Invariant Submanifolds of Sasakian Manifolds Admitting Semi-symmetric Metric Connection

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Abstract The object of this paper is to study invariant submanifolds $M$ of Sasakian manifolds $\tilde{M}$ admitting a semi-symmetric metric connection and to show that $M$ admits semi-symmetric metric connection. Further it is proved that the second fundamental forms $\sigma$ and $\tilde{\sigma}$ with respect to Levi-Civita connection and semi-symmetric metric connection coincide. It is shown that if the second fundamental form $\sigma$ is recurrent, 2-recurrent, generalized 2-recurrent and $M$ has parallel third fundamental form with respect to semi-symmetric metric connection, then $M$ is totally geodesic with respect to Levi-Civita connection.

1. Semi-symmetric Metric Connection

The geometry of invariant submanifolds $M$ of Sasakian manifolds $\tilde{M}$ is carried out from 1970's by M. Kon [12], D. Chinea [8], K. Yano and M. Kon [17]. It is proved that invariant submanifold of Sasakian structure also carries Sasakian structure. Also the authors B.S. Anitha and C.S. Bagewadi [1] have studied and the same authors [2] have studied on Invariant submanifolds of Sasakian manifolds admitting semi-symmetric non-metric connection. In this paper we extend the results to invariant submanifolds $M$ of Sasakian manifolds admitting semi-symmetric metric connection.

We know that a connection $\nabla$ on a manifold $M$ is called a metric connection if there is a Riemannian metric $g$ on $M$ if $\nabla g = 0$ otherwise it is non-metric. Further it is said to be semi-symmetric if its torsion tensor $T(X,Y) = 0$, i.e., $T(X,Y) = w(Y)X - w(X)Y$, where $w$ is a 1-form. In 1924, A. Friedmann and J.A. Schouten [10] introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, H.A. Hayden [11] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by K. Yano [16] in

2010 Mathematics Subject Classification. 53D15, 53C21, 53C25, 53C40.

Key words and phrases. Invariant submanifolds; Sasakian manifold; Semi-symmetric metric connection; Totally geodesic.
1970. After that the properties of semi-symmetric metric connection have studied by many authors like K.S. Amur and S.S. Pujar [3], C.S. Bagewadi, D.G. Prakasha and Venkatesha [4, 5], A. Sharfuddin and S.I. Hussain [14], U.C. De and G. Pathak [9] etc. If \( \nabla \) denotes semi-symmetric metric connection on a contact metric manifold, then it is given by [4]

\[
\nabla_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{1.1}
\]

where \( \eta(Y) = g(Y, \xi) \).

The covariant differential of the \( p \)th order, \( p \geq 1 \), of a \((0, k)\)-tensor field \( T \), \( k \geq 1 \), defined on a Riemannian manifold \((M, g)\) with the Levi-Civita connection \( \nabla \), is denoted by \( \nabla^p T \). The tensor \( T \) is said to be recurrent and 2-recurrent [13], if the following conditions hold on \( M \), respectively,

\[
(\nabla T)(X_1, \ldots, X_k; X)T(Y_1, \ldots, Y_k) = (\nabla T)(Y_1, \ldots, Y_k; X)T(X_1, \ldots, X_k), \tag{1.2}
\]

\[
(\nabla^2 T)(X_1, \ldots, X_k; X, Y)T(Y_1, \ldots, Y_k) = (\nabla^2 T)(Y_1, \ldots, Y_k; X, Y)T(X_1, \ldots, X_k),
\]

where \( X, Y, X_1, Y_1, \ldots, X_k, Y_k \in TM \). From (1.2) it follows that at a point \( x \in M \), if the tensor \( T \) is non-zero, then there exists a unique 1-form \( \phi \) and a \((0, 2)\)-tensor \( \psi \), defined on a neighborhood \( U \) of \( x \) such that

\[
\nabla T = T \otimes \phi, \quad \phi = d(\log ||T||) \tag{1.3}
\]

and

\[
\nabla^2 T = T \otimes \psi, \tag{1.4}
\]

hold on \( U \), where \( ||T|| \) denotes the norm of \( T \) and \( ||T||^2 = g(T, T) \). The tensor \( T \) is said to be generalized 2-recurrent if

\[
((\nabla^2 T)(X_1, \ldots, X_k; X, Y) - (\nabla T \otimes \phi)(X_1, \ldots, X_k; X, Y))T(Y_1, \ldots, Y_k)
\]

\[
= ((\nabla^2 T)(Y_1, \ldots, Y_k; X, Y) - (\nabla T \otimes \phi)(Y_1, \ldots, Y_k; X, Y))T(X_1, \ldots, X_k),
\]

hold on \( M \), where \( \phi \) is a 1-form on \( M \). From this it follows that at a point \( x \in M \) if the tensor \( T \) is non-zero, then there exists a unique \((0, 2)\)-tensor \( \psi \), defined on a neighborhood \( U \) of \( x \), such that

\[
\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi, \tag{1.5}
\]

holds on \( U \).

2. Isometric Immersion

Let \( f : (M, g) \rightarrow (\tilde{M}, \tilde{g}) \) be an isometric immersion from an \( n \)-dimensional Riemannian manifold \((M, g)\) into \((n + d)\)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\), \( n \geq 2, d \geq 1 \). We denote by \( \nabla \) and \( \tilde{\nabla} \) as Levi-Civita connection of \( M^n \)
and $\tilde{M}^{n+d}$ respectively. Then the formulas of Gauss and Weingarten are given by
\begin{align}
\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \quad (2.1) \\
\tilde{\nabla}_X N &= -A_N X + \nabla^\bot_X N, \quad (2.2)
\end{align}
for any tangent vector fields $X, Y$ and the normal vector field $N$ on $M$, where $\sigma, A$ and $\nabla^\bot$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form $\sigma$ is identically zero, then the manifold is said to be \textit{totally geodesic}. The second fundamental form $\sigma$ and $A_N$ are related by
\[ \tilde{g}(\sigma(X, Y), N) = g(A_N X, Y), \]
for tangent vector fields $X, Y$. The first and second covariant derivatives of the second fundamental form $\sigma$ are given by
\begin{align}
(\tilde{\nabla}_X \sigma)(Y, Z) &= \nabla^\bot_X (\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \quad (2.3) \\
(\tilde{\nabla}^2 \sigma)(Z, W, X, Y) &= (\tilde{\nabla}_X \tilde{\nabla}_Y \sigma)(Z, W), \quad (2.4) \\
&= \nabla^\bot_X ((\tilde{\nabla}_Y \sigma)(Z, W)) - (\tilde{\nabla}_Y \sigma)(\nabla_X Z, W) \\
&\quad - (\tilde{\nabla}_X \sigma)(Z, \nabla_Y W) - (\tilde{\nabla}_{\nabla_X Y} \sigma)(Z, W)
\end{align}
respectively, where $\tilde{\nabla}$ is called the \textit{vander Waerden-Bortolotti connection} of $M$ [7]. If $\tilde{\nabla}\sigma = 0$, then $M$ is said to have \textit{parallel second fundamental form} [7].

3. \textbf{Sasakian Manifolds}

An $n$-dimensional differential manifold $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$ if it carries a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and 1-form $\eta$ on $M$ respectively such that
\[ \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi \xi = 0. \quad (3.1) \]

Thus a manifold $M$ equipped with this structure is called an almost contact manifold and is denoted by $(M, \phi, \xi, \eta)$. If $g$ is a Riemannian metric on an almost contact manifold $M$ such that
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (3.2) \]
where $X, Y$ are vector fields defined on $M$, then $M$ is said to have an almost contact metric structure $(\phi, \xi, \eta, g)$ and $M$ with this structure is called an almost contact metric manifold and is denoted by $(M, \phi, \xi, \eta, g)$.

If on $(M, \phi, \xi, \eta, g)$ the exterior derivative of 1-form $\eta$ satisfies,
\[ \Phi(X, Y) = d\eta(X, Y) = g(X, \phi Y), \quad (3.3) \]
then $(\phi, \xi, \eta, g)$ is said to be a contact metric structure and together with manifold $M$ is called contact metric manifold and $\Phi$ is a 2-form. The contact metric structure
(\(M, \phi, \xi, \eta, g\)) is said to be normal if
\[
[\phi, \phi](X, Y) + 2d \eta \otimes \xi = 0. \tag{3.4}
\]

If the contact metric structure is normal, then it is called a Sasakian structure and \(M\) is called a Sasakian manifold. Note that an Almost contact metric manifold defines Sasakian structure if and only if
\[
(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.5}
\]
\[
\nabla_X \xi = -\phi X. \tag{3.6}
\]

**Example of Sasakian manifold.** Consider the 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{E_1, E_2, E_3\}\) be linearly independent global frame field on \(M\) given by
\[
E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial y} + 2xe^z \frac{\partial}{\partial z}, \quad E_3 = e^z \frac{\partial}{\partial z}.
\]
Let \(g\) be the Riemannian metric defined by
\[
g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0,
\]
\[
g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.
\]
The \((\phi, \xi, \eta)\) is given by
\[
\eta = -2xdy + e^{-z}dz, \quad \xi = E_3 = \frac{\partial}{\partial z},
\]
\[
\phi E_1 = E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0.
\]
The linearity property of \(\phi\) and \(g\) yields
\[
\eta(E_3) = 1, \quad \phi^2 U = -U + \eta(U)E_3,
\]
\[
g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W), \quad g(U, \xi) = \eta(U),
\]
for any vector fields \(U, W\) on \(M\). By definition of Lie bracket, we have
\[
[E_1, E_2] = 2E_3.
\]
The Levi-Civita connection with respect to above metric \(g\) and be given by Koszula formula
\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))
\]
\[
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]
Then, we have
\[
\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = E_3, \quad \nabla_{E_1} E_3 = -E_2,
\]
\[
\nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = E_1,
\]
\[
\nabla_{E_3} E_1 = -E_2, \quad \nabla_{E_3} E_2 = E_1, \quad \nabla_{E_3} E_3 = 0.
\]
The tangent vectors $X$ and $Y$ to $M$ are expressed as linear combination of $E_1, E_2, E_3$, i.e., $X = a_1 E_1 + a_2 E_2 + a_3 E_3$ and $Y = b_1 E_1 + b_2 E_2 + b_3 E_3$, where $a_i$ and $b_j$ are scalars. Clearly $(\phi, \xi, \eta, g)$ and $X, Y$ satisfy equations (3.1), (3.2), (3.5) and (3.6). Thus $M$ is a Sasakian manifold. Further the following relations hold:

\[
R(X, Y)Z = \{g(Y, Z)X - g(X, Z)Y\},
\]
\[
R(X, Y)\xi = \{\eta(Y)X - \eta(X)Y\},
\]
\[
R(\xi, X)Y = \{g(X, Y)\xi - \eta(Y)X\},
\]
\[
R(\xi, X)\xi = \{\eta(X)\xi - X\},
\]
\[
S(X, \xi) = (n - 1)\eta(X),
\]
\[
Q\xi = (n - 1)\xi,
\]
for all vector fields, $X, Y, Z$ and where $\nabla$ denotes the operator of covariant differentiation with respect to $g$, $\phi$ is a $(1, 1)$ tensor field, $S$ is the Ricci tensor of type $(0, 2)$ and $R$ is the Riemannian curvature tensor of the manifold.

4. Invariant Submanifolds of Sasakian Manifolds admitting Semi-symmetric Metric Connection

If $\tilde{M}$ is a Sasakian manifold with structure tensors $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, then we know that its invariant submanifold $M$ has the induced Sasakian structure $(\phi, \xi, \eta, g)$.

A submanifold $M$ of a Sasakian manifold $\tilde{M}$ with a semi-symmetric metric connection is called an invariant submanifold of $\tilde{M}$ with a semi-symmetric metric connection, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, $\xi$ becomes tangent to $M$. For an invariant submanifold of a Sasakian manifold with a semi-symmetric metric connection, we have

\[
\sigma(X, \xi) = 0,
\]
for any vector $X$ tangent to $M$.

Let $\tilde{M}$ be a Sasakian manifold admitting a semi-symmetric metric connection $\tilde{\nabla}$.

**Lemma 1.** Let $M$ be an invariant submanifold of contact metric manifold $\tilde{M}$ which admits semi-symmetric metric connection $\tilde{\nabla}$ and let $\sigma$ and $\tilde{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and semi-symmetric metric connection, then (a) $M$ admits semi-symmetric metric connection, (b) the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}$ are equal.

**Proof.** We know that the contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on $\tilde{M}$ induces $(\phi, \xi, \eta, g)$ on invariant submanifold. By virtue of (1.1), we get

\[
\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi.
\]
By using (2.1) in (4.2), we get
\[ \nabla_X Y = \nabla_X Y + \sigma(X, Y) + \eta(Y)X - g(X, Y)\xi. \]  
(4.3)

Now Gauss formula (2.1) with respect to semi-symmetric metric connection is given by
\[ \nabla_X Y = \nabla_X Y + \sigma(X, Y). \]  
(4.4)

Equating (4.3) and (4.4), we get (1.1) and
\[ \sigma(X, Y) = \sigma(X, Y). \]  
(4.5)

5. Recurrent Invariant Submanifolds of Sasakian Manifolds Admitting Semi-symmetric Metric Connection

We consider invariant submanifolds of a Sasakian manifold when \( \sigma \) is recurrent, 2-recurrent, generalized 2-recurrent and \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection. We write the equations (2.3) and (2.4) with respect to semi-symmetric metric connection in the form
\[ (\nabla_X \sigma)(Y, Z) = \nabla_X^{\perp}(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \]  
(5.1)
\[ (\nabla^2 \sigma)(Z, W, X, Y) = (\nabla_X \nabla^*_Y \sigma)(Z, W), \]
\[ = \nabla_X^{\perp}(\nabla_Y \sigma)(Z, W) - (\nabla_Y \sigma)(\nabla_X Z, W) - (\nabla_X \sigma)(Z, \nabla_Y W) - (\nabla_X \nabla_Y \sigma)(Z, W). \]  
(5.2)

We prove the following theorems

**Theorem 1.** Let \( M \) be an invariant submanifold of a Sasakian manifold \( \tilde{M} \) admitting semi-symmetric metric connection. Then \( \sigma \) is recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let \( \sigma \) be recurrent with respect to semi-symmetric metric connection. Then from (1.3) we get
\[ (\nabla_X \sigma)(Y, Z) = \phi(X)\sigma(Y, Z), \]
where \( \phi \) is a 1-form on \( M \). By using (5.1) and \( Z = \xi \) in the above equation, we have
\[ \nabla_X^{\perp}\sigma(Y, \xi) - \sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = \phi(X)\sigma(Y, \xi), \]  
(5.3)

which by virtue of (4.1) reduces to
\[ -\sigma(\nabla_X Y, \xi) - \sigma(Y, \nabla_X \xi) = 0. \]  
(5.4)
Using (1.1), (3.1), (3.6) and (4.1) in (5.4), we get
\[ \sigma(Y, \phi X) - \sigma(Y, X) = 0. \]  
(5.5)

Replace \( X \) by \( \phi X \) and by virtue of (3.1) and (4.1) in (5.5), we get
\[ -\sigma(Y, X) - \sigma(Y, \phi X) = 0. \]  
(5.6)

Adding equation (5.5) and (5.6), we obtain \( \sigma(X, Y) = 0 \). Thus \( M \) is totally geodesic. The converse statement is trivial. This proves the theorem. \( \square \)

**Theorem 2.** Let \( M \) be an invariant submanifold of a Sasakian manifold \( \tilde{M} \) admitting semi-symmetric metric connection. Then \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let \( M \) has parallel third fundamental form with respect to semi-symmetric metric connection. Then we have
\[ \overline{\nabla}_X \overline{\nabla}_Y \sigma(Z, W) = 0. \]

Taking \( W = \xi \) and using (5.2) in the above equation, we have
\[ \overline{\nabla}_X ((\overline{\nabla}_Y \sigma)(Z, \xi)) - (\overline{\nabla}_Y \sigma)(\overline{\nabla}_X Z, \xi) - (\overline{\nabla}_X \sigma)(Z, \overline{\nabla}_Y \xi) - (\overline{\nabla}_{\overline{\nabla}_X \xi} \sigma)(Z, \xi) = 0. \]  
(5.7)

By using (4.1) and (5.1) in (5.7), we get
\[ 0 = -\overline{\nabla}_X \{ \sigma(\overline{\nabla}_Y Z, \xi) + \sigma(Z, \overline{\nabla}_Y \xi) \} - \overline{\nabla}_Y \sigma(\overline{\nabla}_X Z, \xi) + \sigma(\overline{\nabla}_Y \overline{\nabla}_X Z, \xi) \]
\[ + 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) - \overline{\nabla}_X \sigma(Z, \overline{\nabla}_Y \xi) + \sigma(Z, \overline{\nabla}_X \overline{\nabla}_Y \xi) + \sigma(\overline{\nabla}_{\overline{\nabla}_X \xi} \sigma)(Z, \xi) \]
\[ + \sigma(Z, \overline{\nabla}_{\overline{\nabla}_X \xi} \xi). \]  
(5.8)

In view of (1.1), (3.1), (3.6) and (4.1) the above result (5.8) gives
\[ 0 = 2\overline{\nabla}_X \sigma(Z, \phi Y) - 2\overline{\nabla}_X \sigma(Z, Y) - \sigma(Z, \overline{\nabla}_X \phi Y) - \sigma(Z, \overline{\nabla}_X \eta(Y)) \xi \]
\[ - \sigma(Z, \phi \overline{\nabla}_X Y) - \eta(Y)\sigma(Z, \phi X) + 2\sigma(Z, \overline{\nabla}_X Y) + \eta(Y)\sigma(Z, X) \]
\[ - 2\sigma(\overline{\nabla}_X Z, \phi Y) + 2\sigma(\overline{\nabla}_X Z, Y) - 2\eta(Z)\sigma(X, \phi Y) + 2\eta(Z)\sigma(X, Y). \]  
(5.9)

Put \( Y = \xi \) and using (3.1), (3.6), (4.1) in (5.9), we get
\[ 0 = -2\sigma(Z, \phi X). \]  
(5.10)

Replacing \( X \) by \( \phi X \) and using (3.1) and (4.1) in (5.10) to obtain \( \sigma(X, Z) = 0 \). Thus \( M \) is totally geodesic. The converse statement is trivial. This proves the theorem. \( \square \)

**Corollary 1.** Let \( M \) be an invariant submanifold of a Sasakian manifold \( \tilde{M} \) admitting semi-symmetric metric connection. Then \( \sigma \) is 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.
Proof. Let \( \sigma \) be 2-recurrent with respect to semi-symmetric metric connection. From (1.4), we have
\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, W) = \sigma(Z, W)\phi(X, Y).
\]
Taking \( W = \xi \) and using (5.2) in the above equation, we have
\[
\overline{\nabla}^\perp_X ((\overline{\nabla}_Y \sigma)(Z, \xi)) - (\overline{\nabla}_Y \sigma)(\overline{\nabla}_X Z, \xi) - (\overline{\nabla}_X \sigma)(Z, \overline{\nabla}_Y \xi) - (\overline{\nabla}^\perp_{\overline{\nabla}_X Y} \sigma)(Z, \xi)
= \sigma(Z, \xi)\phi(X, Y). \tag{5.11}
\]
In view of (4.1) and (5.1) we write (5.11) in the form
\[
0 = -\overline{\nabla}^\perp_X \{\sigma(\overline{\nabla}_Y Z, \xi) + \sigma(Z, \overline{\nabla}_Y \xi)\} - \overline{\nabla}^\perp_Y \sigma(\overline{\nabla}_X Z, \xi) + \sigma(\overline{\nabla}_Y \overline{\nabla}_X Z, \xi)
+ 2\sigma(\overline{\nabla}_X Z, \overline{\nabla}_Y \xi) - \overline{\nabla}^\perp_X \sigma(Z, \overline{\nabla}_Y \xi) + \sigma(Z, \overline{\nabla}_X \overline{\nabla}_Y \xi)
+ \sigma(\overline{\nabla}^\perp_{\overline{\nabla}_X Y} Z, \xi) + \sigma(Z, \overline{\nabla}^\perp_{\overline{\nabla}_X Y} \xi). \tag{5.12}
\]
Using (1.1), (3.1), (3.6) and (4.1) in (5.12), we get
\[
0 = 2\overline{\nabla}^\perp_X \sigma(Z, \phi Y) - 2\overline{\nabla}^\perp_X \sigma(Z, Y) - \sigma(Z, \overline{\nabla}_X \phi Y)
- \sigma(Z, \overline{\nabla}_X \eta(Y)\xi) - \sigma(Z, \phi \overline{\nabla}_X Y) - \eta(Y)\sigma(Z, \phi X) + 2\sigma(Z, \overline{\nabla}_X Y)
+ \eta(Y)\sigma(Z, X) - 2\sigma(\overline{\nabla}_X Z, \phi Y) + 2\sigma(\overline{\nabla}_X Z, Y)
- 2\eta(Z)\sigma(X, \phi Y) + 2\eta(Z)\sigma(X, Y). \tag{5.13}
\]
Taking \( Y = \xi \) and using (3.1), (3.6), (4.1) in (5.13), we get
\[
0 = -2\sigma(Z, \phi X). \tag{5.14}
\]
Replacing \( X \) by \( \phi X \) and using (3.1) and (4.1) in (5.14) to obtain \( \sigma(X, Z) = 0 \). Thus \( M \) is totally geodesic. The converse statement is trivial. This proves the theorem. \( \square \)

**Theorem 3.** Let \( M \) be an invariant submanifold of a Sasakian manifold \( \tilde{M} \) admitting semi-symmetric metric connection. Then \( \sigma \) is generalized 2-recurrent with respect to semi-symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.

**Proof.** Let \( \sigma \) be generalized 2-recurrent with respect to semi-symmetric metric connection. From (1.5), we have
\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, W) = \psi(X, Y)\sigma(Z, W) + \phi(X)(\overline{\nabla}_Y \sigma)(Z, W), \tag{5.15}
\]
where \( \psi \) and \( \phi \) are 2-recurrent and 1-form respectively. Taking \( W = \xi \) in (5.15) and using (4.1), we get
\[
(\overline{\nabla}_X \overline{\nabla}_Y \sigma)(Z, \xi) = \phi(X)(\overline{\nabla}_Y \sigma)(Z, \xi).
\]
Using (5.2) and (4.1) in above equation, we get
\[
\nabla^\perp_X ((\nabla_Y \sigma)(Z, \xi)) - (\nabla_Y \sigma)(\nabla_X Z, \xi) = -\phi(X)[\sigma(\nabla_Y Z, \xi) + \sigma(Z, \nabla_Y \xi)].
\]
In view of (4.1) and by virtue of (5.1), the above result gives (5.16), we get
\[
-\nabla^\perp_X \{\sigma(\nabla_Y Z, \xi) + \sigma(Z, \nabla_Y \xi)\} - \nabla_Y^\perp \sigma(\nabla_X Z, \xi) + 2\sigma(\nabla_X Z, \nabla_Y \xi)
\]

Choosing \( Y = \xi \) and using (3.1), (4.1) in (5.18), we get
\[
0 = -2\sigma(Z, \phi X).
\]
Replacing \( X \) by \( \phi X \) and using (3.1) and (4.1) in (5.19) to obtain \( \sigma(X, Z) = 0 \).
Thus \( M \) is totally geodesic. The converse statement is trivial. This proves the theorem. \( \square \)

Using Theorems 5.1 to 5.3 and Corollary 5.1, we have the following result

**Corollary 2.** Let \( M \) be an invariant submanifold of a Sasakian manifold \( \tilde{M} \) admitting semi-symmetric metric connection. Then the following statements are equivalent

(i) \( \sigma \) is recurrent.

(ii) \( \sigma \) is 2-recurrent.

(iii) \( \sigma \) is generalized 2-recurrent.

(iv) \( M \) has parallel third fundamental form.

**References**


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Received May 18, 2012

Accepted July 14, 2012