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Research Article

Coefficient Bounds for Bi-Univalent Functions With Ruscheweyh Derivative and Sălăgean Operator

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Abstract. This paper inaugurate two subclasses of bi-univalent functions on open unit disk Δ and obtain estimates on the initial coefficient for the functions in these subclasses by using Sălăgean and Ruscheweyh differential operators.

Keywords. Univalent functions, Bi-univalent function, Starlike and convex functions

Mathematics Subject Classification (2020). 30C45, 30C50

1. Introduction

Class of regular function is $\mathbb{M}$ with normalized condition $f(0) = 0 = f'(0) - 1$ on Δ and it is defined as $\Delta = \{z \in \mathbb{C}/|z| < 1\}$. Let $\mathcal{F}$ be the class of all functions, $f \in \mathbb{M}$ which are regular in Δ. Let $f(f^{-1}(w)) = w$, $\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$

$$f \in \mathcal{F}, \quad f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in \Delta. \tag{1.1}$$

Inverse of $f(z)$ is $f^{-1}(z)$ and defined as $f^{-1}(f(z))=z$, $z\in\Delta$ and $f(f^{-1})=w$, $\big(|w|<r_0(f);$ $r_0(f)\ge\frac{1}{4}\big)$, where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\dots, \tag{1.2}$$

$f\in\mathcal{F}$ is called as bi-univalent in the unit disk if f and f^{-1} are univalent in unit disk Δ.

Many authors worked on bi-univalent functions subclasses and obtained bounds, e.g., Bulut [2], Lewin [8], Porwal and Darus [9], Srivastava *et al*. [10], Xu *et al*. [12], and it is motivated from the work of Darus and Singh [4].

Definition 1.1. Let $\alpha\ge 0$, $n\in\mathbb{N}$. Denote by $\mathbb{L}_\alpha^n$ the operator given by $\mathbb{L}_\alpha^n f(z)=(1-\alpha)R^nf(z)+\alpha S^nf(z)$, $z\in\Delta$.

Remark 1.2. If $f(z)\in\mathbb{M}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, $z\in\Delta$ then

$$\mathbb{L}_\alpha^n f(z)=z+\sum_{j=2}^{\infty}\alpha j^n+(1-\alpha)C_{n+j-1}^{n}a_jz^j,\quad z\in\Delta.$$

This operator was studied by Frasin and Aouf [7].

Remark 1.3. If $f\in\mathbb{M}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then

$$\mathbb{S}^nf(z)=z+\sum_{j=2}^{\infty}j^na_jz^j,\quad z\in\Delta.$$

Remark 1.4. If $f\in\mathbb{M}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then

$$\mathbb{R}^nf(z)=z+\sum_{j=2}^{\infty}C_{n+j-1}^{n}a_jz^j,\quad z\in\Delta.$$

Definition 1.5. Let f defined by (1.1) is belongs to the class $\wp_\Sigma(n,\gamma,j)$ comply with the below mentioned criteria:
The subclass $\wp_\Sigma$ (n,γ,j) for $n\in\mathbb{Z}$, $0\le\gamma<1$, $\beta\ge1$, $\alpha\ge0$ of $\mathcal{F}$ for the function f of the form (1.1) satisfying the conditions:

$$f\in\Sigma\ \text{ and }\ \left|\arg\left(\frac{(1-\beta)L_\alpha^nf(z)+\beta L_\alpha^{n+1}f(z)}{z}\right)\right|<\frac{\gamma\pi}{2},\quad z\in\Delta, \tag{1.3}$$

$$f\in\Sigma\ \text{ and }\ \left|\arg\left(\frac{(1-\beta)L_\alpha^ng(w)+\alpha L_\alpha^{n+1}g(w)}{z}\right)\right|<\frac{\gamma\pi}{2},\quad z\in\Delta, \tag{1.4}$$

where

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\dots$$

and

$$L_\alpha^nf(z)=z+\sum_{j=2}^{\infty}\alpha j^n+(1-\alpha)C_{n+j-1}^{n}a_jz^j,\quad z\in\Delta,\alpha\ge0,n\in\mathbb{Z}.$$

This paper is sequel to some of the aforecited works (Darus and Singh [4], Porwal and Darus [9], Srivastava *et al*. [10], and Xu *et al*. [12]). Here, we introduce the new subclass $\wp_\Sigma(n,\gamma,j)$, $(0\le\gamma<1,\ \beta\ge1,\ \alpha\ge0,\ n\in\mathbb{Z})$ of analytic function class $\mathbb{M}$ with Ruscheweyh derivative and Sălăgean operator on the initial coefficients.

Lemma 1.6. *If $l \in \mathbb{L}$ then $|c_k| \le 2$ for each l, where $\mathbb{L}$ is the family of all functions $l(z)$ regular in Δ for which $Re\ l(z) > 0$, $l(z) = 1 + c_1 z + c_2 z^2 + \dots$ for $z \in \Delta$.*

2. Coefficient Estimates for $\wp_\Sigma(n,\gamma,j)$

Theorem 2.1. *Let $f(z)$ defined by (1.1) belongs to $\wp_\Sigma(n,\gamma,j)$, $j \in \mathbb{N}$, $n \in \mathbb{Z}$, $0 \le \gamma < 1$, $\beta \ge 1$, $\alpha \ge 0$ then*

$$|a_2| \le \frac{2\gamma}{\sqrt{\begin{pmatrix} 2\gamma[3^n\alpha(1+2\beta)+(1-\alpha)((1-\beta)C^n_{n+2}+\beta C^{n+1}_{n+3})] \\ \cdot(\gamma-1)[2^n\alpha(1+\beta)+(1-\gamma)((1-\beta)C^n_{n+1}+\beta C^{n+1}_{n+2})] \end{pmatrix}}}$$

and

$$|a_3| \le \frac{2\gamma}{(1-\beta)(3^n\alpha+(1-\alpha)C^n_{n+2})+\beta(3^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+3})} + \frac{4\gamma^2}{[(1-\beta)(2^n\alpha+(1-\alpha)C^n_{n+1})+\beta(2^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+2})]^2}.$$

Proof. From equation (1.3) and (1.4),

$$\frac{(1-\beta)L^n_\alpha f(z)+\beta L^{n+1}_\alpha f(z)}{z} = (b(z))^\gamma, \tag{2.1}$$

where $b(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots$ in $\mathcal{F}$. Now,

$$\frac{(1-\beta)L^n_\alpha g(w)+\beta L^{n+1}_\alpha g(w)}{w} = (h(w))^\gamma, \tag{2.2}$$

where $h(w) = 1 + h_1 w + h_2 w^2 + h_3 w^3 + \dots$ in $\mathbb{B}$

$$[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]a_2 = \gamma b_1 \tag{2.3}$$

$$[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]a_3 = \gamma b_2 + \frac{\gamma(\gamma-1)}{2}b_1^2, \tag{2.4}$$

$$-[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]a_2 = \gamma h_1, \tag{2.5}$$

$$[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})](2a_2^2-a_3) = \gamma h_2 + \frac{\gamma(\gamma-1)}{2}h_1^2. \tag{2.6}$$

From equation (2.3) and (2.5)

$$b_1 = -h_1, \tag{2.7}$$

$$2[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]^2 a_2^2 = \gamma^2(b_1^2+h_1^2). \tag{2.8}$$

From (2.4), (2.6) and (2.8)

$$2\gamma[3^n(1+2\beta)+(1-\alpha)((1-\beta)C^n_{n+2}+\beta C^{n+1}_{n+3})]a_2^2 \\ -(\gamma-1)[2^n\alpha(1+\beta)+(1-\alpha)((1-\beta)C^n_{n+1}+\beta C^{n+1}_{n+2})]^2 a_2^2 = \gamma^2(b_2+h_2),$$

$$a_2^2 = \frac{\gamma^2(b_2+h_2)}{\begin{pmatrix} 2\gamma[3^n(1+2\beta)+(1-\alpha)((1-\beta)C^n_{n+2}+\beta C^{n+1}_{n+3})] \\ -(\gamma-1)[2^n\alpha(1+\beta)+(1-\alpha)((1-\beta)C^n_{n+1}+\beta C^{n+1}_{n+2})]^2 \end{pmatrix}}. \tag{2.9}$$

Applying Lemma 1.6 for equation (2.9), we get

$$|a_2| \le \frac{2\gamma}{\sqrt{\begin{pmatrix} 2\gamma[3^n(1+2\beta)+(1-\alpha)((1-\beta)C^n_{n+2}+\beta C^{n+1}_{n+3})] \\ -(\gamma-1)[2^n\alpha(1+\beta)+(1-\alpha)((1-\beta)C^n_{n+1}+\beta C^{n+1}_{n+2})] \end{pmatrix}}}. \tag{2.10}$$

Now, subtracting (2.6) from (2.4)

$$2[(1-\beta)(3^n\alpha+(1-\alpha)C^n_{n+2})+\beta(3^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+3})](a_3-a_2^2)$$
$$=\gamma(b_2-h_2)+\frac{\gamma(\gamma-1)}{2}(b_1^2-h_1^2),$$

$$a_3=\frac{\gamma^2(b_1^2+h_1^2)}{2[(1-\beta)(2^n\alpha+(1-\alpha)C^n_{n+1})+\beta(2^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+2}]^2}$$
$$+\frac{\gamma(b_2-h_2)}{2[(1-\beta)(3^n\alpha+(1-\alpha)C^n_{n+2})+\beta(3^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+3})]}. \tag{2.11}$$

Applying Lemma 1.6 for (2.11), we get

$$|a_3| \le \frac{2\gamma}{(1-\beta)(3^n\alpha+(1-\alpha)C^n_{n+2})+\beta(3^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+3})}$$
$$+\frac{4\gamma^2}{[(1-\beta)(2^n\alpha+(1-\alpha)C^n_{n+1})+\beta(2^{n+1}\alpha+(1-\alpha)C^{n+1}_{n+2})]^2}. \qquad \square$$

3. Coefficient Estimates for $\xi_\Sigma(n,\gamma,j)$

Definition 3.1. Let f defined by (1.1) is belongs $\xi_\Sigma(n,\gamma,j)$ comply with the below mentioned criteria:

The subclass $\xi_\Sigma(n,\gamma,j)$ for $n\in\mathbb{Z}$, $0\le\lambda<1$, $\beta\ge1$, $\alpha\ge0$ of $\mathcal{F}$ for the function f of the form (1.1) satisfying the conditions:

$$f\in\Sigma \text{ and } Re\left(\frac{(1-\beta)L^n_\alpha f(z)+\beta L^{n+1}_\alpha f(z)}{z}\right)>\lambda, \quad z\in\Delta, \tag{3.1}$$

$$f\in\Sigma \text{ and } Re\left(\frac{(1-\beta)L^n_\alpha g(w)+\beta L^{n+1}_\alpha g(w)}{w}\right)>\lambda, \quad z\in\Delta, \tag{3.2}$$

where

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\dots$$

and

$$L^n_\alpha f(z)=z+\sum_{j=2}^{\infty}\alpha j^n+(1-\alpha)C^n_{n+j-1}a_jz^j, \quad z\in\Delta,\ \alpha\ge0,\ n\in\mathbb{Z}.$$

Theorem 3.2. *Let $f(z)$ defined by (1.1) belongs to the class $\xi_\Sigma(n,\gamma,j)$, $n\in\mathbb{Z}$, $0\le\lambda<1$, $\beta\ge1$, $\alpha\ge0$. Then*

$$|a_2|\le\sqrt{\frac{2(1-\lambda)}{(1-\beta)(\alpha3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha3^{n+1}+(1-\alpha)C^{n+1}_{n+3})}}$$

and

$$|a_3| \leq \frac{4(1-\lambda)^2}{[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]^2} + \frac{2(1-\lambda)}{(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})}.$$

Proof. From (3.1) and (3.2),

$$\frac{(1-\beta)L^n_\alpha f(z)+\beta L^{n+1}_\alpha f(z)}{z} = \lambda+(1-\lambda)b(z), \tag{3.3}$$

where $b(z) = 1+b_1z+b_2z^2+b_3z^3+\ldots$ in $\mathcal{F}$,

$$\frac{(1-\beta)L^n_\alpha g(w)+\beta L^{n+1}_\alpha g(w)}{w} = \lambda+(1-\lambda)h(w), \tag{3.4}$$

where $h(w) = 1+h_1w+h_2w^2+h_3w^3+\ldots$ in $\mathbb{B}$.
Comparing coefficients,

$$[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]a_2 = (1-\lambda)b_1, \tag{3.5}$$

$$[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]a_3 = (1-\lambda)b_2, \tag{3.6}$$

$$-[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]a_2 = h_1(1-\lambda), \tag{3.7}$$

$$[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})](2a_2^2-a_3) = h_2(1-\lambda). \tag{3.8}$$

From (3.5) and (3.7)

$$b_1 = -h_1. \tag{3.9}$$

Squaring and adding (3.5) and (3.7)

$$2[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]^2a_2^2 = (1-\lambda)^2(b_1^2+h_1^2). \tag{3.10}$$

From (3.6) and (3.8)

$$2[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]a_2^2 = (1-\lambda)(b_2+h_2), \tag{3.11}$$

$$a_2^2 = \frac{(1-\lambda)(b_2+h_2)}{2[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]}, \tag{3.12}$$

$$|a_2^2| = \frac{4(1-\lambda)}{2[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]},$$

$$|a_2| \leq \sqrt{\frac{2(1-\lambda)}{[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]}}. \tag{3.13}$$

Subtracting (3.8) from (3.6)

$$2[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})](a_3-a_2^2) = (1-\lambda)(b_2-h_2), \tag{3.14}$$

$$a_3 = \frac{(1-\lambda)(b_2-h_2)}{2[(1-\beta)(\alpha 3^n+(1-\alpha)C^n_{n+2})+\beta(\alpha 3^{n+1}+(1-\alpha)C^{n+1}_{n+3})]}+a_2^2. \tag{3.15}$$

On applying Lemma 1.6 we get

$$|a_3| \leq \frac{4(1-\lambda)^2}{[(1-\beta)(\alpha 2^n+(1-\alpha)C^n_{n+1})+\beta(\alpha 2^{n+1}+(1-\alpha)C^{n+1}_{n+2})]^2}$$

$$+\frac{2(1-\lambda)}{(1-\beta)(\alpha 3^n+(1-\alpha)C_{n+2}^n)+\beta(\alpha 3^{n+1}+(1-\alpha)C_{n+3}^{n+1})}.$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Mathematical Analysis and its Applications* (Proceedings of the International Conference on Mathematical Analysis and its Applications, Kuwait, 1985), 1988, pp. 53 – 60, DOI: 10.1016/B978-0-08-031636-9.50012-7.

[2] S. Bulut, A new subclass of analytic functions defined by generalized Ruscheweyh differential operator, *Journal of Inequalities and Applications* **2008** (2008), Article number: 134932, DOI: 10.1155/2008/134932.

[3] L.-I. Cotîrlă, New classes of analytic and bi-univalent functions, *AIMS Mathematics* **6**(10) (2012), 10642 – 10651, DOI: 10.3934/math.2021618.

[4] M. Darus and S. Singh, On some new classes of bi-univalent functions, *Journal of Applied Mathematics, Statistics and Informatics* **14**(2) (2018), 19 – 26, DOI: 10.2478/jamsi-2018-0010.

[5] S. S. Ding, Y. Ling and G. J. Bao, Some properties of a class of analytic functions, *Journal of Mathematical Analysis and Applications* **195**(1) (1995), 71 – 81, DOI: 10.1006/jmaa.1995.1342.

[6] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).

[7] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters* **24**(9) (2011), 1569 – 1573, DOI: 10.1016/j.aml.2011.03.048.

[8] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American Mathematical Society* **18** (1967), 63 – 68, DOI: 10.2307/2035225.

[9] S. Porwal and M. Darus, On a new subclass of bi-univalent functions, *Journal of the Egyptian Mathematical Society* **21**(3) (2013), 190 – 193, DOI: 10.1016/j.joems.2013.02.007.

[10] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Applied Mathematics Letters* **23**(10) (2010), 1188 – 1192, DOI: 10.1016/j.aml.2010.05.009.

[11] H. Tang, N. Magesh, V. K. Balaji and C. Abirami, Coefficient inequalities for a comprehensive class of bi-univalent functions related with bounded boundary variation, *Journal of Inequalities and Applications* **2019** (2019), Article number: 237, DOI: 10.1186/s13660-019-2193-5.

[12] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Applied Mathematics and Computation* **218**(23) (2012), 11461 – 11465, DOI: 10.1016/j.amc.2012.05.034.