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Research Article

On the Structure of Interior KU-algebras and KU-ideals

Areej Almuhaimeed

Department of Mathematics, College of Science, Taibah University, Madinah, Saudi Arabia
aamuhaimeed@taibahu.edu.sa

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Abstract. In this paper, the notation of interior KU-algebra and some related definitions are introduced. Various characterisations for the interior KU-algebra have been proved. The concept of interior positive implicative KU-algebra and prove some results regarding this concept are also presented. In addition, the interior KU-ideals and several properties of these ideals are presented.

Keywords. KU-algebra, Interior algebra, Interior ideals, Positive implicative

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1. Introduction

There are several algebraic structures which defined with a standard operator, for instance, BCK-algebra [10] and BCI-algebra. Prabpayak and Leerawat [6, 7] introduced the algebraic structure KU-algebra and studied some of its properties. Vorster [9] considered interior operators in category theory. This leads to fundamental applications in category theory and topology [8], [2] and [1].

In this paper, the notation of an interior KU-algebra is introduced. A number of questions regarding this notation with the composition are investigated. Moreover, a relation between two interior KU-algebras is introduced and provided a connection between this relation and composition. Also, some properties of interior KU-algebras are presented. Then, an interior positive implicative KU-algebra is defined and proved several related results. After that, the concept of interior ideals in KU-algebras and some of its properties are presented.

2. Preliminaries

In this section, some definitions and results are introduced that will be used in later sections.

Recall that a nonempty set X with a binary operation $*$ is called a KU-algebra if, for $r,s,t \in X$, the following conditions hold:

(KU1) $(r*s)*[(s*t)*(r*t)] = 0$.

(KU2) $r*0 = 0$.

(KU3) $0*r = r$.

(KU4) $r*s = 0 = s*r$ implies that $r = s$.

A relation $\leq$ defined on X as follows:

$$r \leq s \text{ if and only if } s*r = 0.$$

Thus, the above conditions can be rewritten as follows:

(KU1′) $(s*t)*(r*t) \leq r*s$.

(KU2′) $0 \leq r$.

(KU3′) $0*r = r$.

(KU4′) $r \leq s$, $s \leq r$ implies that $r = s$.

Any KU-algebra satisfies the following properties:

Theorem 1 ([3]). *If X is a KU-algebra, then, for $r,s,t \in X$, the following axioms hold:*

(i) *if $r \leq s$, then $s*t \leq r*t$.*

(ii) $r*(s*t) = s*(r*t)$.

(iii) $((s*r)*r) \leq s$.

(iv) $((s*r)*r)*r = s*r$.

Definition 1 ([6, 7]). A nonempty subset M of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:

(i) $0 \in M$.

(ii) $s*t \in M$, $s \in M$ implies that $t \in M$, for all $r,s,t \in M$.

3. Interior KU-algebras

In this section, the main concept in this paper and some related results are introduced.

Definition 2. Let X be a KU-algebra and $I : X \to X$ be a map. The pair (X, I) is called an interior KU-algebra if it satisfies the following conditions:

(i) $I(r) \leq r$, for all $r \in X$.

(ii) $I^2(r) = I(r)$, for all $r \in X$.

(iii) $r \leq s$ implies that $I(r) \leq I(s)$, for all $r,s \in X$.

A straightforward example of an interior KU-algebra is the identity map. We present the following examples:

Example 1. (i) Consider the KU-algebra $(X,*)$ which presented by Table 1:

Table 1

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

Define the map $I_1 : X \to X$ by

$$I_1(0) = 0, \quad I_1(1) = 2, \quad I_1(2) = 2, \quad I_1(3) = 3.$$

Then (X, I_1) is an interior KU-algebra.

(ii) Consider the KU-algebra $(X,*)$ presented by Table 1. Define the map $I_2 : X \to X$ by

$$I_2(0) = 0, \quad I_2(1) = 1, \quad I_2(2) = 1, \quad I_2(3) = 3.$$

Then (X, I_2) is not an interior KU-algebra as $I_2(2) \not\leq 2$.

Consider a KU-algebra X. Let $I(X)$ be the set of all interior KU-algebras. Define an operation as following:

$$(X, I_j) \circledast (X, I_k) = (X, I_j \circledast I_k), \quad \text{for all } (X, I_j), (X, I_k) \in I(X),$$

such that $(I_j \circledast I_k)(r) = I_j(r) * I_k(r)$ for all $r \in X$. Note that $I_0(r) = 0$ for all $r \in X$.

We are able to prove the following theorem:

Theorem 2. *Let X be a KU-algebra. Then $(I(X), \circledast, (X, I_0))$ is a KU-algebra.*

Proof. Straightforward of the above definition. □

Now, a natural question appears: Is the composition of two interior KU-algebra an interior KU-algebra? However, the statement need not be true and we provide a counterexample:

Example 2. Let X be the KU-algebra described in Example 1. Consider the two interior KU-algebras I_1, I_2 defined as following:

$$I_1(0) = 0, \quad I_1(1) = 2, \quad I_1(2) = 2, \quad I_1(3) = 3.$$

$$I_2(0) = 0, \quad I_2(1) = 1, \quad I_2(2) = 0, \quad I_2(3) = 3.$$

Then $I_1 \circ I_2$ is not an interior KU-algebra as $(I_2 \circ I_1)^2(3) = 0 \neq (I_2 \circ I_1)(3) = 2$.

Consider the interior KU-algebra I_1 described above. If i denotes the identity map, then

Table 2

$*$	0	1	2	3
$I_1 \circ i$	0	2	2	3
$i \circ I_1$	0	2	2	3

Note that the composition of two interior KU-algebra need not be commutative in general. For instance, see Example 2.

In the case that the composition is commutative, we can prove the following theorem:

Theorem 3. *Let (X, I_j) and (X, I_k) be two interior KU-algebras in which $I_j \circ I_k = I_k \circ I_j$. Then $(X, I_j \circ I_k)$ is an interior KU-algebra.*

Proof. (i) Since I_j, I_k are interior KU-algebras, we obtain

$$I_j(r) \le r \text{ and } I_k(r) \le r, \quad \text{for all } r \in X.$$

Thus, we have

$$(I_j \circ I_k)(r) = I_j(I_k(r)) \le I_k(r) \le r.$$

(ii) By hypothesis

$$I_j \circ I_k = I_k \circ I_j, \ \ I_j^2(r) = I_j(r), \ \ I_k^2(r) = I_k(r).$$

This implies that

$$\begin{aligned}
(I_j \circ I_k)^2(r) &= ((I_j \circ I_k) \circ (I_j \circ I_k))(r) \\
&= (I_j \circ (I_k \circ I_j) \circ I_k)(r) \\
&= (I_j \circ (I_j \circ I_k) \circ I_k)(r) \\
&= ((I_j \circ I_j) \circ (I_k \circ I_k))(r) \\
&= (I_j^2 \circ I_k^2)(r) \\
&= I_j^2(I_k^2(r)) \\
&= I_j^2(I_k(r)) \\
&= I_j(I_k(r)) \\
&= ((I_j \circ I_k)(r).
\end{aligned}$$

(iii) If $r, s \in X$ such that $r \le s$, then

$$\begin{aligned}
(I_j \circ I_k)(r) &= I_j(I_k(r)) \\
&\le I_j(I_k(s)) \\
&\le (I_j \circ I_k)(s).
\end{aligned}$$

Hence $(X, I_j \circ I_k)$ is an interior KU-algebra. □

Now, we define a relation on interior KU-algebras as follows:

Consider two interior KU-algebras (X, I_j) and (X, I_k). Write $I_j \lessdot I_k$, if $I_j(r) \leq I_k(r)$ for all $r \in X$. Now, prove the following theorem:

Theorem 4. *Let (X, I_j) and (X, I_k) be two interior KU-algebras. Then*

$$I_j \lessdot I_k \text{ if and only if } I_j \circ I_k = I_j.$$

Proof. Suppose that $I_j \lessdot I_k$. Then, by definition, $I_j(r) \leq I_k(r)$ for all $r \in X$ that I_j is an interior KU-algebra, it implies that

$$\begin{aligned} I_j(r) &= I_j^2(r) \\ &= I_j(I_j(r)) \\ &\leq I_j(I_k(r)) \\ &= (I_j \circ I_k)(r). \end{aligned}$$

We also have

$$\begin{aligned} (I_j \circ I_k)(r) &= I_j(I_k(r)) \\ &\leq I_k(I_k(r)) \\ &= I_k^2(r) \\ &= I_k(r) \\ &\leq r. \end{aligned}$$

Thus

$$\begin{aligned} (I_j \circ I_k)(r) &= I_j(I_k(r)) \\ &= I_j^2(I_k(r)) \\ &= I_j(I_j(I_k^2(r))) \\ &\leq I_k(r). \end{aligned}$$

Therefore,

$$I_j \circ I_k = I_j.$$

Now, assume that $I_j \circ I_k = I_j$. It implies that

$$\begin{aligned} I_j(r) &= (I_j \circ I_k)(r) \\ &= I_j(I_k(r)) \\ &\leq I_k(r) \end{aligned}$$

and so $I_j \lessdot I_k$ as required. □

For a KU-algebra X, we introduce the following definition:

Definition 3. A KU-algebra X is called a bounded KU-algebra if there exists an element $1 \in X$ in which $r \leq 1$ for all $r \in X$. We can write $r * 1 = \neg r$.

Now, we are able to present some properties of interior KU-algebras.

Theorem 5. *Let X be a KU-algebra, (X,I) an interior KU-algebra and $r,s,t \in X$. Then*

(i) $r * I(s) \leq I(r) * I(s)$.

(ii) $I(r * s) \leq I(r) * s$.

(iii) $I(0) = 0$.

(iv) $(r * s) * I(t) \leq I(r * s) * I(t)$.

(v) $(I(r) * s) * I(t) \leq (r * s) * I(t)$.

(vi) $I(\neg r) \leq \neg I(r)$.

(vii) $I(\neg r * \neg s) \leq s * r$.

Proof. (i) By Theorem 1.

(ii) $I(r * s) \leq r * s \leq I(r) * s$.

(iii) Since $I(0) \leq 0$, then we obtain

$$0 * I(0) = 0 = I(0) * 0.$$

Thus by (KU4), $I(0) = 0$.

(iv) Using by Theorem 1.

(v) Since $I(r) \leq r$, by Theorem 1 we have, $r * s \leq I(r) * s$. Hence

$$(I(r) * s) * I(t) \leq (r * s) * I(t)$$

by Theorem 1.

(vi) By definition, $I(\neg r) = I(r * 1)$. Then by the second property, we obtain

$$I(r * 1) \leq I(r) * 1 = \neg I(r).$$

Hence $I(\neg r) \leq \neg I(r)$.

(vii) $I(\neg r * \neg s) \leq \neg r * \neg s = (r * 1) * (s * 1) \leq s * r$. □

We define a positive implicative KU-algebra as follows:

Definition 4 ([4,5])**.** A KU-algebra X is called positive implicative if the following condition satisfied:

$$(t * r) * (t * s) = t * (r * s), \quad \text{for all } r,s,t \in X.$$

Example 3. Consider the KU-algebra $(X,*)$ presented in Example 1. Let I be the interior KU-algebra defined as follows:

$$I_3(0) = I_3(1) = I_3(2) = 0, \quad I_1(3) = 3.$$

Then, it clear that I_3 is a positive implicative interior KU-algebra.

Note that if (X,I) is an interior positive implicative KU-algebra, then X need not be positive implicative. As an example, consider I_3 in Example 3. Then I_3 is positive implicative while X is not as

$$(2*1)*(2*3) = 2 \neq 0 = 2*(1*3).$$

Remark 1. Let X be a KU-algebra.

(i) It is not true in general that for $r,s,t \in X$:

$$I((t*r)*(t*s)) = I(t*(r*s)).$$

For instance, check the interior KU-algebra I_1 in Example 1. We have

$$I_1((2*1)*(2*3)) \neq I(2*(1*3)).$$

(ii) It is not true in general that for $r,s \in X$:

$$I(r*s) = I(r*(r*s)).$$

In Example 1, we obtain

$$I(2*1) = 2 \neq 0 = I(2*(2*1)).$$

We provide now a condition which makes the above statements true.

Theorem 6. *Let (X,I) be a positive implicative KU-algebra. Then for $r,s,t \in X$, we have*

$$I((t*r)*(t*s)) = I(t*(r*s))$$

and

$$I(r*s) = I(r*(r*s)).$$

Proof. Since (X,I) is a positive implicative KU-algebra, the result hold. □

4. Interior Ideals

This section begins with the following definition:

Definition 5. Let (X,I) be an interior KU-algebra. Then a subset M of X is called an interior ideal in (X,I) if M is an ideal of X that satisfies:

$$I(r) \in M \Rightarrow r \in M, \quad \text{for all } r \in X.$$

Example 4. Consider the KU-algebra $(X,*)$ which presented by Table 3:

Table 3

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Define the map $I : X \to X$ by

$$I(0) = I(1) = I(2) = I(3) = 0, \quad I(4) = 4.$$

Then (X, I) is an interior KU-algebra. Let $M = \{0, 1, 2, 3\}$. Then M is an ideal of X. It is clear that M is an interior ideal in (X, I). However, the zero ideal is not an interior ideal.

Theorem 7. *The intersection of interior ideals in an interior KU-algebra is an interior ideal.*

Proof. The intersection of interior ideals satisfies the condition in Definition 5. □

Note that the union of interior ideals in an interior KU-algebra need not be an interior ideal. This is clear from the fact that the union of ideals need not be an ideal.

Theorem 8. *Let (X, I) be an interior KU-algebra. If M is a subset of X in which*

$$0 \in M,$$

$$r * (r * s) \in M_t \Rightarrow r * s \in M, \quad \text{for all } r, s \in X, \text{ for all } t \in M,$$

$$I(r) \in M \Rightarrow r \in M, \quad \text{for all } r \in X,$$

then M is a positive implicative interior ideal in (X, I).

Proof. Let $r, s, t \in X$ such that $s * t \in M$ and $s \in M$. Then

$$t = 0 * (0 * t) \in M_t.$$

By hypothesis, $t = 0 * t \in M$. Thus M is an interior ideal in (X, I). In order to see that M is positive implicative, we may assume that $t * (r * s) \in M$ and $t * r \in M$. Then, by Theorem 1, we have

$$t * (r * s) = r * (t * s) \in M.$$

Note that $0 * (0 * r) \in M_t$ implies that $r = 0 * r \in M$. As M is an ideal and $r \in M$, we obtain $t * s \in M$. Therefore, M is a positive implicative interior ideal in (X, I). □

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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