here, $H: X\to Y$ represents a smooth Fredholm function with zero index, where X and Y denote the real Banach spaces, as well as O denotes an open set in X. The technique of the finite-dimensional reduction offers a solution approach for this equation. This methodology relies on employing Lyapunov-Schmidt reduction to transform eq. (1.1) to a similar finite-dimensional equation:

$$\Omega(\zeta,\lambda)=\beta, \quad \zeta\in\widehat{M},\ \beta\in\widehat{N}. \tag{1.2}$$

In this context, $\widehat{M}$ and $\widehat{N}$ represent smooth finite-dimensional manifolds. Previous studies of Loginov [15], Sapronov and Zachepa [18], Sapronov [19], and Vainberg and Trenogin [21] have illustrated a transition from eq. (1.1) to eq. (1.2), utilizing a localized variant of the Lyapunov-Schmidt method. This transition ensures that (1.2) maintains all the topological and analytic characteristics of (1.1), encompassing features such as multiplicity and bifurcation diagrams. Scholars like Vainberg and Trenogin [21], Loginov [15], Sapronov and Zachepa [18], and Sapronov [20] have achieved this transformation through the localized Lyapunov-Schmidt approach, guaranteeing that eq. (1.2) preserves the full spectrum of properties possessed by eq. (1.1).

In 2024, Amir *et al*. [2] investigated the heat transfer and dynamics of nanofluids in the framework of fractional calculus, using Riemann-Liouville and Caputo derivatives (see, Ghanim*et al*. [7]). In addition, from 2021 to 2023, many researchers developed various numerical methods to solve single and diverse systems of fractional differential equations (see, AL-Humedi and Hasan [1], Arshad *et al*. [3], Ashraf, *et al*. [4], Khan *et al*. [11]).

There has been a growing interest in fractional differential equations owing to their significance in various domains, including physics, biology, economics, engineering, system identification, control theory, fluid dynamics, fractional dynamics, and signal processing. More and more researchers are focusing on finding exact solutions to these equations through analytical methods, such as the fractional sub equations method (Sapronov [19], Sapronov and Zachepa [18]). Li and He [12, 13] Implemented a fractional complex transformation to convert fractional differential equations into partial differential equations, hence enabling their analysis through existing methodologies.

This research examines the bifurcation of periodic traveling wave solutions of a nonlinear fractional differential equation, utilizing the local Lyapunov-Schmidt method and its fractional derivative. We investigate the division of periodic traveling wave solutions. Our approach involves utilizing the localized Lyapunov-Schmidt method in conjunction with He's fractional derivative.

$$\frac{\partial^\alpha \mathcal{W}}{\partial t^\alpha}+5\mathcal{W}^2\frac{\partial \mathcal{W}}{\partial y}+5\frac{\partial \mathcal{W}}{\partial y}\frac{\partial^2 \mathcal{W}}{\partial y^2}+5\mathcal{W}\frac{\partial^3 \mathcal{W}}{\partial y^3}+\frac{\partial^5 \mathcal{W}}{\partial y^5}=\psi, \quad \mathcal{W}=\mathcal{W}(y,t), \tag{1.3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the fractional derivative of He, and $\alpha \in (0,1]$ and ψ is a continuous function.

Theorem 1.1 ([5])**.** *Suppose that both X and Y denote the Banach spaces as well as $H(x,\lambda)$ represents a C^1 map given in a neighborhood U of (x_0,y_0) in the range subset in Y in a way which $H(x_0,\lambda_0)=0$ as well as $H_x(x_0,\lambda_0)$ represents a linear Fredholm operator. Consequently, each one of the solution sets (x,λ) of $H(x,\lambda)=0$ close to (x_0,λ_0) (with the value of λ fixed) is a one-to-one correspondence to the set of the solutions in the system with a finite dimension of N_1 of variables Finite number of variables in real equations N_0 of real variables. Moreover, $N_0=\dim(\operatorname{Ker}L)$ and $N_1=\dim(\operatorname{Coker}L)$, $(L=H_x(x_0,\lambda_0))$.*

Definition 1.1 ([18])**.** The discriminant set Σ for (1.1) is characterized as the collection of all values of $\lambda=\overline{\lambda}$ for which (1.1) shows a solution that is degenerate, $\overline{x}\in O$:

$$G(\overline{x},\overline{\lambda})=b, \quad \operatorname{codim}\left(\operatorname{Im}\frac{\partial H}{\partial x}(\overline{x},\overline{\lambda})\right)>0,$$

where Im depicts the operator $\frac{\partial H}{\partial x}(\overline{x},\overline{\lambda})$.

Definition 1.2 ([14])**.** The derivative expressed as a fraction by Li and He [12] can be represented as the following

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^t (s-t)^{n-\alpha-1}(f_0(s)-f(s))ds,$$

where $f_0(x)$ is a known function.

To initiate the application of the Lyapunov-Schmidt method to analyze eq. (1.3), the initial step involves transforming the equation into a partial differential equation. This transformation is achieved through the fractional complex transformation described as follows [12]:

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{1.4}$$

The transformation of eq. (1.3) yields the subsequent partial differential equation:

$$\frac{\partial \mathcal{W}}{\partial T} + 5\mathcal{W}^2 \frac{\partial \mathcal{W}}{\partial y} + 5\frac{\partial \mathcal{W}}{\partial y}\frac{\partial^2 \mathcal{W}}{\partial y^2} + 5\mathcal{W}\frac{\partial^3 \mathcal{W}}{\partial y^3} + \frac{\partial^5 \mathcal{W}}{\partial y^5} = \psi, \quad \mathcal{W} = \mathcal{W}(y,t). \tag{1.5}$$

For examining the traveling wave solutions of eq. (1.5), we employ the subsequent transformation.

$$u(x) = \mathcal{W}(y,t), \;\; x = y - cT, \;\; c = \kappa\alpha, \tag{1.6}$$

where κ stands as the constant, α represents a parameter. By inserting (1.6) in (1.5), we transform (1.5) into a fourth order nonlinearity ordinary differential equation.

$$u'''' + 5uu'' + \frac{5}{3}u - cu = \psi, \quad c = \lambda, \;\; ' = \frac{d}{dx}. \tag{1.7}$$

This study posits that u is a periodic function, $u(x) = u(x+2\pi)$ (Hussain [9, 10], Mizeal and Hussain [17]) examined the bifurcation solutions of eq. (1.7) utilizing the classical Lyapunov-Schmidt approach and an adapted version of the Lyapunov-Schmidt method. In the subsequent section, we will employ the Local approach of Lyapunov-Schmidt to transform eq. (1.7) into a corresponding finite-dimensional system of nonlinear algebraic equations.

2. Bifurcation Equation Simplification (Reduction Technique)

The objective is to examine the bifurcation of moving wave solutions that are periodic to (1.5), it's advantageous to express (1.7) as an operator equation.

$$H(u,\lambda) = u'''' + 5uu'' + \frac{5}{3}u^3 - cu, \tag{2.1}$$

where $H : E \to M$ represents a nonlinear Fredholm operator with zero index, where $E = \Pi_4([0,2\pi],\mathbb{R})$ denotes the space containing each function having derivatives of order up to 4, and $M = \Pi_0([0,2\pi],\mathbb{R})$ represents all continuous functions that are periodic comprise the space. Here, $\mathbb{R}$ denotes the real space, and $u = u(x)$, where $x \in [0,2\pi]$. Therefore, the solution for bifurcation in eq. (2.1) the solution of the operator equation matches exactly with the bifurcation solution

$$H(u,\lambda) = \psi. \tag{2.2}$$

According to Theorem 1.1, the solutions of eq. (1.5) are interchangeable with a system of finite dimensions solved comprising 2 variables and 2 equations, where 2 represents the dimension of both $\operatorname{Ker} H_u(0,\lambda)$ and $\operatorname{Coker} H_u(0,\lambda)$. The initial step in this simplification process involves deriving the linearized equation corresponding to eq. (2.2), expressed as follows:

$$Ah = 0, \quad h \in E,$$

$$A = \frac{\partial F}{\partial u}(0,\lambda) = \frac{d^4}{dx^4} - \lambda.$$

The periodic solution of the equation that got linearized can be represented as:

$$e_p(x) = a_p \sin(px) + b_p \cos(px), \quad p = 1,2,\dots.$$

Thus, the characteristic equation that is identical to this solution can be stated as:

$$p^4 - \lambda = 0.$$

A small change in the parameter λ, lead to bifurcation along to the two modes

$$e_1 = a_1 \sin(x), \quad e_2 = a_2 \cos(x)$$

such that $\|e_i\|_{\mathcal{H}} = 1$ as well as $a_i = \sqrt{2}$, $i = 1,2$, ($\mathcal{H}$ is Hilbert space $L^2([0,2\pi],R)$). Suppose that $N = \mathrm{Ker}(A) = \mathrm{Span}\{e_1, e_2\}$, then the space E decompose into two subspaces, each forming a direct sum, N and $N^\perp$ the orthogonal complement to N,

$$E = N \oplus N^\perp, \quad N^\perp = \{v \in E : v \perp N\}.$$

Similarly, the space M Two subspaces can be combined using a direct sum to represent the original, N and $\widehat{N}^\perp$ perpendicular complement to $\widehat{N}$,

$$M = N \oplus \widehat{N}^\perp, \quad \widehat{N}^\perp = \{\omega \in M : \omega \perp \widehat{N}\}.$$

From the above decomposition of E this implies the existence of two projections $P : E \to N$ and $I - P : E \to N^\perp$ where (I is the identity operator),

$$Pu = \mathcal{W}, \quad (I-P)u = v,$$

it follows that every element $u \in E$ has a unique form,

$$u = z + v, \; z = \sum_{i=1}^{2} \zeta_i e_i \in N, \quad v \in N^\perp, \; \zeta_i = \langle u, e_i \rangle.$$

Similarly, the decomposition of M entails the existence of two projections, $Q : M \to N$ and $I - Q : M \to \widehat{N}^\perp$ such that

$$QH(u,\lambda) = H_1(u,\lambda), \quad (I-Q)H(u,\lambda) = H_2(u,\lambda).$$

Therefore, each and every item can be inferred $H(u,\lambda) \in M$ a unique form can represent it,

$$\begin{aligned} H(u,\lambda) &= H_1(u,\lambda) + H_2(u,\lambda) \\ &= QH(u,\lambda) + (I-Q)H(u,\lambda) = \psi. \end{aligned}$$

Since $\psi \in M$ implies that $\psi = \psi_1 + \psi_2$, $\psi_1 = t_1 e_1 + t_2 e_2 \in N$, $\psi_2 \in \widehat{N}^\perp$,

$$H_1(u,\lambda) = \sum_{i=1}^{2} v_i(u,\lambda) e_i \in N, \quad H_2(u,\lambda) \in \widehat{N}^\perp,$$

$$v_i(u,\lambda) = \langle H(u,\lambda), e_i \rangle_{\mathcal{H}}.$$

Consequently, eq. (2.2) become as,

$$QH(u,\lambda) = \psi_1, \quad (I-Q)H(u,\lambda) = \psi_2$$

or

$$QH(z+v,\lambda) = \psi_1, \quad (I-Q)H(z+v,\lambda) = \psi_2.$$

A smooth map can be explicitly represented using the implicit function theorem $\varphi : N \to N^\perp$ such that $\varphi(u,\lambda) = v$ and

$$(I-Q)H\left(z + \varphi(u,\lambda), \lambda\right) = \psi_2.$$

To solve eq. (2.2) in the area surrounding the point$u = 0$, it is sufficient to solve the equation:

$$QH(u + \varphi(u,\lambda), \lambda) = \psi_1. \tag{2.3}$$

The bifurcation solutions of eq. (2.3) has the form,

$$\phi(\zeta, q) = \psi_1, \quad \zeta = (\zeta_1, \zeta_2),$$

where

$$\phi(\zeta,\lambda) = H_1(z+\varphi(u,\lambda),\lambda).$$

Equation (2.2) can be written in the form of,

$$H(z+v,\lambda) = A(z+v) + T(z+v) = Az + \mathcal{N}(u) + \dots,$$

$$\mathcal{N}(u) = 5uu'' + \frac{3}{5}u^3,$$

where $\mathcal{N}$ and the dots denote the terms of element v. Thus,

$$\phi(\zeta,q) = H_1(z+v,\lambda) = \sum_{i=1}^{2} \langle Au + \omega(u), e_i \rangle_{\mathcal{H}} e_i + \dots = t_1 e_1 + t_2 e_2. \tag{2.4}$$

Upon performing some calculations on eq. (2.4) and by using the property $Ae_1 = q(\lambda)e_1$, $Ae_2 = q(\lambda)e_2$, we have the following system

$$\begin{cases} \frac{5}{2}\zeta_1(\zeta_1^2+\zeta_2^2) + q\zeta_1 = t_1, \\ \frac{5}{2}\zeta_2(\zeta_1^2+\zeta_2^2) + q\zeta_2 = t_2, \end{cases}$$

where $q = 1 - \kappa\alpha$, $\kappa \in R$ and

$$\langle \mathcal{N}(u), h(x) \rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{N}(x)h(x)dx.$$

The symmetry of the function $\psi(x)$ in relation to the variable representing involution, $\psi(x) \to \psi(\pi + x)$ implies that $t_1 = q_1$, $t_2 = 0$, then we obtain:

$$\begin{cases} \frac{5}{2}\zeta_1(\zeta_1^2+\zeta_2^2) + q\zeta_1 = q_1, \\ \frac{5}{2}\zeta_2(\zeta_1^2+\zeta_2^2) + q\zeta_2 = 0. \end{cases} \tag{2.5}$$

In the vicinity of point zero, the bifurcation equation of eq. (2.5) is identical to the mathematical expression,

$$\phi_1(\zeta,q) = \begin{cases} \frac{5}{2}\zeta_1(\zeta_1^2+\zeta_2^2) + q\zeta_1 - q_1, \\ \frac{5}{2}\zeta_2(\zeta_1^2+\zeta_2^2) + q\zeta_2, \end{cases} \tag{2.6}$$

where $q, q_1 \in R$, a solution to eq. (2.6) yields the parameters' equation in terms of ζ_1, ζ_2.

We identified a selective collection of bifurcation sets for eq. (2.6) and constructed the q_1q-plane graph.

By decomposing the solution set into regions, S_1, S_2 the discriminate set yields three genuine solutions; but for S_3, S_4, we have only one real solution.

In set S_1, we choose $q_1 = -10.1$, $q = -30.1$. These numbers yield three genuine solutions to the system (2.6),

$$P_1 = (0.3387775393, 0), \quad P_2 = (3.288055711, 0), \quad P_3 = (-3.626833250, 0).$$

The planar graph illustrating the decomposition of the solution regions is shown in Figure 1, while the corresponding phase portraits associated with the equilibrium points P_1, P_2, and P_3 are presented in Figures 2, 3 and 4, respectively.

In set S_3, we choose $q_1 = 10.1$, $q = 30.1$. Considering these variables, we obtain a single genuine solution of the system (2.6), i.e.,

$$P_1 = (0.3387775393, 0).$$

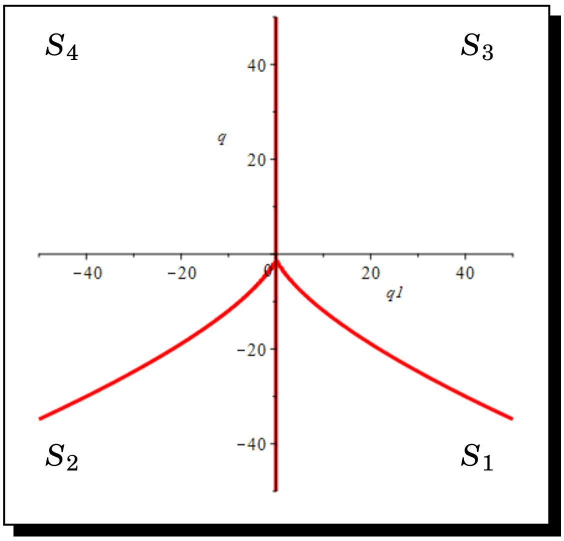

Figure 1. Discriminate set of systems (2.6)

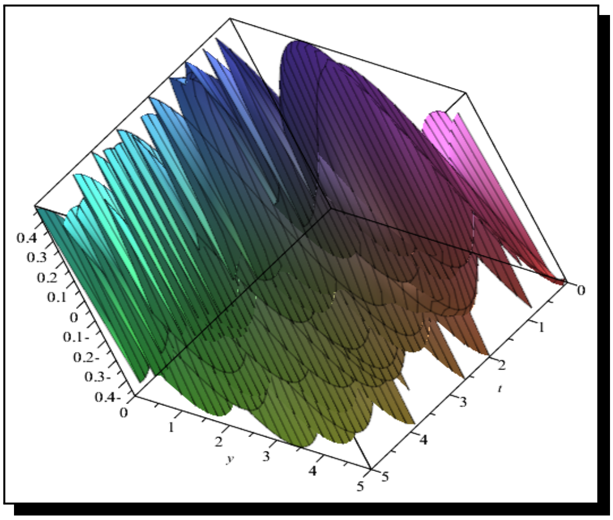

Figure 2. Geometric representation of P_1

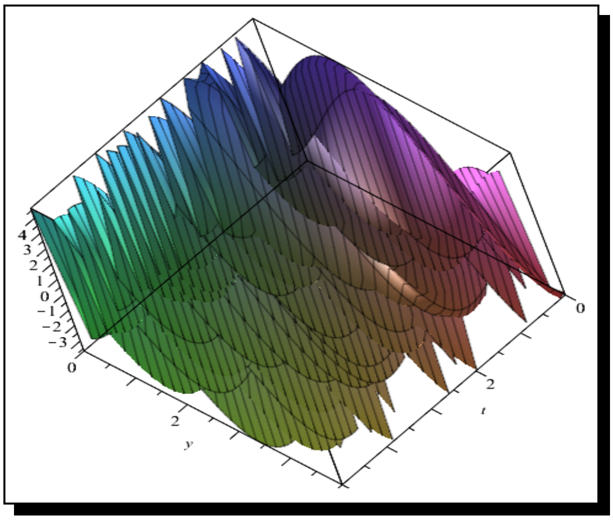

Figure 3. Geometric representation of P_2

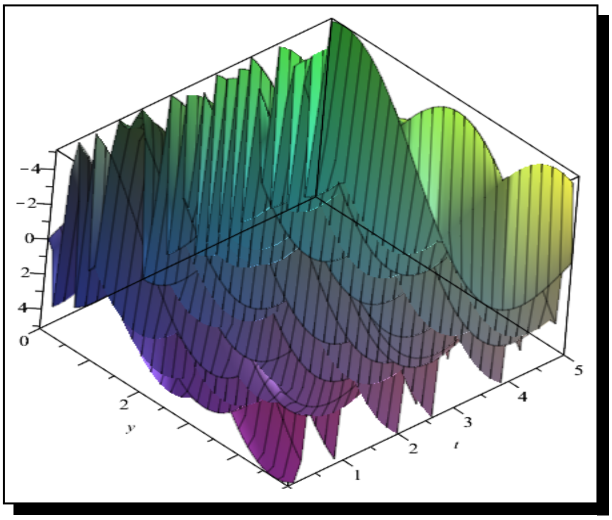

Figure 4. Geometric representation of P_3

3. Conclusion

In our research, we study the bifurcation of solutions of periodic traveling waves in nonlinear fractional differential equations using its fractional derivative and Lyapunov-Schmitt reduction. Converting a fractional differential equation to a partial differential equation, we employed the fractional complex transformation, followed by the traveling wave transform to reduce the partial differential equation to an *Ordinary Differential Equation* (ODE). This process led to the discovery of a simplified ODE, which consists of two nonlinear algebraic equations. We also gave geometric descriptions of solutions to nonlinear fractional differential equations. Linear approximation. Finally, we demonstrate the applicability of the examination of nonlinear fractional differential equations using topological techniques, offering the possibility of obtaining deeper insights into their behaviour.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. O. AL-Humedi and F. L. Hasan, The numerical solutions of nonlinear time-fractional differential equations by LMADM, *Iraqi Journal of Science* **2021**(Special issue 2) (2021), pp. 17 – 26, DOI: 10.24996/ijs.2021.si.2.2.

[2] M. Amir, Q. Ali, A. Raza, M. Y. Almusawa, W. Hamali and A. H. Ali, Computational results of convective heat transfer for fractionalized Brinkman type tri-hybrid nanofluid with ramped temperature and non-local kernel, *Ain Shams Engineering Journal* **15**(3) (2024), 102576, DOI: 10.1016/j.asej.2023.102576.

[3] U. Arshad, M. Sultana, A. H. Ali, O. Bazighifan, A. A. Al-Moneef and K. Nonlaopon, Numerical solutions of fractional-order electrical RLC circuit equations via three numerical techniques, *Mathematics* **10**(17) (2022), 3071, DOI: 10.3390/math10173071.

[4] R. Ashraf, R. Nawaz, O. Alabdali, N. Fewster-Young, A. H. Ali, F. Ghanim and A. A. Lupaş, A new hybrid optimal auxiliary function method for approximate solutions of non-linear fractional partial differential equations, *Fractal and Fractional* **7**(9) (2023), 673, DOI: 10.3390/fractalfract7090673.

[5] M. S. Berger, *Nonlinearity and Functional Analysis: Lectures on Nonlinear Problem in Mathematical Analysis*, 1st edition, Academic Press, Inc., 418 pages (1977).

[6] F. Ghanim, F. S. Khan, A. H. Ali and A. Atangana, Generalized Mittag-Leffler-confluent hypergeometric functions in fractional calculus integral operator with numerical solutions, *Journal of Mathematical Analysis and Applications* **543**(2) (Part 2) (2025), 128917, DOI: 10.1016/j.jmaa.2024.128917.

[7] F. Ghanim, F. S. Khan, H. F. Al-Janaby and A. H. Ali, A new hybrid special function class and numerical technique for multi-order fractional differential equations, *Alexandria Engineering Journal* **104** (2024), 603 – 613, DOI: 10.1016/j.aej.2024.08.009.

[8] S. Guo, L. Q. Mei, Y. Li and Y. F. Sun, The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics, *Physics Letters A* **376**(4) (2012), 407 – 411, DOI: 10.1016/j.physleta.2011.10.056.

[9] M. A. A. Hussain, Bifurcation solutions of elastic beams equation with small perturbation, *International Journal of Mathematical Analysis* **3**(18) (2009), 879 – 888.

[10] M. A. A. Hussain, Nonlinear Ritz approximation for Fredholm functionals, *Electronic Journal of Differential Equations* **2015**(294) (2015), 1 – 11, URL: https://ejde.math.txstate.edu/Volumes/2015/294/abdul.pdf.

[11] F. S. Khan, M. Khalid, A. A. Al-Moneef, A. H. Ali and O. Bazighifan, Freelance model with Atangana-Baleanu Caputo fractional derivative, *Symmetry* **14**(11) (2022), 2424, DOI: 10.3390/sym14112424.

[12] Z.-B. Li and J.-H. He, Converting fractional differential equations into partial differential equations, *Thermal Science* **16**(2) (2012), 331 – 334, DOI: 10.2298/tsci110503068h.

[13] Z.-B. Li and J.-H. He, Fractional complex transform for fractional differential equations, *Mathematical and Computational Applications* **15**(5) (2010), 970 – 973, DOI: 10.3390/mca15050970.

[14] F.-J. Liu, Z.-B. Li, S. Zhang and H.-Y. Liu, He's fractional derivative for heat conduction in a fractal medium arising in silkworm cocoon hierarchy, *Thermal Science* **19**(4) (2015), 1155 – 1159, DOI: 10.2298/tsci1504155l.

[15] B. V. Loginov, *Branching Theory of Solutions of Nonlinear Equations under Group Invariance Conditions*, Fan, Tashkent, (1985).

[16] B. Lu, The first integral method for some time fractional differential equations, *Journal of Mathematical Analysis and Applications* **395**(2) (2012), 684 – 693, DOI: 10.1016/j.jmaa.2012.05.066.

[17] A. A. Mizeal and M. A. A. Hussain, Two-mode bifurcation in solution of a perturbed nonlinear fourth order differential equation, *Archivum Mathematicum* **48**(1) (2012), 27 – 37, DOI: 10.5817/am2012-1-27.

[18] Yu. I. Sapronov and V. R. Zachepa, *Local Analysis of Fredholm Equation*, Voronezh University, Voronezh, Russia (2002).

[19] Yu. I. Sapronov, Finite-dimensional reductions of smooth extremal problems, *Russian Mathematical Surveys* **51**(1) (1996), 97 – 127, DOI: 10.1070/RM1996v051n01ABEH002741.

[20] Yu. I. Sapronov, Regular perturbation of Fredholm maps and theorem about odd field, Works Department of Mathematics, Voronezh University, Voronezh, V. 10 (1973), 82.88.

[21] M. M. Vainberg and V. A. Trenogin, *Theory of Branching Solutions of Nonlinear Equations*, Wolters-Noordhoff B.V., 510 pages (1974).

[22] S. Zhang and H.-Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Physics Letters A* **375**(7) (2011), 1069 – 1073, DOI: 10.1016/j.physleta.2011.01.029.