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Research Article

The Correspondence Between Graphs and Alexandroff Spaces

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Abstract. In this paper, we study the correspondence between graphs and Alexandroff spaces. It is shown that a topological space X is Alexandroff if and only if X is a graph equipped with the X-right topology.

Keywords. Graph, Spectral, Prime spectrum, Ring, Alexandroff space

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1. Introduction

In a discussion of an arbitrary topological space, an important axiom for a topology is that a finite intersection of open sets must result in an open set. However, what if we discuss a topology such that an arbitrary intersection of open sets is open? Examples of this kind of space are often far from the well known topologies (the usual topology on the real line, the co-finite topology). One can give examples of such space: finite spaces and the discrete space. We call such spaces Alexandroff spaces.

According to [1], an *Alexandroff space* (or an space with the property of Alexandroff) is a topological space such that every point has a minimal neighborhood, or equivalently, has unique minimal base. This is also equivalent to the fact that the intersection of every family of open sets is open.

The interest in Alexandroff spaces was a consequence of the very important role of finite spaces in digital topology and the fact that these spaces have all the properties of finite spaces relevant for such theory (see [5], [2]).

In Section 2 we shall show that we can view each topological space as a preorder and that, in fact, topological spaces and preorders are, essentially, the same. This correspondence permit us to find a very interesting way to analyze topology: as a preorder. We can even use this to find equivalences between ideas grounded in preorders and ideas in topology, and how an ordering-based concept can imply a topological concept (and vice versa). We also give some properties of Alexandroff spaces.

A graph G is an ordered pair (V,E) consisting of a set V of vertices and a set E of edges. Any graph G can be represented by a topological space in the following sense:

(i) V is represented by a collection of distinct point in $\mathbb{R}^3$;

(ii) E is represented by a collection of distinct, internally disjoint arcs, homeomorphic to the closed interval $[0,1]$ such that boundary points of the arcs represent the endpoints of the corresponding edge.

In this paper we give new topologies on a graph and we show the correspondence between Alexandroff spaces and graphs.

2. Separation Axioms, Quasi-Compact Subset and Irreducible Subset

- A topological space X is a T_0-*space* (or Kolmogorov space) if for every pair of distinct points x and y, there exists a neighborhood containing one of them but not the other; which is equivalent to the following implication ($\overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y$).
- A topological space X is a T_1-*space* (or accessible space) if for every pair of distinct points x and y, there exists a neighborhood containing x (resp. y) and not containing y (resp. x); which is equivalent to the fact that every singleton is a closed set of X.

A topological space X is said quasi-compact if it satisfies the property of Borel-Lebesgue but it is not necessarily a Hausdorff space. A subset A of X is said quasi-compact if is a quasi-compact space equipped with the induced topology of X. We have the following properties:

(i) The quasi-compactness is invariant under continuous map.

(ii) Each closed subset of a quasi-compact space is quasi-compact.

(iii) The union of finitely many quasi-compact subsets is quasi-compact.

The intersection of tow quasi-compact open subsets is not necessarily quasi-compact [3]. The following example confirm this result:

Example 2.1. In the two Euclidean space we consider the following points: $C(0,1)$, $A(-1,0)$, $B(1,0)$, $A_n\left(-1,\frac{-1}{n}\right)$ and $B_n\left(1,\frac{-1}{n}\right)$. Let X be the set $\{C,A,B,A_n,B_n : n \geq 1\}$ equipped with the following topology: $\{\emptyset, X, U = \{A, A_n : n \geq 1\}, V = \{B, B_n : n \geq 1\}, U_n = \{A_p : n \geq p \geq 1\}, V_n = \{B_p :$

$n \geq p \geq 1\}\}$. U and V are two quasi-compact open subsets but $U \cap V$ is not quasi-compact because $U \cap V = \bigcup_n U_n \cup V_n$ and $U_n \cup V_n$ is an increasing sequence of open subsets.

A closed subset C is irreducible if it is not the union of two proper closed subsets or if the intersection of two nonempty open subsets is nonempty. An element x of C is called a generic point if the closure of the singleton $\{x\}$ is equal to $C : \overline{\{x\}} = C$.

3. Alexandroff Spaces

3.1 Topology and Preorder

Topological spaces and preordered sets are basically the same objects considered from different perspectives. The correspondence between spaces and preordered sets can be described as follows. Given a topology τ on X. Associated to τ, there is a preorder structure on X (i.e., a reflexive and transitive relation), defined by $x \leq y$ if $x \in \overline{\{y\}}$. Conversely, if a preorder $\leq$ on the set X is given, we define for each $x \in X$ the subset $U_x = \{y \in X \mid y \leq x\} =]\leftarrow, x]$. It is easy to see that these subsets form a basis for a topology on X, which is the topology associated to the preorder $\leq$ [4]. Note that, in [3] it was shown that an ordered set $(X, \leq)$ is order-isomorphic to the prime spectrum of a unitary commutative ring equipped with the inclusion.

The applications described above define a one-to-one correspondence between topological structures and preorders on X. Moreover, the T_0 separation axiom is equivalent to the antisymmetry of the associated preorder and therefore, T_0-topologies on X correspond to order relations. Having this equivalence, we will regard T_0-spaces as posets and vice-versa. We will use both structures according to convenience.

Let $(X, \leq)$ be an ordered set and $\mathcal{T}$ be a topology on X. We say that $\mathcal{T}$ is *compatible with* $\leq$ if, for each element $x \in X$, $\overline{\{x\}} = \{y \in X : x \leq y\} = [x, \rightarrow[$ ($\overline{\{x\}}$ is the closure of $\{x\}$) [6].

Proposition 3.1. *Let $(X, \leq)$ be an ordered set and $\mathcal{T}$ be a topology on X. If $\mathcal{T}$ is compatible with $\leq$, then $(X, \mathcal{T})$ is a T_0-space.*

Proof. Let x and y be two points of X.

- If $x < y$, then $x \in X - [y, \rightarrow[$ and so the open set $X - [y, \rightarrow[$ contains x and not contains y.
- If x and y are not comparable, then the open set $X - [x, \rightarrow[$ contains y and not contains x. □

Remark 3.2. If $(X, \mathcal{T})$ is a T_0-space, then X is an ordered set by the order defined by $x \leq_{\mathcal{T}} y$ if and only if $x \in \overline{\{y\}}$.

We have the following proposition:

Proposition 3.3. *If $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ are two homeomorphic T_0-spaces, then the ordered sets $(X, \leq_{\mathcal{T}})$ and $(X', \leq_{\mathcal{T}'})$ are isomorphic.*

Proof. Let h be a homeomorphism between $(X,\mathfrak{T})$ and $(X',\mathfrak{T}')$. If $x \leq_{\mathfrak{T}} y$, then $x \in \overline{\{y\}}^{\mathfrak{T}}$. Since h is continuous, $h(x) \in \overline{\{h(y)\}}^{\mathfrak{T}'}$ and so $h(x) \leq_{\mathfrak{T}'} h(y)$. If now $h(x) \leq_{\mathfrak{T}'} h(y)$, then, by the continuity of h^{-1}, $x \leq_{\mathfrak{T}} y$. Therefore, h is an isomorphism. □

The converse of Proposition 3.3 is false. Indeed, all the compatible topologies with an order $\leq$ induce the same order $\leq$ but are not necessarily homeomorphic. Note that, the (COP)-topology[1] and the left topology[2] of an ordered set $(X,\leq)$ are compatible with $\leq$ but not homeomorphic.

3.2 Properties of Alexandroff Spaces

Proposition 3.4. *Let (X,τ) be an Alexandroff space and let $V(x)$ be the intersection of all open sets containing x. Let $W(x) = \{y \in X : x \in V(y)\}$. Let $\mathcal{F}$ be the set of closed sets of τ. In the following we give some properties of Alexandroff spaces.*

(i) *$\{V(x) : x \in X\}$ is the only minimal base of τ;*

(ii) *The closure $\overline{\{x\}} = W(x)$;*

(iii) *$(X,\mathcal{F})$ is an Alexandroff space and $\{W(x) : x \in X\}$ is the only minimal base of $\mathcal{F}$. The closure for $\mathcal{F}$ $\overline{\{x\}}$ is equal to $V(x)$;*

(iv) *U is an open quasi-compact subset if and only if there exist a finitely many elements $x_1,\dots,x_n$ of U such that $U = \bigcup_{i=1}^{n} V(x_i)$;*

(v) *If (X,τ) is a T_0-space, then $W(x) = W(y)$ if and only if $x = y$.*

(vi) *If (X,τ) is a T_1-space, then τ is the discrete topology.*

Proof. (i) From [1];

(ii) Let $y \in \overline{\{x\}}$. Then every open set $V(y)$ of y contains x and so $y \in W(x)$. Conversely, let $y \in W(x)$, then $x \in V(y)$. Therefore, every open set containing y contains x. Thus $y \in \overline{\{x\}}$.

(iii) By duality;

(iv) By the definition of quasi-compactness and item (i);

(v) From [1];

(vi) If (X,τ) is a T_1-space, then $W(x) = \{x\}$, therefore τ is the discrete topology. □

By Proposition 3.4, we get immediately.

Proposition 3.5. *Let (X,τ) be an topological space. The following properties are equivalent:*

(i) *(X,τ) is an Alexandroff space;*

(ii) *$V(x)$ is an open set of τ, for all $x \in X$;*

(iii) *$(X,\mathcal{F})$ is an Alexandroff space;*

(iv) *$(X,\mathcal{F})$ is a topological space.*

[1]The (COP)-topology is generated by the family $\{X - [x, \to[: x \in X\}$.

[2]The left topology is generated by the family $\{[x, \to[: x \in X\}$.

Proposition 3.6. *Let (X,τ) be an Alexandroff space and F be an irreducible closed subset. If F is a quasi-compact subset of $(X,\mathfrak{F})$, then there exits $x \in F$ such that $\overline{\{x\}} = F$ (x is called a generic point of F).*

Proof. Since F is a closed subset, $F = \bigcup_{x \in F} \overline{\{x\}}$. According to the facts that $\overline{\{x\}}$ is open and F is quasi-compact in $(X,\mathfrak{F})$, there exist a finitely many elements $x_1,\dots,x_n$ of U such that $F = \bigcup_{i=1}^{n} \overline{\{x_i\}}$. Since F is irreducible, there exits $1 \le i \le n$ such that $F = \overline{\{x_i\}}$. □

4. Correspondence Between Alexandroff Spaces and Graphs

Let $G = (V,E)$ be a graph (finite or infinite) and let $u,v \in V$. A path from u to v in G is a sequence of edges $e_1,\dots,e_n$ of E for which there exists a sequence $x_0 = u$, $x_1,\dots,x_{n-1}$, $x_n = v$ of vertices such that e_i has, for $i = 1,\dots,n$, the endpoints x_{i-1} and x_i. We denote by

$$R(u) = \{u\} \cup \{v : \text{if there exists a path from } u \text{ to } v\},$$

$$L(u) = \{u\} \cup \{v : \text{if there exists a path from } v \text{ to } u\}.$$

The family $\{R(u) : u \in G\}$ (respectively $\{L(u) : u \in G\}$) forms a base of a topology on G called the G-right $\tau(G^R)$ (respectively, G-left $\tau(G^L)$) topology.

Two vertices a and b in a graph G are called adjacent in G if a and b are endpoints of an edge e of G. The graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism.

Definition 4.1. The graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are homeomorphic if $(G_1,\tau(G_1^R))$ and $(G_2,\tau(G_2^R))$ are homeomorphic.

Immediately, we obtain the following proposition.

Proposition 4.2. *Two isomorphic graphs are homeomorphic.*

Note that the converse of Proposition 4.2 is not true in general; indeed, one can see [7, Example 10, p. 673].

Proposition 4.3. *Suppose that the graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are reflexive. If G_1 and G_2 are homeomorphic, then they are isomorphic.*

Proof. Let $f : (G_1,\tau(G_1^R)) \to (G_2,\tau(G_2^R))$ be a homeomorphism. Let a and b be two adjacent vertices of G_1. Then $b \in R(a)$. Since f is continuous, $f(b) \in R(f(a))$. Since G_2 is transitive, $f(a)$ and $f(b)$ are two adjacent vertices of G_2. Since f^{-1} is continuous and G_1 is transitive, if $f(a)$ and $f(b)$ are two adjacent vertices of G_2, then a and b are adjacent vertices of G_1. Therefore, G_1 and G_2 are isomorphic. □

We can now state and prove the main result of this paper.

Theorem 4.4. *Let (X,τ) be a topological space. (X,τ) is an Alexandroff space if and only if X has the structure of a graph and τ is the right topology $\tau(X^R)$.*

Proof. *Sufficiency.* Let U be an open set of $(X,\tau(X^R))$ containing a point x. Then there exists $yinX$ such that $x\in R(y)\subset U$. Thus $R(x)\subset U$. Since $R(x)$ is an open set of $(X,\tau(X^R))$ containing x, $R(x)$ is the intersection of all open sets of $(X,\tau(X^R))$ containing x. Consequently, $(X,\tau(X^R))$ is an Alexandroff space (Proposition 3.5 (ii)).

Necessity. Let (X,τ) be an Alexandroff space. If $x\in X$, we denote by $V(x)$ be the intersection of all open sets containing x. We define on X the following graph structure: the vertices set $V=X$ and a and b are endpoints of an edge $e\in E$ (the set of edges) if $b\in R(a)$. Note that $R(a)=\{a\}\cup\{b:\text{if there exists a path from } a \text{ to } b\}$ which is equal to $V(a)$. Therefore, the family $(V(a))_{a\in X}$ is a base of open sets of (X,τ) and the family $(R(a))_{a\in X}$ is a base of open sets of $(X,\tau(X^R))$. Consequently, $\tau=\tau(X^R)$. □

5. Conclusion

We studied the correspondence between graphs and Alexandroff spaces. It is shown that a topological space X is Alexandroff if and only if X is a graph equipped with the X-right topology. In a future work, these correspondences permit us to find some applications of Alexandroff space in information retrieval theory.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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