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Interpolating Some Classes of Operators between Families

Research Article

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Abstract. Many operator ideals (single operator ideals and chains, see inside) possess the strong property of interpolation for the J and K methods of Lions-Peetre, Sparr, Fernández and Cobos-Peetre. That is, let $\mathfrak{I}$ be one of the ideals considered here, let $\overline{A}$ and $\overline{B}$ be interpolation families and $T:\overline{A}\to\overline{B}$ a bounded linear operator then, the interpolated operator $T_{J,K}:J(\overline{A})\to K(\overline{B})$ belongs to $\mathfrak{I}$ if and only if the induced operator $T_{\mathscr{J}\mathscr{S}}$ from the intersection space $\mathscr{J}(\overline{A})$ into the sum space $\mathscr{S}(\overline{B})$ belongs to $\mathfrak{I}$.

Keywords. Real methods of interpolation between families; Operator ideals

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1. Introduction

Let $\mathfrak{I}$ be an operator ideal, let $\overline{A}$ and $\overline{B}$ be finite interpolation families (n-tuples of Banach spaces), let $J(\overline{A})$ and $K(\overline{B})$ be the spaces obtained from $\overline{A}$ and $\overline{B}$ by the J and K methods of interpolation respectively. The ideal $\mathfrak{I}$ possesses the (so called here for the lack of a better name) *Strong Property of Interpolation*, SPI in short, with respect to the J and K methods for families if the interpolated operator $T_{J,K}:J(\overline{A})\to K(\overline{B})$ belongs to $\mathfrak{I}$ when the induced operator $T_{\mathscr{J}\mathscr{S}}:\mathscr{J}(\overline{A})\to\mathscr{S}(\overline{B})$ from the intersection into the sum spaces is in $\mathfrak{I}$.

The ideals of separable operators, Rosenthal, weakly compact, Banach-Saks, alternate signs Banach-Saks, decomposing, their dual ideals and chains of them, possess the strong property of interpolation with respect to the J and K methods for finite families of Lions-Peetre, Sparr, Fernández and Cobos-Peetre.

In memory of my beloved Uncle, Diógenes

The paper is firstly concerned with the work of M. J. Carro ([3], [4], [5]) and that of S. Heinrich [11]. The roots (of the paper) go back to the famous and celebrated work of W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński [9].

The main result is achieved in §5, Theorem 5.4, after a necessary preamble. This result applies better when the J and K methods are equivalent, as in the case of Lions-Peetre method for pairs and as in the many examples proposed by Fernández and Sparr where their J and K methods are equivalent (see [10] and [19]).

The notation is standard; unexplained symbols, terms or concepts can be found in [14] and [17].

2. Preliminaries

Let $\mathfrak{L}(E,F)$ be the space of all bounded linear operators. An operator ideal $\mathfrak{I}$ is a class of bounded linear operators, such that the components $\mathfrak{I}(E,F) = \mathfrak{I} \bigcap \mathfrak{L}(E,F)$ satisfy the following conditions: (i) $\mathfrak{I}(E,F)$ is a linear subspace of $\mathfrak{L}(E,F)$, (ii) $\mathfrak{I}(E,F)$ contains the finite rank operators and (iii) if $R \in \mathfrak{L}(X,E), S \in \mathfrak{I}(E,F)$ and $T \in \mathfrak{L}(F,Y)$ then, $TSR \in \mathfrak{I}(X,Y)$ (see [17] and [11]).

The operator ideal is injective if for every isomorphic embedding $J \in \mathfrak{L}(F,Y)$ one has that $T \in \mathfrak{L}(E,F)$ and $JT \in \mathfrak{I}(E,Y)$ implies $T \in \mathfrak{I}(E,F)$; it is surjective if for every surjection $Q \in \mathfrak{L}(X,E)$ one has that $T \in \mathfrak{L}(E,F)$ and $TQ \in \mathfrak{I}(X,F)$ implies $T \in \mathfrak{I}(E,F)$. The ideal is closed if the components $\mathfrak{I}(E,F)$ are closed subspaces of $\mathfrak{L}(E,F)$ (see [17, Chapter 4] and [11]).

Every operator ideal $\mathfrak{I}$ defines a class of Banach spaces, *Space*$(\mathfrak{I})$, in the following way: $E \in$ *Space*$(\mathfrak{I})$ if and only if $1_E \in \mathfrak{I}(E,E)$.

Let S be a countable set and $(X_\alpha)_{\alpha\in S}$ a family of Banach spaces; denote by $\left(\sum_{\alpha\in S} X_\alpha\right)_p$, with $1 \le p < \infty$, the space of all the maps $(x_\alpha)_{\alpha\in S}$, such that $x_\alpha \in X_\alpha$ with the norm $\|(x_\alpha)_{\alpha\in S}\| = \left(\sum_{\alpha\in S} \|x_\alpha\|^p_{X_\alpha}\right)^{\frac{1}{p}} < \infty$.

For every $i,j \in S$ denote by J_i the natural embedding of X_i into $\left(\sum_{\alpha\in S} X_\alpha\right)_p$ and by Q_j the natural projection of $\left(\sum_{\alpha\in S} X_\alpha\right)_p$ onto X_j.

Definition 2.1 ([11])**.** The ideal $\mathfrak{I}$ satisfies the $\sum_p$-condition for $1 \le p < \infty$, if for any two families $(E_\alpha)_{\alpha\in S}$ and $(F_\alpha)_{\alpha\in S}$ of Banach spaces the following holds: if $T \in \mathfrak{L}\left(\left(\sum_{\alpha\in S} E_\alpha\right)_p, \left(\sum_{\alpha\in S} F_\alpha\right)_p\right)$ and $Q_j T J_i \in \mathfrak{I}(E_i,F_j)$ for every $i,j \in S$, then, $T \in \mathfrak{I}\left(\left(\sum_{\alpha\in S} E_\alpha\right)_p, \left(\sum_{\alpha\in S} F_\alpha\right)_p\right)$.

3. Real Methods of Interpolation for Finite Families

Beside the Lions-Peetre method of interpolation for pairs of Banach spaces, there are three methods of interpolation for finite families of Banach spaces. They are, in chronological order: the Sparr method (see [19]), the Fernández method for 2^d Banach spaces (see [10]) and the Cobos-Peetre method associated with the vertices of a convex polygon in $\mathbb{R}^2$ (see [7]). In all of them, both, the K and J functionals are defined by introducing a positive weight factor $\overline{\omega}$ (tuple of positive real numbers) in the norms of the sum and intersection spaces, being $\overline{\omega}$ chosen in a different way for each method.

3.1. María J. Carro in [3] and [4], proposed a method of real interpolation for families of Banach spaces that, beside the fact of being the natural setting to compare with the complex method of interpolation for families proposed by the St. Louis group in [8], provides a unified approach for the aforesaid real methods of interpolation for finite families. (For a comparison with other real methods of interpolation for families of Banach spaces see [6]). For a thorough study of her method we refer the reader to her nice and masterful work [3]. Here, we only give a brief description of her method.

Let D denote the unit disk, $D = \{z \in \mathbb{C} : |z| < 1\}$ and Γ its boundary. The family $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathscr{A}, \mathscr{U}\}$ is a complex *interpolation family* (i.f.) on Γ with $\mathscr{U}$ as the containing space and $\mathscr{A}$ as the log-intersection space if:

(a) for each γ, the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathscr{U}$; $\|\cdot\|_\gamma$ is the norm on $A(\gamma)$ and $\|\cdot\|_{\mathscr{U}}$ that on $\mathscr{U}$;

(b) for every $a \in \bigcap_{\gamma\in\Gamma} A(\gamma)$ the application $\gamma \to \|a\|_\gamma$ is a measurable function on Γ;

(c) If $\mathscr{A}$ is the *log-intersection* linear space

$$\mathscr{A} = \left\{a \in A(\gamma) \text{ for a.e. } \gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma d\gamma < \infty\right\}$$

with $\log^+ = \max(\log, 0)$, then, there exists a measurable function P on Γ such that

$$\int_\Gamma \log^+ P(\gamma) d\gamma < \infty \quad \text{and} \quad \|a\|_{\mathscr{U}} \le P(\gamma)\|a\|_\gamma \text{ for a.e. } \gamma, (a \in \mathscr{A}).$$

Let $\mathscr{L}$ be the multiplicative group defined by

$$\mathscr{L} = \{\alpha : \Gamma \to \mathbb{R}^+ ; \alpha \text{ is measurable with } \log \alpha \in L^1(\Gamma)\}$$

and $\mathscr{G}$ be the space of all $\mathscr{A}$-valued, simple and measurable functions on Γ. The space $\overline{\mathscr{G}}$ is that of all Bochner integrable (in $\mathscr{U}$) functions $a(\cdot)$ such that $a(\gamma) \in A(\gamma)$ for a.e. $\gamma \in \Gamma$ and such that $a(\cdot)$ can be a.e. approximated in the $A(\gamma)$-norm by a sequence of functions from $\mathscr{G}$.

For $\alpha \in \mathscr{L}$ and $a \in \mathscr{U}$, with $a = \int_\Gamma a(\gamma) d\gamma$, define the K-functional with respect to the i.f. $\overline{A}$ by:

$$K(\alpha, a) = \inf\left\{\int_\Gamma \alpha(\gamma)\|a(\gamma)\|_\gamma d\gamma\right\},$$

where the infimum is taken over all representations $a = \int_\Gamma a(\gamma) d\gamma$ (convergence in $\mathscr{U}$), with $a(\cdot) \in \overline{\mathscr{G}}$. And, for $a \in \mathscr{A}$ define the J-functional by

$$J(\alpha, a) = \operatorname*{ess\,sup}_{\gamma\in\Gamma} (\alpha(\gamma)\|a\|_\gamma).$$

For $\alpha \in \mathscr{L}$ and $z \in D$, define

$$\alpha(z) = \exp\left(\int_\Gamma \log \alpha(\gamma) P_z(\gamma) d\gamma\right),$$

where P_z is the Poisson kernel at $z \in D$, see [12].

Let $\overline{A}$ be an i.f., $S \subset \mathscr{L}$ a subgroup of $\mathscr{L}$ (such as those considered in the Section 3.2) and $1 \le p \le \infty$. Following Carro's notation (see nevertheless [6], Remark 1.1, where a change of notation is proposed), define the K-space $[A]^S_{z_0,p}$, as that of all $a \in \mathscr{U}$ for which

$$\left(\frac{K(\alpha, a)}{\alpha(z_0)}\right)_{\alpha \in S} \in \ell^p(S),$$

endowed with the norm

$$\|a\|_{[A]^S_{z_0,p}} = \left(\sum_{\alpha \in S}\left(\frac{K(\alpha, a)}{\alpha(z_0)}\right)^p\right)^{\frac{1}{p}};$$

(as always for $p = \infty$).

The J-space $(A)^S_{z_0,p}$, is defined as that of all $a \in \mathscr{U}$ for which there exists a map $(u(\alpha))_{\alpha \in S}$, from S into $\mathscr{A}$, so that $a = \sum_{\alpha \in S} u(\alpha)$ (convergence in the $\mathscr{U}$ norm) and

$$\left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)}\right)_{\alpha \in S} \in \ell^p(S), \tag{$*$}$$

endowed with the norm

$$\|a\|_{(A)^S_{z_0,p}} = \inf\left(\sum_{\alpha \in S}\left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)}\right)^p\right)^{\frac{1}{p}},$$

where the infimum extends over all representations of a.

In order to have that the spaces $[A]^S_{z_0,p}$, and $(A)^S_{z_0,p}$ are Banach spaces with the intermediate property ($\mathscr{A} \subset [A]^S_{z_0,p} \subset \mathscr{U}$ and $\mathscr{A} \subset (A)^S_{z_0,p} \subset \mathscr{U}$) and the usual embedding of $(A)^S_{z_0,p}$ into $[A]^S_{z_0,p}$, some natural conditions are necessary on the subgroup S. These conditions are (see [3] and [4, §2, p. 56]):

(i) for every $\alpha \in S$ there exists a constant C_α such that $P(\gamma) \le C_\alpha \alpha(\gamma)$, a.e. γ (see the definition of i.f.);

(ii) for every $z_0 \in D$, there exists a compact $K \subset D$ such that $\sum_{\alpha \in S} \frac{\inf_{z \in K} \alpha(z)}{\alpha(z_0)} < \infty$;

(iii) S is a multiplicative subgroup of $\mathscr{L}$.

Under these conditions, ($*$) implies the absolute convergence of $\sum_{\alpha \in S} u(\alpha)$ in $\mathscr{U}$.

Let $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathscr{A}, \mathscr{U}\}$ and $\overline{B} = \{B(\gamma) : \gamma \in \Gamma; \mathscr{B}, \mathscr{V}\}$ be two i.f.; let $T : \overline{A} \to \overline{B}$ be an interpolation operator, i.e., $T : \mathscr{U} \to \mathscr{V}$ is a bounded linear operator and $T_\gamma : A(\gamma) \to B(\gamma)$ is bounded for each γ with $\|T_\gamma\|_{A(\gamma) \to B(\gamma)} \le M(\gamma) \in \mathscr{L}$. If $\|M\|_\infty < \infty$, then, for every subgroup $S \subset \mathscr{L}$ (see Section 3.2), the interpolated operators:

$$T^S_{z_0,p} : [A]^S_{z_0,p} \to [B]^S_{z_0,p} \quad \text{and} \quad T^S_{z_0,p} : (A)^S_{z_0,p} \to (B)^S_{z_0,p}$$

are bounded with norms $\le \|M\|_\infty$.

In view that $(B)^S_{z_0,p} \subset [B]^S_{z_0,p}$, the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ is also bounded.

3.2. Now, consider finite families of spaces A_i, continuously embedded into the same Hausdorff topological vector space $\mathscr{H}$. Denote by $\mathscr{J}(\overline{A})$ the intersection $\bigcap A_i$ and by $\mathscr{S}(\overline{A})$ the sum $A_0 + A_1 + \ldots + A_n$, with the norms

$$\|a\|_{\mathscr{J}(\overline{A})} = \max\left\{\|a\|_{A_0}, \|a\|_{A_1}, \ldots, \|a\|_{A_n}\right\}$$

and

$$\|a\|_{\mathscr{S}(\overline{A})} = \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} + \ldots + \|a_n\|_{A_n}\right\},$$

where the infimum extends over all representations of $a = a_0 + a_1 + \ldots + a_n$.

Suppose that $\mathscr{J}(\overline{A})$ is dense in every A_i (see [3]), and do $\mathscr{A} = \mathscr{J}(\overline{A})$, $\mathscr{U} = \mathscr{S}(\overline{A})$:

(i) Let $\overline{A} = (A_0, A_1)$; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, $i = 0, 1$, with $\{\Gamma_0, \Gamma_1\}$ a partition of Γ. Do

$$S = \left\{\alpha_m = 1_{\Gamma_0} + 2^m 1_{\Gamma_1}; m \in \mathbb{Z}\right\},$$

to get that $[A]^S_{z_0,p} = (A_0, A_1)_{|\Gamma_1|_{z_0},p} = K_{\theta,p}(\overline{A})$: the K-space of Lions-Peetre with $\theta = |\Gamma_1|_{z_0}$, where $|E|_z$ is the harmonic measure of $E \subset \Gamma$ at $z \in D$, (see [15]).

(ii) Let $\overline{A} = (A_0, A_1, \ldots, A_n)$; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, $i = 0, 1, \ldots, n$ and $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ a partition of Γ. Do

$$S = \left\{\alpha_{\overline{m}} = 1_{\Gamma_0} + \sum_{i=1,n} 2^{m_i} 1_{\Gamma_i}\ ;\ \overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n\right\},$$

to get that $[A]^S_{z_0,p} = (A_0, A_1, \ldots, A_n)_{(|\Gamma_i|_{z_0}, i=1,\ldots,n),p;K}$: the Sparr K-space, see [19].

(iii) Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a family of 2^2 spaces; take $A(\gamma) = A_i$ with $\gamma \in \Gamma_i$, $i = 0, 1, 2, 3$ and $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$ a partition of Γ. Do

$$S = \{\alpha_{\overline{m}} = 1_{\Gamma_0} + 2^k 1_{\Gamma_1} + 2^l 1_{\Gamma_2} + 2^k 2^l 1_{\Gamma_3}\ ;\ \overline{m} = (k, l) \in \mathbb{Z}^2\},$$

to obtain that $[A]^S_{z_0,p} = (A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2),p;K}$, where $\theta_1 = |\Gamma_1 \bigcup \Gamma_3|_{z_0}$, $\theta_2 = |\Gamma_2 \bigcup \Gamma_3|_{z_0}$: Fernández K-space, which can be generalized to families of 2^d spaces, see [10].

(iv) Let $\overline{A} = (A_1, \ldots, A_n)$; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, $i = 0, 1, \ldots, n$ and $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ a partition of Γ. Do

$$S = \left\{\alpha_{(k,l)} = \sum_{i=1,n} 2^{kx_i + ly_i} 1_{\Gamma_i}\ ;\ (k, l) \in \mathbb{Z}^2\right\},$$

where (x_i, y_i) are the vertices of a convex polygon Π in the affine plane $\mathbb{R}^2$, to obtain, for an interior point (α, β) of Π that $[A]^S_{z_0,p} = \overline{A}_{(\alpha,\beta),p;K}$, with $(\alpha, \beta) = \sum_{i=1,n} |\Gamma_i|_{z_0}(x_i, y_i)$: Cobos-Peetre K-space, see [7].

With the same subgroup S in each case, apply the J-method just described to obtain the J-spaces $(A)^S_{z_0,p}$ of Lions-Peetre, Sparr, Fernández and Cobos-Peetre, respectively. The density of $\mathscr{J}(\overline{A})$ in each A_i is not necessary for the J-method.

Despite the fact that the J-space always embeds into the K-space, the J and K-methods are not equivalent. Nevertheless, they are equivalent in the case of Lions-Peetre method for pairs, see [15], and in the many examples proposed by Fernández and Sparr where their J and K methods are equivalent (see [10] and [19]).

4. Some Classes of Operators

4.1. An operator $T \in \mathfrak{L}(E,F)$ is separable if $T(E)$ is a separable subspace of F or, equivalently, if $T(B_E)$ is a separable subset of F. The operator is weakly compact if $T(B_E)$ is a relatively weakly compact set, or, by using the Eberlein-Smulian Theorem, if and only if every sequence (Tx_n) with $x_n \in B_E$ admits a weakly convergent subsequence, see [14] and [17].

$T \in \mathfrak{L}(E,F)$ is a Rosenthal operator if for each $s \in \mathfrak{L}(\ell_1,E)$ the composition Ts is not an isomorphic embedding. Using Rosenthal's Theorem, it is easy to obtain the following characterization of these operators: T is Rosenthal if and only if every bounded sequence (x_n) of E possesses a subsequence (x'_n) such that (Tx'_n) is weak Cauchy, that is, if and only if $T(B_E)$ is weakly pre-compact, see [14].

$T \in \mathfrak{L}(E,F)$ has the Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n), such that (Tx'_n) is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum\limits_{k=1,n} Tx'_k$, converges in F. The operator T has the alternate-signs Banach-Saks property if any bounded sequence (x_n) of E possesses a subsequence (x'_n), such that $((-1)^n Tx'_n)$ is Cesàro convergent, i.e., the sequence of the averages $n^{-1} \sum\limits_{k=1,n} (-1)^k Tx'_k$ converges in F. The operator T has the Banach-Saks-Rosenthal property if any weakly null sequence (x_n) of E possesses a subsequence (x'_n), such that the sequence of the averages $n^{-1} \sum\limits_{k=1,n} Tx'_k$ converges in F. See [1] and [13] for a thorough study of these operators.

Let (Ω,μ) be a probability space. An operator $X \in \mathfrak{L}(L_1(\Omega,\mu),E)$ is right-decomposable, see [17], if there exists a μ-measurable and E-valued kernel $x(\omega)$, with $\omega \in \Omega$, such that for all $f \in L_1(\Omega,\mu)$:

$$X(f) = \int_\Omega f(\omega)x(\omega)d\mu.$$

$T \in \mathfrak{L}(E,F)$ is a Radon-Nikodym operator if TX is right-decomposable for every $X \in \mathfrak{L}(L_1(\Omega,\mu),E)$, see [14] and [17].

An operator $Z \in \mathfrak{L}(F,(L_\infty(\Omega,\mu))$ is left-decomposable if there exists a μ-measurable and F^*-valued kernel $z(\omega)$, such that for all $y \in F$:

$$Zy(\omega) = \langle y, z(\omega)\rangle.$$

$T \in \mathfrak{L}(E,F)$ is a decomposing operator if ZT is left decomposable for every $Z \in \mathfrak{L}(F,(L_\infty(\Omega,\mu))$; see [17, Chapter 24].

The following characterization of decomposing operators is well known, see [17, 24.4.3]: T is decomposing if and only if its dual T^*, is a Radon-Nikodym operator.

The classes of operators defined above are ideals of operators. Following Pietsch, capital gothic letters will denote each one of them. So, $\mathfrak{X}$ will be the ideal of separable operators, $\mathfrak{W}$ that of weakly compact operators, $\mathfrak{R}$ Rosenthal operators, $\mathfrak{Y}$ Radon-Nikodym and $\mathfrak{Q}$ that of decomposing operators; see [17] for a detailed study of these ideals.

Write $\mathscr{BS}$ for the Banach-Saks operator ideal, $\mathscr{ABS}$ for the alternate-signs Banach-Saks and $\mathscr{BSR}$ for the Banach-Saks-Rosenthal operator ideal. These last ideals are not treated by Pietsch; see [1] and [13] for their study.

All of these ideals are closed and injective. Separable, weakly compact, Rosenthal, Banach-Saks, alternate-signs Banach-Saks and decomposing are also surjective operator ideals. Neither $\mathfrak{Y}$ nor $\mathscr{BSR}$ are surjective.

4.2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two operator ideals. The product $\mathfrak{D}\circ\mathfrak{C}$ is a new operator ideal defined as follows: $T \in \mathfrak{L}(E,F)$ belongs to $\mathfrak{D}\circ\mathfrak{C}$ if there exists a Banach space G and operators $U \in \mathfrak{C}(E,G)$, $V \in \mathfrak{D}(G,F)$, such that $T = VU$, see [17] and [11].

Heinrich, in [11, Theorem 1.1], proves that if $\mathfrak{C}$ and $\mathfrak{D}$ are closed then $\mathfrak{D}\circ\mathfrak{C}$ is also closed. It is always true that $\mathfrak{D}\circ\mathfrak{C} \subset \mathfrak{D}\cap\mathfrak{C}$, but the converse inclusion is not valid in general. Nevertheless, see [11, Theorem 1.3], if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective then $\mathfrak{D}\circ\mathfrak{C} = \mathfrak{D}\cap\mathfrak{C}$.

It is clear that if $\mathfrak{C}$ is injective and $\mathfrak{D}$ is surjective, both satisfying the $\sum_p$-condition, $1 \le p < \infty$ then, $\mathfrak{D}\circ\mathfrak{C} = \mathfrak{D}\cap\mathfrak{C}$ also satisfies the $\sum_p$-condition.

Let $\mathfrak{I}$ be an operator ideal. The operator $T \in \mathfrak{L}(E,F)$ belongs to the dual ideal $\mathfrak{I}^{dual}$ if the adjoint operator T^* belongs to $\mathfrak{I}(F^*,E^*)$. For example, $\mathfrak{Q} = \mathfrak{Y}^{dual}$.

If $\mathfrak{I}$ is injective, $\mathfrak{I}^{dual}$ is surjective and if $\mathfrak{I}$ is surjective, $\mathfrak{I}^{dual}$ is injective (see [17, Chapter 4]). If $\mathfrak{I}$ is closed, so is $\mathfrak{I}^{dual}$. If $\mathfrak{I}$ satisfies the $\sum_p$-condition, for all p with $1 < p < \infty$, then $\mathfrak{I}^{dual}$ satisfies $\sum_p$-condition for all p with $1 < p < \infty$.

The product of several operator ideals will be called a *chain*. For example, $\mathfrak{I} = \mathfrak{R}\circ\mathfrak{R}^{dual}$, $\mathfrak{I} = \mathfrak{X}\circ\mathfrak{W}$, $\mathfrak{I} = (\mathfrak{Q}\circ\mathscr{ABS})^{dual}$ or $\mathfrak{I} = \mathfrak{X}\circ\mathfrak{R}^{dual}\circ\mathfrak{Q}$, all them being, certainly, injective, surjective and closed operator ideals, are chains.

Let E and F be Banach spaces. An operator $T \in chain(E,F)$ is a *mixed operator*. For example, $T \in \mathfrak{X}\circ\mathfrak{R}^{dual}\circ\mathfrak{Q}(E,F)$, which means that $T : E \to F$ is, at the same time, a separable, dual Rosenthal and decomposing operator, is a mixed operator of type $\mathfrak{X}\circ\mathfrak{R}^{dual}\circ\mathfrak{Q}$.

4.3. The relationships $\mathscr{BS} \subset \mathscr{ABS} \subset \mathscr{WBS}$, strict inclusions, are well known, see [1] and [13].

The following theorem gives relationships much more precise that will be used in the next sections:

Theorem 4.4. (i) $\mathscr{BS} = \mathfrak{W}\circ\mathscr{BSR}$,

(ii) $\mathscr{BS} = \mathfrak{W}\circ\mathscr{ABS}$, and

(iii) $\mathscr{ABS} = \mathfrak{R}\circ\mathscr{BSR}$ (the products in this order since $\mathscr{BSR}$ is not surjective).

Proof. (i) to prove that $\mathscr{B}\mathscr{S} \subset \mathfrak{W} \circ \mathscr{B}\mathscr{S}\mathscr{R}$, use the well known fact that $\mathscr{B}\mathscr{S} \subset \mathfrak{W}$; for $\mathfrak{W} \circ \mathscr{B}\mathscr{S}\mathscr{R} \subset \mathscr{B}\mathscr{S}$ use the Eberlein-Smulian characterization of weakly compact operators; (ii) follows at once from (i). For the proof of (iii) see [18, Theorem 2.3]. □

Now, the objective will be to interpolate mixed operators.

5. Interpolation of Operators

Interpolation theory is concerned with the following question: let $\overline{A}$ and $\overline{B}$ be interpolation n-tuples and let $T : \overline{A} \to \overline{B}$ be an interpolation operator; if all the extreme operators $T_i : A_i \to B_i$ $(i = 0, 1, \dots, n)$ or some of them, belong to a class $\mathfrak{I}$ of operators, what can we expect from the interpolated operator $T^S_{z_0,p}$?.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator, denote by $T_{\mathscr{J}\mathscr{S}}$ the induced operator from $\mathscr{J}(\overline{A})$ into $\mathscr{S}(\overline{B})$.

Definition 5.1. An operator ideal $\mathfrak{I}$, possesses the *Strong Property of Interpolation* (SPI, in short), with respect to Carro Method, i.e., the real method for families depending on the parameters $z_0 \in D$, $1 < p < \infty$ and the subgroup $S \subset \mathscr{L}$, if the following holds: the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ belongs to $\mathfrak{I}$ if and only if $T_{\mathscr{J}\mathscr{S}} \in \mathfrak{I}$.

Theorem 5.2. *Any injective and surjective operator ideal $\mathfrak{I}$, which satisfies the $\sum_p$-condition, possess the SPI with respect to Carro Method.*

Proof. Let $\overline{A}$ and $\overline{B}$ be interpolation families of finite tuples. In order to avoid a complicated notation, write A for the intersection $\mathscr{J}(\overline{A})$ and B for the sum $\mathscr{S}(\overline{B})$. Define on A and B the following equivalent norms (equivalent to the norms of intersection and sum spaces, respectively):

$$\|x\|_\alpha = \frac{J(\alpha, x)}{\alpha(z_0)} \quad \text{for } x \in A \text{ and } \alpha \in S,$$
$$\|y\|_\alpha = \frac{K(\alpha, y)}{\alpha(z_0)} \quad \text{for } y \in B \text{ and } \alpha \in S,$$

where S is the corresponding subgroup of $\mathscr{L}$ for each one of the methods described in the Section 3.2.

Denote by A_α the space $(A, \| \quad \|_\alpha)$ and by B_α the space $(B, \| \quad \|_\alpha)$. For each $(x_\alpha)_{\alpha\in S} \in \left(\sum_{\alpha\in S} A_\alpha\right)_p$, the sum $\sum_{\alpha\in S} x_\alpha$ converges (absolutely) in $\mathscr{S}(\overline{A})$. Then, there is a surjection Q from $\left(\sum_{\alpha\in S} A_\alpha\right)_p$ onto the J-space $(A)^S_{z_0,p}$:

$$Q(x_\alpha)_{\alpha\in S} = \sum_{\alpha\in S} x_\alpha \quad (\text{convergence in } \mathscr{S}(\overline{A}))$$

and an isomorphic embedding J from the K-space $[B]^S_{z_0,p}$ into $\left(\sum_{\alpha\in S} B_\alpha\right)_p$ defined by $J(y) = (y_\alpha)_{\alpha\in S}$ where $y_\alpha = y$ for all α.

Let $T : \overline{A} \to \overline{B}$ be an interpolation operator and assume that $T_{\mathscr{J}\mathscr{S}} \in \mathfrak{I}$. Denote by J_i the natural embedding of A_i into $\left(\sum_{\alpha\in S} A_\alpha\right)_p$ and by Q_j the natural projection of $\left(\sum_{\alpha\in S} B_\alpha\right)_p$ onto B_j. The operator $Q_j J T^S_{z_0,p} Q J_i$ is just $T_{\mathscr{J}\mathscr{S}}$. It is, then, an operator of the class $\mathfrak{I}$ and, since $\mathfrak{I}$ satisfies the $\sum_p$-condition, the operator $JT^S_{z_0,p}Q$ belongs to $\mathfrak{I}\left(\left(\sum_{\alpha\in S} A_\alpha\right)_p, \left(\sum_{\alpha\in S} B_\alpha\right)_p\right)$. Now, injectivity and surjectivity of $\mathfrak{I}$ imply that $T^S_{z_0,p} \in \mathfrak{I}((A)^S_{z_0,p}, [B]^S_{z_0,p})$. Converse is clear. □

Theorem 5.3. *The single ideals $\mathfrak{X}$, $\mathfrak{W}$, $\mathfrak{R}$, $\mathscr{BSR}$, $\mathscr{BS}$, $\mathscr{ABS}$ and $\mathfrak{Q}$ satisfy the $\sum_p$-condition for $1 < p < \infty$.*

Proof. It is well known that $\mathfrak{W}$ and $\mathfrak{R}$ satisfy the condition. Heinrich proved that $\mathscr{BS}$ and $\mathfrak{Q}$ satisfy it, see [11]. It is clear that $\mathfrak{X}$ also satisfies it. For the proof that $\mathscr{BSR}$ also possess the $\sum_p$-condition see [11, Theorem 2.3, p. 407] or [18, Lemma 3.3]). Finally, to prove that $\mathscr{ABS}$ possess it, use Theorem 4.4.

Except for the case of separable operators, the assumption that $1 < p < \infty$ is necessary. □

It is clear that all the chains proposed as examples in §4.2, satisfy $\sum_p$-condition with $1 < p < \infty$. Then, we have:

Theorem 5.4. *The single ideals $\mathfrak{X}$, $\mathfrak{W}$, $\mathfrak{R}$, $\mathscr{BS}$, $\mathscr{ABS}$, $\mathfrak{Q}$, dual ideals $\mathfrak{X}^{\mathrm{dual}}$, $\mathfrak{R}^{\mathrm{dual}}$, $\mathscr{BS}^{\mathrm{dual}}$, $\mathscr{ABS}^{\mathrm{dual}}$, $\mathfrak{Q}^{\mathrm{dual}}$ and chains as $\mathfrak{I} = \mathfrak{R}\circ\mathfrak{R}^{\mathrm{dual}}$, $\mathfrak{I} = \mathfrak{X}\circ\mathfrak{W}$, $\mathfrak{I} = (\mathfrak{X}\circ\mathscr{ABS})^{\mathrm{dual}}$ or $\mathfrak{I} = \mathfrak{X}\circ\mathfrak{R}^{\mathrm{dual}}\circ\mathfrak{Q}$, all of them possess the SPI with respect to Carro Method.*

Proof. They are injective, surjective and satisfy the $\sum_p$-condition. Conclude by using Theorem 5.2. □

Since all the operator ideals from Theorem 5.4 satisfy the $\sum_p$-condition we have:

Corollary 5.5. *Let $\mathfrak{I}$ be an operator ideal as those of Theorem 5.4 and $(X_m)_{m\in\mathbb{Z}}$ a family of Banach spaces then, the sum space $\left(\sum_{m\in\mathbb{Z}} X_m\right)_p$ with $1 < p < \infty$ belongs to Space$(\mathfrak{I})$ if and only if $X_m \in$ Space$(\mathfrak{I})$ for each m.*

Proof. The identity operator 1 from $\left(\sum_{m\in\mathbb{Z}} X_m\right)_p$ into itself is in $\mathfrak{I}\left(\left(\sum_{m\in\mathbb{Z}} X_m\right)_p, \left(\sum_{m\in\mathbb{Z}} X_m\right)_p\right)$ since $Q_j 1 J_i \in \mathfrak{I}(X_i, X_j)$ for every $i, j \in \mathbb{Z}$. □

For example, for $\mathfrak{I} = \mathscr{ABS}$ we have that $\left(\sum_{m\in\mathbb{Z}} X_m\right)_p \in$ *Space*$(\mathscr{ABS})$ if and only if $X_m \in$ *Space*$(\mathscr{ABS})$ for each $m \in \mathbb{Z}$ (cf. [16]).

See [5], for a thorough study of the SPI for weakly compact operators between infinite families (in general, Theorem 5.1.1), finite families (in particular) and for interpolation of limited operators.

In the case of $\mathscr{ABS}$, Theorem 5.4. generalizes that result obtained by A. Kryczka in [13], Corollary 4.2.

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