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The k-Lucas Hyperbolic Functions

Research Article

Dursun Taşci and Huriye Azman*

Department of Mathematics, Gazi University, Faculty of Science, Teknikokullar, 06500 Ankara, Turkey
***Corresponding author:** huriyeazman@gazi.edu.tr

Abstract. In this paper, we studied and introduced an extension of the classical hyperbolic functions. Namely, we defined k-Lucas hyperbolic functions and studied their hyperbolic and recurrence properties, and looked at relationship this new k-Lucas hyperbolic functions between k-Fibonacci hyperbolic functions, which were studied before by Falcon and Plaza. We gave the definition of the quasi-sine k-Lucas function and some of the features associated with it.

Keywords. Hyperbolic functions; Lucas numbers; k-Lucas numbers

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1. Introduction

The Lucas sequence $\{L_n\} = \{2, 1, 3, 4, 7, 11, 18, \dots\}$ is the simplest and most well-known integer sequence, each term is equal to sum of previous two terms, beginning with the values $L_0 = 2$, $L_1 = 1$. Furthermore the ratio of two consecutive Lucas numbers converges to the Golden Mean (Golden Ratio), $\phi = \frac{1+\sqrt{5}}{2}$.

Stakhov and Rozin defined the symmetrical hyperbolic functions [11]. Later, Falcon and Plaza introduced a new class of hyperbolic functions, which are called k-Fibonacci hyperbolic functions [5]. Also, they studied hyperbolic and recurrence properties of these new type functions [5]. Then, Falcon introduced the k-Lucas numbers [2].

In this paper, we intrudeced a new class of hyperbolic functions, which we have named "the k-Lucas hyperbolic functions". Additionally, we studied hyperbolic and recurrence properties of these functions, and looked at the relationship between k-Fibonacci hyperbolic numbers. Finally, we mentioned the quasi-sine k-Lucas functions and some relations about them.

1.1 The k-Lucas Numbers

Falcon [2] defined the k-Lucas numbers that are given for any positive real number k by the following recurrence relation for $n \geq 1$:

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \tag{1.1}$$

with the initial values

$$L_{k,0} = 2, \quad L_{k,1} = k.$$

First k-Lucas numbers are:

$$L_{k,0} = 2,$$
$$L_{k,1} = k,$$
$$L_{k,2} = k^2 + 2,$$
$$L_{k,3} = k^3 + 3k,$$
$$L_{k,4} = k^4 + 4k^2 + 2,$$
$$L_{k,5} = k^5 + 5k^3 + 5k,$$
$$L_{k,6} = k^6 + 6k^4 + 9k^2 + 2.$$

Particularly, for the case $k = 1$, we get the classical Lucas numbers $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$, with the recurrence relation

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1} \quad \text{for } n \geq 1.$$

For the case $k = 2$, we get the classical Pell Lucas numbers $\{2, 2, 6, 14, 34, 82, 198, \ldots\}$, with the recurrence relation

$$P_0 = 2,\ P_1 = 2,\ P_{n+1} = 2P_n + P_{n-1} \quad \text{for } n \geq 1.$$

The characteristic equation for the recurrence relation of the k-Lucas numbers (1.1) is:

$$\sigma^2 = k\sigma + 1. \tag{1.2}$$

This characteristic equation (1.2) has two real root:

$$\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}, \qquad \sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}.$$

In particular, if we take $k = 1$ in σ_1, we have $\sigma_1 = \frac{1+\sqrt{5}}{2}$ known as the golden ratio ϕ. If we take $k = 2$ in σ_1, we have $\sigma_1 = 1 + \sqrt{2}$ known as the silver ratio. Finally, if we take $k = 3$ in σ_1, we have $\sigma_1 = \frac{3+\sqrt{13}}{2}$ known as the bronze ratio.

The Binet's formula for k-Lucas numbers defined by (see [2]):

$$L_{k,n} = \sigma_1^n + \sigma_2^n \tag{1.3}$$

where σ_1, σ_2 are the roots of the characteristic equation (1.2).

By using $\sigma_2 = -\frac{1}{\sigma_1}$, we can write the formula (1.3) as follows:

$$L_{k,n} = \sigma_1^n + (-1)^n \sigma_1^{-n}. \tag{1.4}$$

In [2], Falcon proved that: $\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = \sigma_1$ where σ_1 is the positive root of Eq. (1.2). It is obvious that, for the case $k = 1$, we get $\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = \phi$.

2. k-Lucas Hyperbolic Functions

The classical hyperbolic functions are defined by:

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Moreover, we know that the Lucas hyperbolic sine and cosine functions are respectively given by [11, 13, 15]:

$$sLh(x) = \phi^{(2x+1)} - \phi^{-(2x+1)},$$

$$cLh(x) = \phi^{2x} + \phi^{-2x}.$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

We can expand these functions to the k-Lucas hyperbolic functions as follows:

$$sL_kh(x) = \sigma_1^{(2x+1)} - \sigma_1^{-(2x+1)},$$

$$cL_kh(x) = \sigma_1^{2x} + \sigma_1^{-2x}.$$

where σ_1 is the positive root of the characteristic equation (1.2), that is $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$.

If we look at the graphics of k-Lucas hyperbolic sine and cosine functions, we see that $sL_kh(x)$ is symmetric with respect to the origin, while graphic of $cL_kh(x)$ presents a symmetry with respect to the axis $x = 0$. For this reason, hence, we will define the k-Lucas hyperbolic sine and cosine functions, respectively:

$$sL_kh(x) = \sigma_1^x - \sigma_1^{-x}, \tag{2.1}$$

$$cL_kh(x) = \sigma_1^x + \sigma_1^{-x},$$

since $\sigma_1 + \sigma_1^{-1} = \sqrt{k^2+4}$.

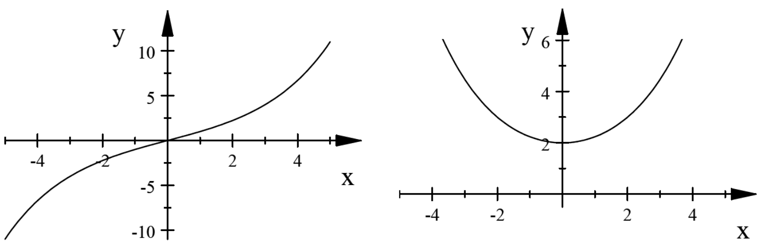

Figure 1

The k-Lucas numbers are determined through the k-Lucas hyperbolic functions as follows:

$$cL_kh(2n) = L_{k,2n},$$
$$sL_kh(2n+1) = L_{k,2n+1}.$$

There are the following relations between the k-Lucas hyperbolic functions and the classical hyperbolic functions:

$$sL_kh(x) = 2\sinh(x\ln\sigma_1),$$
$$cL_kh(x) = 2\cosh(x\ln\sigma_1).$$

Also, the k-Lucas hyperbolic functions and the k-Fibonacci hyperbolic functions are connected among themselves by the following simple correlations:

$$sF_kh(x) = \frac{sL_kh(x)}{\sqrt{k^2+4}},$$
$$cF_kh(x) = \frac{cL_kh(x)}{\sqrt{k^2+4}}.$$

2.1 Properties of the k-Lucas Hyperbolic Functions

Now, we will give some properties about the k-Lucas hyperbolic functions, which are similar to the classical hyperbolic functions.

Proposition 1 (Pythagorean theorem)**.**

$$[cL_kh(x)]^2 - [sL_kh(x)]^2 = 4.$$

Proof.

$$\begin{aligned}[cL_kh(x)]^2 - [sL_kh(x)]^2 &= (\sigma_1^x + \sigma_1^{-x})^2 - (\sigma_1^x - \sigma_1^{-x})^2 \\ &= \sigma_1^{2x} + 2 + \sigma_1^{-2x} - \sigma_1^{2x} + 2 - \sigma_1^{-2x} \\ &= 4.\end{aligned}$$

□

Proposition 2 (Sum and difference)**.**

$$2cL_kh(x \mp y) = cL_kh(x)cL_kh(y) \mp sL_kh(x)sL_kh(y),$$
$$2sL_kh(x \mp y) = sL_kh(x)cL_kh(y) \mp cL_kh(x)sL_kh(y).$$

Proof. Let us prove the first identity:

$$\begin{aligned}&cL_kh(x)cL_kh(y) + sL_kh(x)sL_kh(y) \\ &\quad = (\sigma_1^x + \sigma_1^{-x})(\sigma_1^y + \sigma_1^{-y}) + (\sigma_1^x - \sigma_1^{-x})(\sigma_1^y - \sigma_1^{-y}) \\ &\quad = \sigma_1^{x+y} + \sigma_1^{x-y} + \sigma_1^{-x+y} + \sigma_1^{-x-y} + \sigma_1^{x+y} - \sigma_1^{x-y} - \sigma_1^{-x+y} + \sigma_1^{-x-y} \\ &\quad = 2(\sigma_1^{x+y} + \sigma_1^{-(x+y)}) \\ &\quad = 2cL_kh(x+y).\end{aligned}$$

By doing $y = x$ in the first and third previous formula, we have:

$$cL_kh(2x) = \frac{1}{2}\left[(cL_kh(x))^2 + (sL_kh(x))^2\right],$$

$$sL_kh(2x) = sL_kh(x)cL_kh(x).$$ □

Proposition 3 (nth derivatives)**.**

$$(cL_kh(x))^{(n)} = \begin{cases}(\ln\sigma_1)^{(n)}sL_k(x), & if\ n = 2m+1,\\ (\ln\sigma_1)^{(n)}cL_k(x), & if\ n = 2m,\end{cases}$$

$$(sL_kh(x))^{(n)} = \begin{cases}(\ln\sigma_1)^{(n)}cL_k(x), & if\ n = 2m+1,\\ (\ln\sigma_1)^{(n)}sL_k(x), & if\ n = 2m.\end{cases}$$

2.2 Some Reoccurrences About the k-Lucas Hyperbolic Functions

Now, we will show some identities about the k-Lucas hyperbolic functions, which are related with the k-Lucas numbers.

Proposition 4 (Recursive relations)**.**

$$sL_kh(x+1) = kcL_kh(x) + sL_kh(x-1),$$

$$cL_kh(x+1) = ksL_kh(x) + cL_kh(x-1).$$

Proof. Let us prove the first identity:
Since $\sigma_1^2 = k\sigma_1 + 1$, then $\sigma_1^{x+1} = k\sigma_1^x + \sigma_1^{x-1}$.
In addition $\sigma_1^{-(x-1)} = k\sigma_1^{-x} + \sigma_1^{-(x+1)}$ then $k\sigma_1^{-x}$- $\sigma_1^{-(x-1)} = -\sigma_1^{-(x+1)}$.

$$\begin{aligned}kcL_kh(x) + sL_kh(x-1) &= k(\sigma_1^x + \sigma_1^{-x}) + (\sigma_1^{x-1} + \sigma_1^{-(x-1)})\\ &= k\sigma_1^x + k\sigma_1^{-x} + \sigma_1^{x-1} + \sigma_1^{-x+1}\\ &= (k\sigma_1^x + \sigma_1^{x-1}) + (k\sigma_1^{-x} + \sigma_1^{-x+1})\\ &= \sigma_1^{x+1} - \sigma_1^{-(x+1)}\\ &= sL_kh(x+1).\end{aligned}$$ □

Proposition 5 (Catalan's identities)**.**

$$cL_kh(x-r)cL_kh(x+r) - (cL_kh(x))^2 = (sL_kh(r))^2,$$

$$cL_kh(x-r)cL_kh(x+r) - (sL_kh(x))^2 = (cL_kh(r))^2,$$

$$sL_kh(x-r)sL_kh(x+r) - (sL_kh(x))^2 = -(sL_kh(r))^2,$$

$$sL_kh(x-r)sL_kh(x+r) - (cL_kh(x))^2 = -(cL_kh(r))^2.$$

Proof. Let us prove the first identity:

$$\begin{aligned}
&cL_kh(x-r)cL_kh(x+r)-(cL_kh(x))^2\\
&\quad=(\sigma_1^{x-r}+\sigma_1^{-(x-r)})(\sigma_1^{x+r}+\sigma_1^{-(x+r)})-(\sigma_1^{x}+\sigma_1^{-x})^2\\
&\quad=(\sigma_1^{x-r}+\sigma_1^{-x+r})(\sigma_1^{x+r}+\sigma_1^{-x-r})-(\sigma_1^{2x}+2+\sigma_1^{-2x})\\
&\quad=\sigma_1^{2x}+\sigma_1^{-2r}+\sigma_1^{2r}+\sigma_1^{-2x}-\sigma_1^{2x}-2-\sigma_1^{-2x}\\
&\quad=\sigma_1^{2r}-2+\sigma_1^{-2r}\\
&\quad=(sL_kh(r))^2.
\end{aligned}$$

By doing $r=1$ into Catalan's identities, Cassini or Simson's identities appear:

$$cL_kh(x-1)cL_kh(x+1)-(sL_kh(x))^2=k^2+4,$$

$$sL_kh(x-1)sL_kh(x+1)-(cL_kh(x))^2=-(k^2+4).$$

□

Proposition 6.

$$cL_kh(x)cL_kh(x+r)=cL_kh(2x+r)+cL_kh(r),$$

$$sL_kh(x)sL_kh(x+r)=cL_kh(2x+r)-cL_kh(r),$$

$$sL_kh(x)cL_kh(x+r)=sL_kh(2x+r)-sL_kh(r),$$

$$cL_kh(x)sL_kh(x+r)=sL_kh(2x+r)+sL_kh(r).$$

Proof. Let us prove the first identity:

$$\begin{aligned}
cL_kh(x)cL_kh(x+r)&=(\sigma_1^{x}+\sigma_1^{-x})(\sigma_1^{x+r}+\sigma_1^{-(x+r)})\\
&=\sigma_1^{2x+r}+\sigma_1^{-r}+\sigma_1^{r}+\sigma_1^{-2x-r}\\
&=(\sigma_1^{2x+r}+\sigma_1^{-(2x+r)})+(\sigma_1^{r}+\sigma_1^{-r})\\
&=cL_kh(2x+r)+cL_kh(r).
\end{aligned}$$

By doing $r=0$ in the previous equations it is obtained:

$$(cL_kh(x))^2=cL_kh(2x)+2,$$

$$(sL_kh(x))^2=cL_kh(2x)-2,$$

$$sL_kh(x)cL_kh(x)=sL_kh(2x).$$

□

2.3 Some Relations Between the k-Lucas Hyperbolic Numbers and the k-Fibonacci Hyperbolic Numbers

Now, we will give some correlations between the k-Lucas and the k-Fibonacci hyperbolic numbers, like previously shown between the k-Lucas and the k-Fibonacci numbers [2].

Proposition 7.

$$(sL_kh(x))^2 = (k^2+4)(cF_kh(x))^2 - 4,$$
$$(cL_kh(x))^2 = (k^2+4)(sF_kh(x))^2 + 4.$$

Proof. Let us prove the first identity:

$$\begin{aligned}(k^2+4)(cF_kh(x))^2 - 4 &= (k^2+4)\left(\frac{\sigma_1^x+\sigma_1^{-x}}{\sqrt{k^2+4}}\right)^2 - 4\\ &= \sigma_1^{2x} - 2 + \sigma_1^{-2x}\\ &= (sL_kh(x))^2.\end{aligned}$$

□

Proposition 8.

$$cL_kh(x) = cF_kh(x-1) + cF_kh(x+1),$$
$$sL_kh(x) = sF_kh(x-1) + sF_kh(x+1).$$

Proof. Let us prove the first identity:

$$\begin{aligned}cF_kh(x-1)+cF_kh(x+1) &= \left(\frac{\sigma_1^{x-1}+\sigma_1^{-(x-1)}}{\sqrt{k^2+4}}\right)+\left(\frac{\sigma_1^{x+1}+\sigma_1^{-(x+1)}}{\sqrt{k^2+4}}\right)\\ &= \frac{\sigma_1^{x-1}+\sigma_1^{-(x-1)}+\sigma_1^{x+1}+\sigma_1^{-(x+1)}}{\sqrt{k^2+4}}\\ &= \frac{(\sigma_1^{x-1}+\sigma_1^{x+1})+(\sigma_1^{-(x-1)}+\sigma_1^{-(x+1)})}{\sqrt{k^2+4}}\\ &= \frac{\sigma_1^{x}(\sigma_1^{-1}+\sigma_1)+\sigma_1^{-x}(\sigma_1+\sigma_1^{-1})}{\sqrt{k^2+4}}\\ &= cL_kh(x).\end{aligned}$$

□

Proposition 9.

$$(cL_kh(x))^2 + (sL_kh(x+1))^2 = (k^2+4)cF_kh(2x+1),$$
$$(sL_kh(x))^2 + (cL_kh(x+1))^2 = (k^2+4)cF_kh(2x+1).$$

Proof. Let us prove the first identity:

$$\begin{aligned}(cL_kh(x))^2+(sL_kh(x+1))^2 &= (\sigma_1^x+\sigma_1^{-x})^2+(\sigma_1^{x+1}-\sigma_1^{-(x+1)})^2\\ &= \sigma_1^{2x}+2+\sigma_1^{-2x}+\sigma_1^{2x+2}-2+\sigma_1^{-(2x+2)}\\ &= \sigma_1^{2x}(1+\sigma_1^2)+\sigma_1^{-2x}(1+\sigma_1^{-2})\end{aligned}$$

$$
\begin{aligned}
&= \sigma_1^{2x}\sigma_1\sqrt{k^2+4}+\sigma_1^{-2x}\left(\frac{\sqrt{k^2+4}}{\sigma_1}\right) \\
&= \sigma_1^{2x+1}\sqrt{k^2+4}+\sigma_1^{-2x-1}\sqrt{k^2+4} \\
&= \sqrt{k^2+4}(\sigma_1^{2x+1}+\sigma_1^{-(2x+1)}) \\
&= (k^2+4)\left(\frac{\sigma_1^{2x+1}+\sigma_1^{-(2x+1)}}{\sqrt{k^2+4}}\right) \\
&= (k^2+4)cF_kh(2x+1). \qquad \square
\end{aligned}
$$

Proposition 10.

$$sL_kh(x)cF_kh(x) = sF_kh(2x),$$

$$cL_kh(x)sF_kh(x) = sF_kh(2x).$$

Proof. Let us prove the first identity:

$$
\begin{aligned}
cL_kh(x)sF_kh(x) &= (\sigma_1^x+\sigma_1^{-x})\left(\frac{\sigma_1^x-\sigma_1^{-x}}{\sqrt{k^2+4}}\right) \\
&= \frac{\sigma_1^{2x}-\sigma_1^{-2x}}{\sqrt{k^2+4}} \\
&= sF_kh(2x). \qquad \square
\end{aligned}
$$

3. The Quasi-sine k-Lucas Function

In Eq. (1.4), Binet's formula for the k-Lucas sequence was written as follows:

$$L_{k,n} = \sigma_1^n + (-1)^n\sigma_1^{-n}.$$

In Eq. (2.1), we defined the k-Lucas hyperbolic sine function as:

$$sL_kh(x) = \sigma_1^x - \sigma_1^{-x}.$$

Furthermore, we know that, where $x \in \mathbb{R}$, if we take x as an odd number, $sL_kh(x)$ takes the value which correspond to the k-Lucas sequence, that is $sL_kh(x) = L_{k,n}$.

With them, using the well-known identity $\cos(n\pi) = (-1)^n$, we can give the following definition.

Definition 11. *The quasi-sine k-Lucas function defined by:*

$$Q_{L,k}(x) = \sigma_1^x + \cos(\pi x)\sigma_1^{-x}.$$

Note that $Q_{L,k}(x) = L_{k,n}$ for all integer n.

The graphics of $Q_{L,k}(x)$ for $k = 1,2,3$ are given in Figure 2.

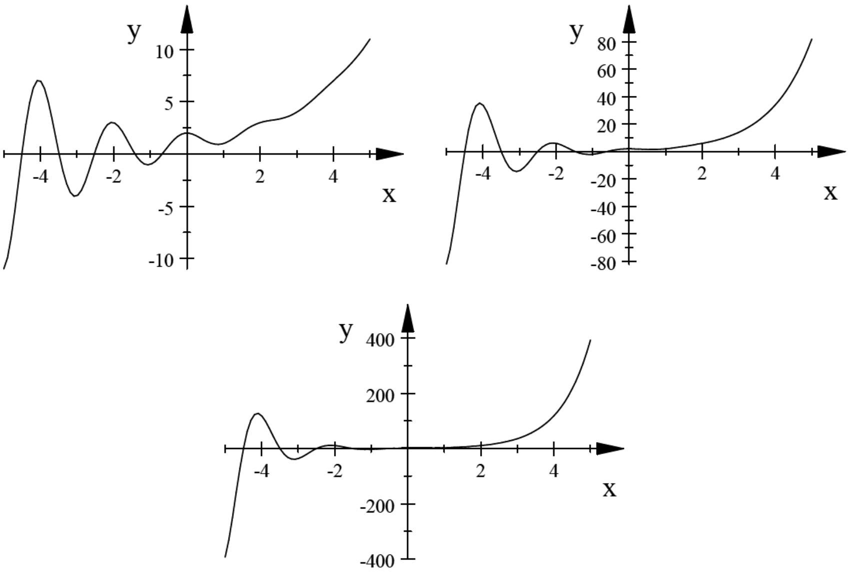

Figure 2

Figure 3. shows that the graphics of $Q_{L,k}(x)$ for $k = 1,2$ along with their evolving tangent curves which are the k-Lucas cosine and sine hyperbolic functions.

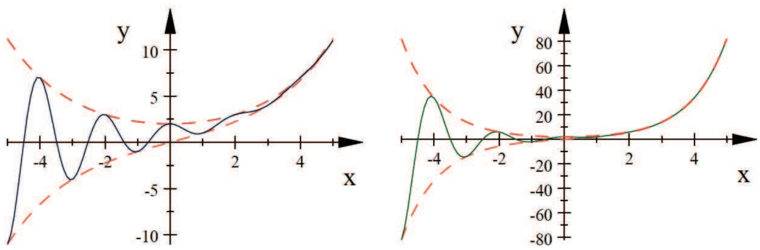

Figure 3

3.1 The Quasi-sine k-Lucas Functions and the k-Lucas Numbers

Now, we will give some relations about the quasi-sine k-Lucas functions, which are similar to the k-Lucas numbers.

Theorem 12 (Recursive relation)**.**

$$Q_{L,k}(x+2) = k\ Q_{L,k}(x+1) + Q_{L,k}(x).$$

Proof.

$$\begin{aligned} k\ Q_{L,k}(x+1) + Q_{L,k}(x) &= k\left[\sigma_1^{x+1} + \cos\pi(x+1)\sigma_1^{-x-1}\right] + \left[\sigma_1^{x} + \cos(\pi x)\sigma_1^{-x}\right] \\ &= k\left[\sigma_1^{x+1} - \cos(\pi x)\sigma_1^{-x-1}\right] + \left[\sigma_1^{x} + \cos(\pi x)\sigma_1^{-x}\right] \\ &= k\sigma_1^{x+1} - k\cos(\pi x)\sigma_1^{-x-1} + \sigma_1^{x} + \cos(\pi x)\sigma_1^{-x} \end{aligned}$$

$$
\begin{aligned}
&= \sigma_1^x(k\sigma_1+1)-\cos(\pi x)\sigma_1^{-x-2}(k\sigma_1-\sigma_1^2)\\
&= \sigma_1^{x+2}-\cos(\pi x)\sigma_1^{-x-2}(k\sigma_1-k\sigma_1-1)\\
&= \sigma_1^{x+2}-\cos(\pi x+2\pi)\sigma_1^{-x-2}(-1)\\
&= \sigma_1^{x+2}+\cos\pi(x+2)\sigma_1^{-(x+2)}\\
&= Q_{L,k}(x+2).
\end{aligned}
$$

□

Theorem 13 (Catalan's identity)**.**

$$Q_{L,k}(x-r)Q_{L,k}(x+r)-(Q_{L,k}(x))^2=(-1)^r\cos(\pi x)(Q_{L,k}(r))^2-4\cos(\pi x).$$

Proof.

$$
\begin{aligned}
&Q_{L,k}(x-r)Q_{L,k}(x+r)-(Q_{L,k}(x))^2\\
&\quad= \left[\sigma_1^{x-r}+\cos\pi(x-r)\sigma_1^{-x+r}\right]\left[\sigma_1^{x+r}+\cos\pi(x+r)\sigma_1^{-x-r}\right]-\left[\sigma_1^x+\cos(\pi x)\sigma_1^{-x}\right]^2\\
&\quad= \left[\sigma_1^{x-r}+(-1)^r\cos(\pi x)\sigma_1^{-x+r}\right]\left[\sigma_1^{x+r}+(-1)^r\cos(\pi x)\sigma_1^{-x-r}\right]\\
&\qquad-[\sigma_1^{2x}+2\cos(\pi x)+\cos^2(\pi x)\sigma_1^{-2x}]\\
&\quad= \sigma_1^{2x}+(-1)^r\cos(\pi x)\sigma_1^{-2r}+(-1)^r\cos(\pi x)\sigma_1^{2r}+(-1)^{2r}\cos^2(\pi x)\sigma_1^{-2x}\\
&\qquad-\sigma_1^{2x}-2\cos(\pi x)-\cos^2(\pi x)\sigma_1^{-2x}\\
&\quad= (-1)^r\cos(\pi x)[\sigma_1^r+\cos(\pi r)\sigma_1^{-r}]^2-4\cos(\pi x)\\
&\quad= (-1)^r\cos(\pi x)(Q_{L,k}(r))^2-4\cos(\pi x).
\end{aligned}
$$

□

Theorem 14. *For any integer r,*

$$\lim_{x\to\infty}\frac{Q_{L,k}(x+r)}{Q_{L,k}(x)}=\sigma_1^r.$$

Proof.

$$
\begin{aligned}
\lim_{x\to\infty}\frac{Q_{L,k}(x+r)}{Q_{L,k}(x)} &= \lim_{x\to\infty}\frac{\sigma_1^{x+r}+\cos\pi(x+r)\sigma_1^{-x-r}}{\sigma_1^x+\cos(\pi x)\sigma_1^{-x}}\\
&= \lim_{x\to\infty}\frac{\sigma_1^{x+r}+(-1)^r\cos(\pi x)\sigma_1^{-x-r}}{\sigma_1^x+\cos(\pi x)\sigma_1^{-x}}\\
&= \lim_{x\to\infty}\frac{\sigma_1^x(\sigma_1^r+(-1)^r\cos(\pi x)\frac{1}{\sigma_1^{2x}}}{\sigma_1^x(1+\cos(\pi x)\frac{1}{\sigma_1^{2x}}}\\
&= \sigma_1^r.
\end{aligned}
$$

In particular, taking $k=1$, we have that $\sigma_1=\phi$ and $\lim_{x\to\infty}\frac{Q_{L,k}(x+1)}{Q_{L,k}(x)}=\phi$. □

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