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The Quasi-Hyperbolic Tribonacci and Quasi-Hyperbolic Tribonacci-Lucas Functions

Research Article

Dursun Taşci and Huriye Azman*
Department of Mathematics, Gazi University, Faculty of Science, Teknikokullar, 06500 Ankara, Turkey
***Corresponding author:** huriyeazman@gazi.edu.tr

Abstract. In the present paper, we studied an extension of the classical hyperbolic functions. We wrote a new relation that is equal to the Binet formula of the Tribonacci-Lucas numbers. We defined the quasi-hyperbolic Tribonacci and quasi-hyperbolic Tribonacci-Lucas functions. Finally, we investigated the recurrence and hyperbolic properties of these new hyperbolic functions.

Keywords. Hyperbolic functions; Binet's formula; Tribonacci numbers; Tribonacci-Lucas numbers

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1. Introduction

Spickerman defined the Binet's formula for Tribonacci sequence [5]. In [4], Stakhov and Rozin introduced the symmetrical hyperbolic functions. Then, Falcon and Plaza defined a new class of hyperbolic functions called k-Fibonacci hyperbolic functions [2]. Also, Kocer, Tuglu and Stakhov obtained the hyperbolic functions with second order recurrence sequences [3].

In this paper, we studied, in a sense, an extension of the classical hyperbolic functions. Our main goal is to get the continuous versions of Tribonacci and Tribonacci-Lucas numbers. For this, by using the same way with Spickerman [5], we wrote a new relation that is equal to the Binet formula of the Tribonacci-Lucas numbers. We defined the quasi-hyperbolic Tribonacci and quasi-hyperbolic Tribonacci-Lucas functions which we named it as quasi-hyperbolic because of it is non-hyperbolic, but it provides some properties of the classical hyperbolic functions. Finally, we investigated the hyperbolic and recurrence properties of these new functions.

1.1 The Tribonacci and Tribonacci-Lucas numbers

The Tribonacci numbers $\{U_n\}_{n\in\mathbb{N}} = \{0,1,1,2,4,7,13,24,\ldots\}$ is defined by [1]

$$U_{n+1} = U_n + U_{n-1} + U_{n-2}, \quad n \geq 2 \tag{1.1}$$

for the initial conditions

$$U_0 = 0, \quad U_1 = U_2 = 1.$$

The Tribonacci-Lucas numbers $\{V_n\}_{n\in\mathbb{N}} = \{3,1,3,7,11,21,\ldots\}$ is defined by [1]

$$V_{n+1} = V_n + V_{n-1} + V_{n-2}, \quad n \geq 2 \tag{1.2}$$

with the initial conditions

$$V_0 = 3, \quad V_1 = 1, \quad V_3 = 3.$$

The characteristic equation of these reoccurrence relations (1.1) and (1.2) is

$$x^3 - x^2 - x - 1 = 0. \tag{1.3}$$

This characteristic equation (1.3) has three roots

$$x_1 = \rho = \frac{1}{3}\left(\sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}} + 1\right), \tag{1.4}$$

$$x_2 = \sigma = \frac{1}{6}\left(2 - \sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}} + \sqrt{3}i\sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}}\right),$$

$$x_3 = \omega = \overline{\sigma}\ .$$

The Binet formula for the Tribonacci numbers is

$$U_n = \frac{\rho^{n+2}}{(\rho-\sigma)(\rho-\omega)} + \frac{\sigma^{n+2}}{(\sigma-\rho)(\sigma-\omega)} + \frac{\omega^{n+2}}{(\omega-\rho)(\omega-\sigma)}. \tag{1.5}$$

In [5], Spickerman defined a new representation which is equivalent to the Binet formula (1.5) after some operations

$$U_n = \lambda\rho^n + r^n(\psi\cos n\theta + \gamma\sin n\theta) \tag{1.6}$$

where the constants λ, ρ, r, ψ, γ, θ have the following approximate values

$$\lambda = 0,6184, \quad \rho = 1,8393,$$

$$r = 0,7374, \quad \psi = 0,3816,$$

$$\gamma = 0,0374, \quad \theta = 124,69^\circ$$

and the Binet formula for the Tribonacci-Lucas numbers is

$$V_n = \rho^{n+2} + \sigma^{n+2} + \omega^{n+2} \qquad [1]. \tag{1.7}$$

Now we will follow the same way with Spickerman to have a new relation that is equal to the Binet formula (1.7).

Definition 1. By using the relations

$$\sigma = r(\cos\theta + i\sin\theta),$$

$$\sigma^n = r^n(\cos n\theta + i\sin n\theta); \quad \theta = \tan^{-1}(I(\sigma)/R(\sigma))$$

where $I(\sigma)$ is the imaginary part of the σ, and $R(\sigma)$ is the real part of the σ, we have

$$V_n = \rho^2\rho^n + r^n\cos n\theta[2r^2(1-2\sin^2\theta)] + r^n\sin n\theta[-4r^2\sin\theta\cos\theta].$$

Denoting the coefficients of ρ^n, $r^n\cos n\theta$ and $r^n\sin n\theta$ respectively by λ', ψ' and γ', we write

$$V_n = \lambda'\rho^n + r^n(\psi'\cos n\theta + \gamma'\sin n\theta). \tag{1.8}$$

Approximate values for the constants are

$$\lambda' = 3,383, \qquad \rho = 1,8393,$$

$$r = 0,7374, \qquad \psi' = 0,7353,$$

$$\gamma' = 2,0357, \qquad \theta = 124,69^\circ.$$

2. New Quasi-hyperbolic Tribonacci Functions

Definition 2. Let x be a real number. We define the quasi-hyperbolic Tribonacci sine and cosine functions $sTh(x)$ and $cTh(x)$ by, respectively

$$sTh(x) := \lambda\rho^x - r^x(\psi\cos x\theta + \gamma\sin x\theta), \tag{2.1}$$

$$cTh(x) := \lambda\rho^x + r^x(\psi\cos x\theta + \gamma\sin x\theta). \tag{2.2}$$

The graphics of these functions are given in Figure 1.

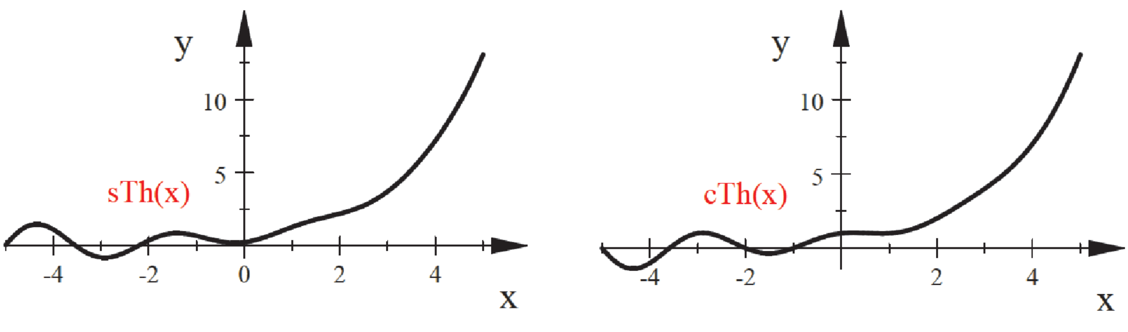

Figure 1. The quasi-hyperbolic Tribonacci sine and cosine.

2.1 Properties of the Quasi-hyperbolic Tribonacci Functions

Now, we will give some properties about the quasi-hyperbolic Tribonacci functions, which are similar to the classical hyperbolic functions. Let be $sTh(x)$ and $cTh(x)$ as in (2.1) and (2.2).

Proposition 3 (Pythagorean theorem)**.**

$$[cTh(x)]^2 - [sTh(x)]^2 = 4\lambda(\rho r)^x[\psi\cos x\theta + \gamma\sin x\theta].$$

Proof.

$$\begin{aligned}
&[cTh(x)]^2-[sTh(x)]^2\\
&\quad=[\lambda\rho^x+r^x(\psi\cos x\theta+\gamma\sin x\theta)]^2-[\lambda\rho^x-r^x(\psi\cos x\theta+\gamma\sin x\theta)]^2\\
&\quad=(\lambda\rho^x)^2+2(\lambda\rho^x)\cdot r^x(\psi\cos x\theta+\gamma\sin x\theta)+[r^x(\psi\cos x\theta+\gamma\sin x\theta)]^2\\
&\qquad-[(\lambda\rho^x)^2-2(\lambda\rho^x)\cdot r^x(\psi\cos x\theta+\gamma\sin x\theta)+[r^x(\psi\cos x\theta+\gamma\sin x\theta)]^2]\\
&\quad=4(\lambda\rho^x)r^x[\psi\cos x\theta+\gamma\sin x\theta]\\
&\quad=4\lambda(\rho r)^x[\psi\cos x\theta+\gamma\sin x\theta].
\end{aligned}$$

□

Proposition 4 (De Moivre)**.**

$$[cTh(x)+sTh(x)]^n=(2\lambda)^{n-1}[[cTh(nx)+sTh(nx)].$$

Proof. If we look at the LHS of the identity, we have

$$\begin{aligned}
[cTh(x)+sTh(x)]^n&=[\lambda\rho^x+r^x(\psi\cos x\theta+\gamma\sin x\theta)+\lambda\rho^x-r^x(\psi\cos x\theta+\gamma\sin x\theta)]^n\\
&=(2\lambda\rho^x)^n\\
&=(2\lambda)^n\rho^{nx}.
\end{aligned}$$

If we look at the RHS of the identity, we have

$$\begin{aligned}
&(2\lambda)^{n-1}[[cTh(nx)+sTh(nx)]\\
&\quad=(2\lambda)^{n-1}[\lambda\rho^{nx}+r^{nx}(\psi\cos nx\theta+\gamma\sin nx\theta)+\lambda\rho^{nx}-r^{nx}(\psi\cos nx\theta+\gamma\sin nx\theta)]\\
&\quad=(2\lambda)^{n-1}[2\lambda\rho^{nx}]\\
&\quad=(2\lambda)^n\rho^{nx}.
\end{aligned}$$

So the proof is complete. □

Proposition 5 (Sum)**.**

$$2\psi(cTh(x+y))=cTh(x)\cdot cTh(y)+sTh(x)\cdot sTh(y)+2\rho^{x+y}(\lambda\psi-\lambda^2)-2r^{x+y}\sin x\theta\sin y\theta(\psi^2-\gamma^2),$$
$$2\psi(sTh(x+y))=sTh(x)\cdot cTh(y)+cTh(x)\cdot sTh(y)+2\gamma^{x+y}\sin x\theta\sin y\theta(\psi^2-\gamma^2).$$

Proof. Firstly, let prove the first identity.

$$\begin{aligned}
&cTh(x)\cdot cTh(y)+sTh(x)\cdot sTh(y)+2\rho^{x+y}(\lambda\psi-\lambda^2)-2r^{x+y}\sin x\theta\sin y\theta(\psi^2-\gamma^2)\\
&\quad=[\lambda\rho^x+r^x(\psi\cos x\theta+\gamma\sin x\theta)][\lambda\rho^y+r^y(\psi\cos y\theta+\gamma\sin y\theta)]\\
&\qquad+[\lambda\rho^x-r^x(\psi\cos x\theta+\gamma\sin x\theta)][\lambda\rho^y-r^y(\psi\cos y\theta+\gamma\sin y\theta)]\\
&\qquad+2\rho^{x+y}(\lambda\psi-\lambda^2)-2r^{x+y}\sin x\theta\sin y\theta(\psi^2-\gamma^2)
\end{aligned}$$

$$
\begin{aligned}
&= [2\lambda^2\rho^{x+y} + 2r^{x+y}\psi^2\cos x\theta\cos y\theta + 2r^{x+y}\gamma^2\sin x\theta\sin y\theta \\
&\quad + 2r^{x+y}\beta\gamma\sin x\theta\cos y\theta + 2r^{x+y}\beta\gamma\cos x\theta\sin y\theta] \\
&\quad + 2\rho^{x+y}(\lambda\psi - \lambda^2) - 2r^{x+y}\sin x\theta\sin y\theta(\psi^2 - \gamma^2) \\
&= 2\psi[\lambda\rho^{x+y} + r^{x+y}(\psi(\cos x\theta\cos y\theta - \sin x\theta\sin y\theta) \\
&\quad + \gamma(\sin x\theta\cos y\theta + \cos x\theta\sin y\theta)] \\
&= 2\psi[\lambda\rho^{x+y} + r^{x+y}(\psi\cos(x+y)\theta + \gamma\sin(x+y)\theta)] \\
&= 2\psi(cTh(x+y)).
\end{aligned}
$$

Now, let prove the second identity.

$$
\begin{aligned}
&sTh(x)\cdot cTh(y) + cTh(x)\cdot sTh(y) + 2\gamma^{x+y}\sin x\theta\sin y\theta(\psi^2 - \gamma^2) \\
&\quad = [\lambda\rho^x - r^x(\psi\cos x\theta + \gamma\sin x\theta)][\lambda\rho^y + r^y(\psi\cos y\theta + \gamma\sin y\theta)] \\
&\qquad + [\lambda\rho^x + r^x(\psi\cos x\theta + \gamma\sin x\theta)][\lambda\rho^y - r^y(\psi\cos y\theta + \gamma\sin y\theta)] \\
&\qquad + 2\gamma^{x+y}\sin x\theta\sin y\theta(\psi^2 - \gamma^2) \\
&\quad = [2\lambda^2\rho^{x+y} - 2r^{x+y}\psi^2\cos x\theta\cos y\theta - 2r^{x+y}\gamma^2\sin x\theta\sin y\theta \\
&\qquad - 2r^{x+y}\beta\gamma\sin x\theta\cos y\theta - 2r^{x+y}\beta\gamma\cos x\theta\sin y\theta] + 2\gamma^{x+y}\sin x\theta\sin y\theta(\psi^2 - \gamma^2) \\
&\quad = 2\psi[\lambda\rho^{x+y} - r^{x+y}(\psi(\cos x\theta\cos y\theta - \sin x\theta\sin y\theta) + \gamma(\sin x\theta\cos y\theta + \cos x\theta\sin y\theta)] \\
&\quad = 2\psi[\lambda\rho^{x+y} - r^{x+y}(\psi\cos(x+y)\theta + \gamma\sin(x+y)\theta)] \\
&\quad = 2\psi(sTh(x+y)).
\end{aligned}
$$

□

Corollary 6 (Double argument)**.** *If we take $y = x$ in the previous formulas, then we have*

$$2\psi(cTh(2x)) = [cTh(x)]^2 + [sTh(x)]^2 + 2\rho^{2x}(\lambda\psi - \lambda^2) - 2r^{2x}\sin^2 x\theta(\psi^2 - \gamma^2),$$

$$2\psi(sTh(2x)) = 2sTh(x).cTh(x) + 2\gamma^{2x}\sin^2 x\theta(\psi^2 - \gamma^2).$$

Now we will study the Tribonacci's properties of the quasi-hyperbolic Tribonacci numbers.

Proposition 7 (Recursive relations)**.**

$$sTh(x+1) = sTh(x) + sTh(x-1) + sTh(x-2),$$

$$cTh(x+1) = cTh(x) + cTh(x-1) + cTh(x-2).$$

Proof. Let us prove the first equation. Let look at the LHS and RHS of the identity, respectively

$$
\begin{aligned}
sTh(x+1) &= \lambda\rho^{x+1} - r^{x+1}[\psi\cos(x+1)\theta + \gamma\sin(x+1)\theta] \\
&= \lambda\rho^{x+1} - r^{x+1}[\psi(\cos x\theta\cos\theta - \sin x\theta\sin\theta) + \gamma(\sin x\theta\cos\theta + \cos x\theta\sin\theta) \\
&= \lambda\rho^{x+1} - r^{x+1}\psi\cos x\theta\cos\theta + r^{x+1}\psi\sin x\theta\sin\theta - r^{x+1}\gamma\sin x\theta\cos\theta - r^{x+1}\gamma\sin x\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
&sTh(x)+sTh(x-1)+sTh(x-2)\\
&\quad=[\lambda\rho^x-r^x(\psi\cos x\theta+\gamma\sin x\theta)]+[\lambda\rho^{x-1}-r^{x-1}(\psi\cos(x-1)\theta+\gamma\sin(x-1)\theta)]\\
&\qquad+[\lambda\rho^{x-2}-r^{x-2}(\psi\cos(x-2)\theta+\gamma\sin(x-2)\theta)]\\
&\quad=\lambda[\rho^x+\rho^{x-1}+\rho^{x-2}]-r^x[\psi\cos x\theta+\gamma\sin x\theta]\\
&\qquad-r^{x-1}[\psi(\cos x\theta\cos\theta+\sin x\theta\sin\theta)+\gamma(\sin x\theta\cos\theta-\cos x\theta\sin\theta)]\\
&\qquad-r^{x-2}[\psi(\cos x\theta\cos 2\theta+\sin x\theta\sin 2\theta)+\gamma(\sin x\theta\cos 2\theta-\cos x\theta\sin 2\theta)]\\
&\quad=\lambda\rho^{x-2}(\rho^2+\rho+1)-r^x\psi\cos x\theta(1+r^{-1}\cos\theta+r^{-2}\cos 2\theta)-r^x\psi\sin x\theta(r^{-1}\sin\theta\\
&\qquad+r^{-2}\sin 2\theta)-r^x\gamma\sin x\theta(1+r^{-1}\cos\theta+r^{-2}\cos 2\theta)\\
&\qquad+r^x\gamma\cos x\theta(r^{-1}\sin\theta+r^{-2}\sin 2\theta)\\
&\quad=\lambda\rho^{x+1}-r^x\psi\cos x\theta(1+r^{-1}\cos\theta+r^{-2}\cos 2\theta)\\
&\qquad-r^x\psi\sin x\theta(r^{-1}\sin\theta+r^{-2}\sin 2\theta)-r^x\gamma\sin x\theta(1+r^{-1}\cos\theta+r^{-2}\cos 2\theta)\\
&\qquad+r^x\gamma\cos x\theta(r^{-1}\sin\theta+r^{-2}\sin 2\theta)
\end{aligned}
$$

By using $1+r^{-1}\cos\theta+r^{-2}\cos 2\theta=r\cos\theta$ and $r^{-1}\sin\theta+r^{-2}\sin 2\theta=-r\sin\theta$, we obtain the result which we look for. □

3. New Quasi-hyperbolic Tribonacci-Lucas Functions

Definition 8. Let x be a real number. We define the quasi-hyperbolic Tribonacci-Lucas sine and cosine functions $sTLh(x)$ and $cTLh(x)$ by, respectively

$$sTLh(x):=\lambda'\rho^x-r^x(\psi'\cos x\theta+\gamma'\sin x\theta), \tag{3.1}$$

$$cTLh(x):=\lambda'\rho^x+r^x(\psi'\cos x\theta+\gamma'\sin x\theta). \tag{3.2}$$

The graphics of the quasi-hyperbolic Tribonacci-Lucas sine and cosine are in Figure 2.

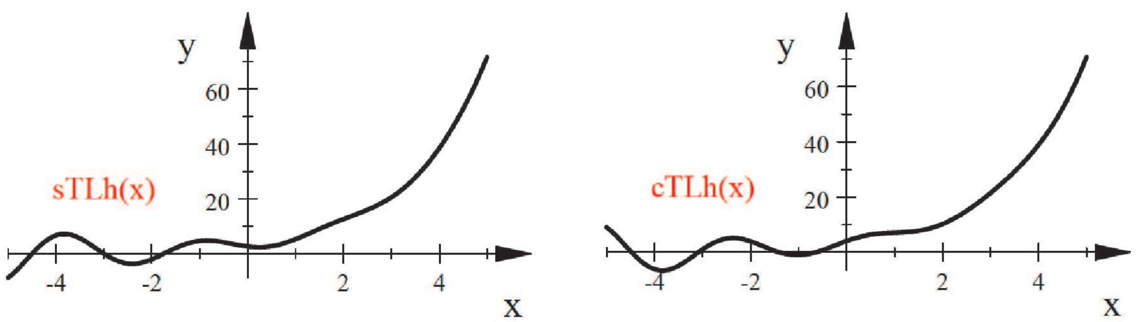

Figure 2. The quasi-hyperbolic Tribonacci-Lucas sine and cosine.

3.1 Properties of the Quasi-hyperbolic Tribonacci-Lucas Functions

Now, we will give some properties about the quasi-hyperbolic Tribonacci-Lucas functions, which are similar to the classical hyperbolic functions. Let be $sTLh(x)$ and $cTLh(x)$ as in (3.1) and (3.2).

Proposition 9 (Pythagorean theorem)**.**

$$[cTLh(x)]^2-[sTLh(x)]^2=4\lambda'(\rho r)^x[\psi'\cos x\theta+\gamma'\sin x\theta].$$

Proof.

$$\begin{aligned}
&[cTLh(x)]^2-[sTLh(x)]^2\\
&\quad=[\lambda'\rho^x+r^x(\psi'\cos x\theta+\gamma'\sin x\theta)]^2-[\lambda'\rho^x-r^x(\psi'\cos x\theta+\gamma'\sin x\theta)]^2\\
&\quad=(\lambda'\rho^x)^2+2(\lambda'\rho^x).r^x(\psi'\cos x\theta+\gamma'\sin x\theta)+[r^x(\psi'\cos x\theta+\gamma'\sin x\theta)]^2\\
&\qquad-[(\lambda'\rho^x)^2-2(\lambda'\rho^x).r^x(\psi'\cos x\theta+\gamma'\sin x\theta)+[r^x(\psi'\cos x\theta+\gamma'\sin x\theta)]^2]\\
&\quad=4(\lambda'\rho^x)r^x[\psi'\cos x\theta+\gamma'\sin x\theta]\\
&\quad=4\lambda'(\rho r)^x[\psi'\cos x\theta+\gamma'\sin x\theta].
\end{aligned}$$

□

Proposition 10 (De Moivre)**.**

$$[cTLh(x)+sTLh(x)]^n=(2\lambda')^{n-1}[[cTLh(nx)+sTLh(nx)].$$

Proof. If we look at the LHS of the identity, we have

$$\begin{aligned}
&[cTLh(x)+sTLh(x)]^n\\
&\quad=[\lambda'\rho^x+r^x(\psi'\cos x\theta+\gamma'\sin x\theta)+\lambda'\rho^x-r^x(\psi'\cos x\theta+\gamma'\sin x\theta)]^n\\
&\quad=(2\lambda'\rho^x)^n\\
&\quad=(2\lambda')^n\rho^{nx}.
\end{aligned}$$

If we look at the RHS of the identity, we have

$$\begin{aligned}
(2\lambda')^{n-1}[[cTLh(nx)+sTLh(nx)]&=(2\lambda')^{n-1}[\lambda'\rho^{nx}+r^{nx}(\psi'\cos nx\theta+\gamma'\sin nx\theta)\\
&\quad+\lambda'\rho^{nx}-r^{nx}(\psi'\cos nx\theta+\gamma'\sin nx\theta)]\\
&=(2\lambda')^{n-1}[2\lambda'\rho^{nx}]\\
&=(2\lambda')^n\rho^{nx}.
\end{aligned}$$

So the proof is complete. □

Proposition 11 (Sum)**.**

$$\begin{aligned}
2\psi'(cTLh(x+y))&=cTLh(x)\cdot cTLh(y)+sTLh(x)\cdot sTLh(y)+2\rho^{x+y}(\lambda'\psi'-\lambda'^2)\\
&\quad-2r^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
2\psi'(sTLh(x+y))&=sTLh(x)\cdot cTLh(y)+cTLh(x)\cdot sTLh(y)+2\gamma'^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2).
\end{aligned}$$

Proof. Let prove the first identity.

$$\begin{aligned}
&cTLh(x)\cdot cTLh(y)+sTLh(x)\cdot sTh(y)+2\rho^{x+y}(\alpha\beta-\alpha^2)-2r^{x+y}\sin x\theta\sin y\theta(\beta^2-\gamma'^2)\\
&\quad=[\lambda'\rho^x+r^x(\psi'\cos x\theta+\gamma'\sin x\theta)][\lambda'\rho^y+r^y(\psi'\cos y\theta+\gamma'\sin y\theta)]\\
&\qquad+[\lambda'\rho^x-r^x(\psi'\cos x\theta+\gamma'\sin x\theta)][\lambda'\rho^y-r^y(\psi'\cos y\theta+\gamma'\sin y\theta)]\\
&\qquad+2\rho^{x+y}(\lambda'\psi'-\lambda'^2)-2r^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
&\quad=[2\lambda'^2\rho^{x+y}+2r^{x+y}\psi'^2\cos x\theta\cos y\theta+2r^{x+y}\gamma'^2\sin x\theta\sin y\theta\\
&\qquad+2r^{x+y}\beta\gamma'\sin x\theta\cos y\theta+2r^{x+y}\beta\gamma'\cos x\theta\sin y\theta]\\
&\qquad+2\rho^{x+y}(\lambda'\psi'-\lambda'^2)-2r^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
&\quad=2\psi'[\lambda'\rho^{x+y}+r^{x+y}(\psi'(\cos x\theta\cos y\theta-\sin x\theta\sin y\theta)+\gamma'(\sin x\theta\cos y\theta+\cos x\theta\sin y\theta)]\\
&\quad=2\psi'[\lambda'\rho^{x+y}+r^{x+y}(\psi'\cos(x+y)\theta+\gamma'\sin(x+y)\theta)]\\
&\quad=2\psi'(cTLh(x+y)).
\end{aligned}$$

Now, let prove the second one

$$\begin{aligned}
&sTh(x)\cdot cTh(y)+cTh(x)\cdot sTh(y)+2\gamma'^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
&\quad=[\lambda'\rho^x-r^x(\psi'\cos x\theta+\gamma'\sin x\theta)][\lambda'\rho^y+r^y(\psi'\cos y\theta+\gamma'\sin y\theta)]\\
&\qquad+[\lambda'\rho^x+r^x(\psi'\cos x\theta+\gamma'\sin x\theta)][\lambda'\rho^y-r^y(\psi'\cos y\theta+\gamma'\sin y\theta)]\\
&\qquad+2\gamma'^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
&\quad=[2\lambda'^2\rho^{x+y}-2r^{x+y}\psi'^2\cos x\theta\cos y\theta-2r^{x+y}\gamma'^2\sin x\theta\sin y\theta\\
&\qquad-2r^{x+y}\beta\gamma'\sin x\theta\cos y\theta-2r^{x+y}\beta\gamma'\cos x\theta\sin y\theta]\\
&\qquad+2\gamma'^{x+y}\sin x\theta\sin y\theta(\psi'^2-\gamma'^2)\\
&\quad=2\psi'[\lambda'\rho^{x+y}-r^{x+y}(\psi'(\cos x\theta\cos y\theta-\sin x\theta\sin y\theta)+\gamma'(\sin x\theta\cos y\theta+\cos x\theta\sin y\theta)]\\
&\quad=2\psi'[\lambda'\rho^{x+y}-r^{x+y}(\psi'\cos(x+y)\theta+\gamma'\sin(x+y)\theta)]\\
&\quad=2\psi'(sTh(x+y)).
\end{aligned}$$

□

Corollary 12 (Double argument)**.** *By doing $y=x$ in the previous formulas, we have*

$$2\psi'(cTLh(2x))=[cTLh(x)]^2+[sTLh(x)]^2+2\rho^{2x}(\lambda'\psi'-\lambda'^2)-2r^{2x}\sin^2 x\theta(\psi'^2-\gamma'^2),$$

$$2\psi'(sTLh(2x))=2sTLh(x)\cdot cTLh(x)+2\gamma'^{2x}\sin^2 x\theta(\psi'^2-\gamma'^2).$$

Now we will study the Tribonacci's properties of the quasi-hyperbolic Tribonacci-Lucas numbers.

Proposition 13 (Recursive relations)**.**

$$sTLh(x+1)=sTLh(x)+sTLh(x-1)+sTLh(x-2),$$

$$cTLh(x+1)=cTLh(x)+cTLh(x-1)+cTLh(x-2).$$

Proof. Let us prove the first identity. So let look at the LHS and RHS of that identity, respectively

$$
\begin{aligned}
&sTLh(x+1)\\
&\quad=\lambda'\rho^{x+1}-r^{x+1}[\psi'\cos(x+1)\theta+\gamma'\sin(x+1)\theta]\\
&\quad=\lambda'\rho^{x+1}-r^{x+1}[\psi'(\cos x\theta\cos\theta-\sin x\theta\sin\theta)+\gamma'(\sin x\theta\cos\theta+\cos x\theta\sin\theta)\\
&\quad=\lambda'\rho^{x+1}-r^{x+1}\psi'\cos x\theta\cos\theta+r^{x+1}\psi'\sin x\theta\sin\theta-r^{x+1}\gamma'\sin x\theta\cos\theta-r^{x+1}\gamma'\sin x\theta\cos\theta,\\
&sTLh(x)+sTLh(x-1)+sTLh(x-2)\\
&\quad=[\lambda'\rho^{x}-r^{x}(\psi'\cos x\theta+\gamma'\sin x\theta)]+[\lambda'\rho^{x-1}-r^{x-1}(\psi'\cos(x-1)\theta+\gamma'\sin(x-1)\theta)]\\
&\qquad+[\lambda'\rho^{x-2}-r^{x-2}(\psi'\cos(x-2)\theta+\gamma'\sin(x-2)\theta)]\\
&\quad=\lambda'[\rho^{x}+\rho^{x-1}+\rho^{x-2}]-r^{x}[\psi'\cos x\theta+\gamma'\sin x\theta]\\
&\qquad-r^{x-1}[\psi'(\cos x\theta\cos\theta+\sin x\theta\sin\theta)+\gamma'(\sin x\theta\cos\theta-\cos x\theta\sin\theta)]\\
&\qquad-r^{x-2}[\psi'(\cos x\theta\cos2\theta+\sin x\theta\sin2\theta)+\gamma'(\sin x\theta\cos2\theta-\cos x\theta\sin2\theta)]\\
&\quad=\lambda'\rho^{x-2}(\rho^2+\rho+1)-r^{x}\psi'\cos x\theta(1+r^{-1}\cos\theta+r^{-2}\cos2\theta)-r^{x}\psi'\sin x\theta(r^{-1}\sin\theta+r^{-2}\sin2\theta)\\
&\qquad-r^{x}\gamma'\sin x\theta(1+r^{-1}\cos\theta+r^{-2}\cos2\theta)+r^{x}\gamma'\cos x\theta(r^{-1}\sin\theta+r^{-2}\sin2\theta)\\
&\quad=\lambda'\rho^{x+1}-r^{x}\psi'\cos x\theta(1+r^{-1}\cos\theta+r^{-2}\cos2\theta)-r^{x}\psi'\sin x\theta(r^{-1}\sin\theta+r^{-2}\sin2\theta)\\
&\qquad-r^{x}\gamma'\sin x\theta(1+r^{-1}\cos\theta+r^{-2}\cos2\theta)+r^{x}\gamma'\cos x\theta(r^{-1}\sin\theta+r^{-2}\sin2\theta).
\end{aligned}
$$

By using $1+r^{-1}\cos\theta+r^{-2}\cos2\theta=r\cos\theta$ and $r^{-1}\sin\theta+r^{-2}\sin2\theta=-r\sin\theta$, we obtain the result which we look for. □

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