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Global Existence and Blow-up of Solutions to a Quasilinear Parabolic Equation with Nonlocal Source and Nonlinear Boundary Condition*

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Abstract. This paper investigates the behavior of positive solution to the following p-Laplacian equation

$$u_t - (|u_x|^{p-2}u_x)_x = \int_0^a u^\alpha(\xi,t)d\xi + ku^\beta(x,t), \quad (x,t) \in [0,a] \times (0,T)$$

with nonlinear boundary condition $u_x|_{x=0} = 0$, $u_x|_{x=a} = u^q|_{x=a}$, where $p \geq 2$, $\alpha, \beta, k, q > 0$. The authors first get the local existence result by a regularization method. Then under appropriate hypotheses, the authors establish that positive weak solution either exists globally or blow up in finite time by using comparison principle.

1. **Introduction**

In this paper, we study the following p-Laplacian equation with nonlocal source and inner absorption:

$$\begin{cases} u_t - (|u_x|^{p-2}u_x)_x = \displaystyle\int_0^a u^\alpha(\xi,t)d\xi + ku^\beta(x,t), & (x,t) \in [0,a] \times (0,T) \\ u_x|_{x=0} = 0,\ u_x|_{x=a} = u^q|_{x=a}, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.1}$$

where $T > 0$, $p > 2$, $\alpha, \beta, k, q > 0$, $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.

Problems of this form arise in mathematical models such as modeling gas or fluid flow through a porous medium and completely turbulent flow and for the spread of certain biological populations, see [1, 2, 3, 4, 5, 6] and the references therein. Parabolic equations involving a nonlocal source, which arise in a population model that communicates through chemical means, were studied

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in [7, 8]. The nonlinear boundary condition in (1.1) can be physically interpreted as a nonlinear radiation law, see [9].

Over the last few years, much effort has been devoted to the study of blow-up properties for nonlocal semilinear parabolic equations with nonlinear boundary conditions. Conditions on blowing up, blow-up set, blow-up rate and asymptotic behavior of solutions are obtained (see [10, 11, 12, 21, 22, 23, 24] and the references therein). The problem concerning (1.1) include the existence and multiplicity of global solutions, blowing-up, blow-up rates and blow-up sets, uniqueness and nonuniqueness etc.

Peng and Yang [12] investigated the blow-up properties of the following problem

$$\begin{cases} x^q u_t - u_{xx} = \int_0^a u^m(x,t)dx - ku^n(x,t), & (x,t) \in (0,a)\times(0,T) \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.2}$$

The motivation for studying problem (1.2) comes from Ockendon's model (see [13]) for the flow in a channel of a fluid whose viscosity depends on temperature

$$xu_t = u_{xx} + e^u, \tag{1.3}$$

where u represents the temperature of the fluid.

In [14], Galaktionov and Levine studied the heat conduction equation with gradient dependent diffusion

$$\begin{cases} u_t = (|u_x|^{m-1}u_x)_x, & x > 0,\ \ 0 < t < T, \\ -|u_x|^{m-1}u_x(0,t) = u^p(0,t), & t \in (0,T), \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.4}$$

where $m \geq 1$ and u_0 has compact support. They proved that the critical global exponent and the critical Fujita exponent for the problem (1.4).

For the p-Laplacian equation, a few authors (see [15, 16]) have investigate the following equation:

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^q. \tag{1.5}$$

with initial and boundary conditions. Roughly speaking, their results are:

(1) the solution u exists globally if $q < p-1$, and
(2) u blows up in finite time if $q > p-1$ and $u_0(x)$ is sufficiently large.

The authors in [17] studied the following equation:

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega u^q(x,t)dx. \tag{1.6}$$

with null Dirichlet conditions and obtain that the solution either exists globally or blows up in finite time. Under appropriate hypotheses, they have local theory of the solution and obtain that the solution either exists globally or blow-up in finite time.

Recently, in [18], the following problem has been intensively studied by X. Wu:

$$\begin{cases} u_t = (|u_x|^{p_1-1}u_x)_x + \int_0^a v^{m_1}(\xi,t)d\xi, & (x,t)\in[0,a]\times(0,T), \\ v_t = (|v_x|^{p_2-1}v_x)_x + \int_0^a u^{m_2}(\xi,t)d\xi, & (x,t)\in[0,a]\times(0,T), \\ u_x|_{x=0}=0, u_x|_{x=a}=u^{q_{11}}v^{q_{12}}|_{x=a}, v_x|_{x=0}=0, v_x|_{x=a}=u^{q_{21}}v^{q_{22}}|_{x=a}, & \\ & t\in(0,T), \\ u(x,0)=u_0(x), v(x,0)=v_0(x), & x\in[0,a], \end{cases} \tag{1.7}$$

where $T>0$, $m_1,m_2,q_{11},q_{12},q_{21},q_{22}>0$, $p_1,p_2>1$. They proved the global existence and blow-up of positive weak solutions of (1.7) by using comparison principle.

However, to the authorąŕs best knowledge, there is little literature on the study of the global existence and blow-up properties for the system (1.1). Motivated by the above works, in this paper, we study global existence or blow-up of weak solutions to (1.1). Note that (1.1) has nonlinear and nonlocal source $\int_0^a u^\alpha(\xi,t)d\xi$, local term $u^\beta(x,t)$ and nonlinear boundary condition u^q, which make the behavior of the solution different from that for that of homogeneous Neumann or Dirichlet boundary value problems. To overcome these difficulties, we used some modification of the technique in [19] so that we can handle the nonlinearities. In this paper, the blow-up means that there exists a $T^*<+\infty$ such that $\|u(\cdot,t)\|_\infty<\infty$ for $t\in(0,T^*)$ and $\lim\limits_{t\to T^*}\|u(\cdot,t)\|_\infty=\infty$.

The outline of this paper is as follows: in the next section, we will prove the local existence results by a regularization method. We will give the proof of a weak comparison principle and discuss the global existence and blow-up of solutions in the third section.

2. **Local existence**

In this section, we study the local existence of (1.1) under appropriate hypotheses. From the point of physics, we need only to consider the nonnegative solutions. Since (1.1) is the degenerate parabolic equations for $|u_x|=0$, one cannot expect the existence of classical solution. As it is now well known that degenerate equations need not posses classical solutions, most of studies of p-Laplacian equations concerned with weak solutions (see [7, 9]). We begin by giving a precise definition of a weak solution for problem (1.1).

Before stating our main results, we make some assumptions. Let $D=(0,a)$ and $\Omega_r=D\times(0,r]$. Let $\overline{D}$ and $\overline{\Omega}_r$ be their respective closures, $z^+=\max\{z,0\}$. Before stating our main results, we make some assumptions on the initial value $u_0(x)$.

(H_1) $u_0(x)\in C^{2+\alpha}([0,a])$ for some $0<\alpha<1$, $u_0(x)\ge\delta>0$;

(H_2) $(|u_{0x}|^{p-1}u_{0x})_x \in L^2([0,a])$;
(H_3) $u_0(x)$ satisfies the compatibility conditions: $u_{0x}(0)=0$, $u_{0x}(a)=u_0^q(a)$.

Definition 2.1. A function $u \in C(\Omega_T)$ is called a supersolution (subsolution) of problem (1.1) in Ω_T if all of the following hold:

(i) $u \in L^\infty(0,T;W^{1,\infty}(0,a)) \cap W^{1,2}(0,T;L^2(0,a))$, $u(x,0) \geq (\leq) u_0(x)$;
(ii) For any nonnegative functions $\varphi \in L^1(0,T;W^{1,2}(0,a)) \cap L^2(\Omega_T)$,

$$\begin{aligned}
&\iint_{\Omega_T} u_t(x,t)\varphi(x,t)dxdt + \iint_{\Omega_T} |u_x(x,t)|^{p-2}u_x(x,t)\varphi_x(x,t)dxdt \\
&\quad \geq (\leq) \int_0^T u^q(a,t)\varphi(a,t)dt + \iint_{\Omega_T}\left(\int_0^a u^\alpha(\xi,t)d\xi\right)\varphi(x,t)dxdt \\
&\qquad + k\iint_{\Omega_T} u^\beta(x,t)\varphi(x,t)dxdt. \qquad (2.1)
\end{aligned}$$

A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1). For every $T < \infty$, if u is a solution of (1.1), we say u is global.

Remark 2.2. Clearly, every nonnegative classical (sub-, super-) solution of (1.1) is a weak (sub-, super-) solution of (1.1) in the sense of Definition 2.1.

Theorem 2.3. *Assume* (H_1)-(H_3)*, then there exists a* $T_0 > 0$ *such that* (1.1) *admits a solution* $u \in C(0,T_0;L^\infty(\Omega_T)) \cap L^p(0,T_0;W_0^{1,p}(\Omega_T))$ *satisfying* $u \geq \delta$ *for some positive constant* δ*. Moreover, if* $T < +\infty$*, then*

$$\lim_{t\to T^-} \sup_{x\in[0,a]} \|u(\cdot,t)\|_{L^\infty} = +\infty.$$

In this case, we say that the solution u blows up in finite time.

Proof. The proof of this theorem basically follows line by line the proof of Theorem 1 in [19]. Consider the following approximate problems for (1.1):

$$\begin{cases} u_{\varepsilon t} - ((|u_{\varepsilon x}|^2+\varepsilon)^{\frac{p-2}{2}}u_{\varepsilon x})_x = F(u_\varepsilon), & (x,t)\in(0,a)\times(0,T), \\ u_{\varepsilon x}|_{x=0} = 0,\ \ u_{\varepsilon x}|_{x=a} = G(u_\varepsilon), & t\in(0,T), \\ u_\varepsilon(x,0) = u_0(x), & x\in[0,a]. \end{cases} \qquad (2.2)$$

We need to control the nonlocal term by applying the technique developed in [19]. Choose the bounded functions: $F(z), G(z) \in C^\infty(R)$ such that: $F(z) = \int_0^a z^\alpha(\xi,t)d\xi - kz^\beta(x,t)$, $G(z) = z^q$ for $\delta \leq z \leq M+1$ where $M = \|u_0(x)\|_\infty$. And we assume that there exist positive constants l and L such that

$$0 < l \leq F(z), G(z) \leq L < +\infty, \quad \frac{\partial G(z)}{\partial z} \geq 0$$

for any $z \in R$.

Equation (2.2) are a nondegenerate problem for each fixed ε. We divide our proof into four steps.

Step 1. There exists a small constant $t_1 > 0$ and a positive constant C independent of ε such that:

$$\|u_{\varepsilon x}\|_\infty \leq C \quad \text{on } \overline{\Omega}_{t_1}. \tag{2.3}$$

To this end, choose bounded functions: $\Phi_\varepsilon(z) \in C^\infty(R)$, $0 < \rho_\varepsilon \leq \Phi'_\varepsilon(z) \leq \rho_\varepsilon^{-1}$ and

$$\Phi_\varepsilon(z) = (|z|^2 + \varepsilon)^{\frac{p-2}{2}} z \quad \text{for } |z| \leq K + L + 1,$$

where K will be determined later. Then consider the following problem:

$$\begin{cases} u_{\varepsilon t} - (\Phi_\varepsilon(u_{\varepsilon x}))_x = F(u_\varepsilon), & (x,t) \in (0,a) \times (0,T), \\ u_{\varepsilon x}|_{x=0} = 0, \ u_{\varepsilon x}|_{x=a} = G(u_\varepsilon), & t \in (0,T), \\ u_\varepsilon(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{2.4}$$

For (2.4), standard parabolic theory (see [20]) shows that there is a solution u_ε in the class $H^{2+\beta,1+\beta/2}(\overline{\Omega}_T)$ for some $\beta \in (0,1)$. Obviously, Comparison principle holds for (2.4). Therefore,

$$u_\varepsilon(x,t) \geq \delta > 0, \quad u_{\varepsilon t}(x,t) \geq 0, \quad (x,t) \in \overline{\Omega}_T$$

and for some constant $c \in R$, $0 < c \leq F'(z) \leq c^{-1}$. Similarly to the proof of Proposition 3.1 in [19], there exists a small constant $t_1 > 0$ such that:

$$\|u_{\varepsilon x}\|_\infty \leq K + L + 1 \quad \text{on } \overline{\Omega}_{t_1},$$

where $K = \|u_{0x}(x)\|_\infty$. Thus u_ε is a solution of (2.2) in $\overline{\Omega}_{t_1}$. Setting $C = K + L + 1$, we get the conclusion.

Step 2. There exist a constant $t_2 > 0$, such that $u_\varepsilon(x,t) \leq M + 1$ on $\overline{\Omega}_{t_2}$.

We may assume that $T \in [0,1)$. Let $h \geq M$. In fact, multiplying (2.2) by $(u_\varepsilon - h)^+$ and integrating over $\overline{\Omega}_T$, we obtain

$$\begin{aligned} &\int_0^a (u_\varepsilon - h)^{+2} dx + \iint_{\overline{\Omega}_T} |u_{\varepsilon x}|^{p-2} (u_\varepsilon - h)_x^{+2} dx dt \\ &\quad \leq c \int_0^T (u_\varepsilon - h)^+|_{x=a} dt + c \iint_{\overline{\Omega}_T} (u_\varepsilon - h)^+ dx dt, \end{aligned}$$

for some positive constant c independent of ε. Then similarly to the proof of Proposition 3.1 in [19], there exists a $t_2 > 0$, independent of ε, such that

$$u_\varepsilon(x,t) \leq M + 1 \text{ on } \overline{\Omega}_{t_2}.$$

Step 3. There exist constants M_1, independent of ε, such that

$$|u_{\varepsilon t}|_{L^2(\overline{\Omega}_T)} \leq M_1 < +\infty,$$

where $T = \min\{t_1, t_2\}$.

To do so, multiplying (2.2) by $u_{\varepsilon t}$ and integrating over $\overline{\Omega}_T$, we have

$$\frac{1}{2}\int_0^a u_{\varepsilon t}^2 dx + \iint_{\overline{\Omega}_T} (u_{\varepsilon x}^2+\varepsilon)^{(p-2)/2-1}(pu_{\varepsilon x}^2+\varepsilon)u_{\varepsilon t}^2 dxdt$$
$$= \frac{1}{2}\int_0^a u_{\varepsilon 0t}^2 dx + \int_0^T ((u_{\varepsilon x}^2+\varepsilon)^{(p-2)/2-1}u_{\varepsilon x})_t u_{\varepsilon t}(a,t)dt + \iint_{\overline{\Omega}_T} F' u_{\varepsilon t}^2 dxdt\,.$$

Similarly to the proof of Proposition 3.1 in [19], we have

$$\int_0^a u_{\varepsilon t}^2 dx \le \int_0^a u_{\varepsilon 0t}^2 dx + C\iint_{\overline{\Omega}_T} u_{\varepsilon t}^2 dxdt + C\iint_{\overline{\Omega}_T} F' u_{\varepsilon t}^2 dxdt.$$

Using Young's inequality, we have

$$\int_0^a u_{\varepsilon t}^2 dx \le \int_0^a u_{\varepsilon 0t}^2 dx + C\iint_{\overline{\Omega}_T} u_{\varepsilon t}^2 dxdt.$$

By the Gronwall's lemma, we obtain the desired results.

Step 4. Therefore, by the Aubin theorem, it follows that (up to extraction of a subsequence):

$$u_\varepsilon \to u, \quad \text{a.e. for } (x,t)\in\Omega_T,$$
$$u_{\varepsilon x} \to u_x, \quad \text{weakly in } L^\infty(\Omega_T),$$
$$u_{\varepsilon t} \to u_t, \quad v_{nt}\to v_t, \quad \text{weakly in } L^2(\Omega_T),$$
$$\Phi_\varepsilon(u_{\varepsilon x})_{x_i} \to |u_x|^{p-2}u_x \text{ weakly in } L^\infty(\Omega_T).$$

From (2.3), we have

$$\lim_{n\to\infty}\iint_{\Omega_T} \psi|u_{\varepsilon x}|^{p-2}u_{\varepsilon x}(u_\varepsilon-u)_x dxdt = 0,$$

where $\psi \in C_0^{1,1}(\Omega_T)$, $\psi \ge 0$. Then similarly to the proof of Theorem 2.1 in [15], we have

$$\iint_{\Omega_T} \psi(|u_x|^{p-2}u_x - |u_{\varepsilon x}|^{p-2}u_{\varepsilon x})dxdt = 0.$$

The proof of Theorem 2.3 is completed by a standard limiting process. □

3. Global Existence and Blow-up

In this section, we shall discuss the global existence and blow-up in finite time of the solution for system (1.1). Throughout this section we denote $\tau = \frac{p-1}{p}$ and choose $\overline{\lambda}$, $\underline{\lambda}$ satisfying $\overline{\lambda} > 1 > \underline{\lambda} > 0$.

Our approach in a combination principle and upper and sub-technique. In order to prove our results, we give the following weak comparison principle.

Lemma 3.1 (Comparison principle). *Assume that u be a weak solution of* (1.1) *in* Ω_T, $\underline{u} \geq \delta$ *and* $\overline{u}$ *a subsolution and a supersolution of* (1.1) *in* Ω_T, *respectively, with nonlinear boundary flux* $\underline{\lambda u}^q$, $\overline{\lambda u}^q$ *and nonlocal terms* $\underline{\lambda}\Big(\int_0^a \underline{u}^\alpha(\xi,t)d\xi + k\underline{u}^\beta(x,t)\Big)$, $\overline{\lambda}\Big(\int_0^a \overline{u}^\alpha(\xi,t)d\xi + k\overline{u}^\beta(x,t)\Big)$. *Then* $\underline{u} \leq u \leq \overline{u}$ *on* $\overline{\Omega}_T$.

Proof. Similarly to the proof of Proposition 2.1 in [19], we have

$$\iint_{\Omega_t} (\underline{u}-u)_t \chi[\underline{u}>u]dxdt \leq \int_0^t f_1(x,t)\chi[\underline{u}>u]|_{x=a}dt + \iint_{\Omega_t} f_2(x,t)\chi[\underline{u}>u]dxdt, \tag{3.1}$$

where

$$f_1(x,t) = (\underline{\lambda u}^q)^p - (u^q)^p,$$

$$f_2(x,t) = \underline{\lambda}\left(\int_0^a \underline{u}^\alpha(\xi,t)d\xi + k\underline{u}^\beta(x,t)\right) - \left(\int_0^a u^\alpha(\xi,t)d\xi + ku^\beta(x,t)\right).$$

Since $0<\delta \leq \underline{u}(x,0) < \overline{u}(x,0)$, $0 \leq x \leq a, \underline{\lambda} < 1$, by the continuity of $\underline{u}, u$, there exists a time $\tau > 0$ such that $f_1(a,t), f_2(a,t) \leq 0$ for all $t \in [0,\tau]$. Therefore, we have that $\underline{u} \leq u$ on $\overline{\Omega}_\tau$.

Define $\tau^* = \sup\{\tau \in [0,T] : \underline{u} \leq u \text{ for all } (x,t) \in \overline{\Omega}_\tau\}$. We claim that $\tau^* = T$. Otherwise, from the continuity of $\underline{u}, u$, there exists a $\varepsilon > 0$ such that $\tau^* + \varepsilon < T$, $f_1(a,t), f_2(a,t) \leq 0$ for all $t \in [0, \tau^*+\varepsilon]$. By (3.1) we have $\underline{u} \leq u$ on $\overline{\Omega}_{\tau^*+\varepsilon}$, which contradicts the definition of τ^*. Hence $\underline{u} \leq u$ on $\overline{\Omega}_T$.

Similarly, we can prove that $u \leq \overline{u}$ on $\overline{\Omega}_T$. This completes the proof of Lemma 3.1. □

Theorem 3.2. *If* $0<\alpha<1$, $(1-\alpha)(q-1)>0$, *then the solution of* (1.1) *blows up in finite time.*

Proof. It is easy to prove that there exist positive constants $m_1 > 0$, $m_2 > 1$ satisfying

$$m_1m_2 + \alpha - 1 = 0, \quad m_1m_2 - m_1 - m_1m_2q \leq 0. \tag{3.2}$$

Set

$$w_1 = [d(1+x^{\frac{1}{\tau}}) + (c-bt)^{-m_1}]^{m_2} = [S_1]^{m_2},$$

where $b,c,d>0$ satisfy

$$b \leq \underline{\lambda} a(m_1m_2)^{-1}2^{1-m_2}, \quad c \geq 2^{\frac{1}{m_1}}\delta^{-\frac{1}{m_1m_2}},$$

$$d \leq \min\left\{\frac{1}{1+a^{\frac{1}{\tau}}}c^{-m_1}, \underline{\lambda}\left(m_2\frac{1}{\tau}a^{\frac{1}{p-1}}2^{m_2-1}\right)^{-1}c^{m_1m_2-m_1-m_1m_2q}\right\}.$$

Computing directly, we obtain

$$w_{1t} = m_2[S_1]^{m_2-1}m_1b(c-bt)^{-m_1-1}, \;\; w_{1x} = m_2[S_1]^{m_2-1}d\frac{1}{\tau}x^{\frac{1}{p-1}}. \tag{3.3}$$

By (3.2) and (3.3) we have

$$\begin{aligned}
&(w_{1x}^{p-1})_x + \lambda\left(\int_0^a w_1^{\alpha}(\xi,t)d\xi + kw_1^{\beta}\right) \\
&\quad > \left(m_2 d\frac{1}{\tau}\right)^{p-1}\{[S_1]^{(p-1)(m_2-1)}x\}_x + \underline{\lambda}\int_0^a [S_1]^{m_2\alpha}(\xi,t)d\xi \\
&\quad \geq \left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} + \underline{\lambda}\int_0^a (c-bt)^{-m_1 m_2\alpha}d\xi \\
&\quad \geq \left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} + \underline{\lambda}a(c-bt)^{-m_1 m_2\alpha} \\
&\quad > \left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} + \underline{\lambda}a(c-bt)^{-m_1(m_2-1)-m_1-1}. \qquad (3.4)
\end{aligned}$$

For the conditions defined before, we can also get

$$\left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} > 0, \quad \underline{\lambda}a \geq m_1 m_2 b 2^{m_2-1},$$

thus

$$\begin{aligned}
&\left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} + \underline{\lambda}a(c-bt)^{-m_1(m_2-1)-m_1-1} \\
&\quad \geq m_1 m_2 b 2^{m_2-1}(c-bt)^{-m_1(m_2-1)-m_1-1}. \qquad (3.5)
\end{aligned}$$

For $0 \leq x \leq a$, $d \leq \frac{1}{1+a^{\frac{1}{\tau}}}c^{-m_1}$, we have

$$\begin{aligned}
[S_1] &= d(1+x^{\frac{1}{\tau}}) + (c-bt)^{-m_1} \\
&\leq (1+x^{\frac{1}{\tau}})\frac{1}{1+a^{\frac{1}{\tau}}}c^{-m_1} + (c-bt)^{-m_1} \\
&< 2(c-bt)^{-m_1},
\end{aligned}$$

also we have

$$\begin{aligned}
w_{1t} &= m_2[S_1]^{m_2-1}m_1 b(c-bt)^{-m_1-1} \\
&\leq m_1 m_2 b 2^{m_2-1}(c-bt)^{-m_1(m_2-1)-m_1-1}. \qquad (3.6)
\end{aligned}$$

From (3.5) and (3.6), the following result can be proved

$$\left(m_2 d\frac{1}{\tau}\right)^{p-1}[S_1]^{(p-1)(m_2-1)} + \underline{\lambda}a(c-bt)^{-m_1(m_2-1)-m_1-1} \geq w_{1t}. \qquad (3.7)$$

By (3.4) and (3.7) we have

$$(w_{1x}^{p-1})_x + \lambda\left(\int_0^a w_1^{\alpha}(\xi,t)d\xi + kw_1^{\beta}\right) \geq w_{1t}. \qquad (3.8)$$

Noting that on the boundary

$$
\begin{aligned}
w_{1x}(a,t) &= m_2[S_1]^{m_2-1} d\frac{1}{\tau} a^{\frac{1}{p-1}} \\
&\leq m_2 d\frac{1}{\tau} a^{\frac{1}{p-1}} 2^{m_2-1}(c-bt)^{-m_1(m_2-1)} \leq \underline{\lambda} w_1^q.
\end{aligned}
\tag{3.9}
$$

Under the assumptions of a, b, m_1, m_2, we have that for $x \in [0,1]$,

$$
w_1(x,0) \leq [d(1+a^{\frac{1}{\tau}})+c^{-m_1}]^{m_2} \leq \delta \leq u_0(x). \tag{3.10}
$$

From (3.8)-(3.10) and the comparison principle, it follows that $\geq w_1$. This shows that u blows up in finite time. □

Theorem 3.3. *If $\alpha = \beta \leq 1$, and $u_0(x)$ satisfies (H_1)-(H_3). Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem*

$$
\begin{cases}
-(|\varphi_x|^{p-2}\varphi_x)_x = 1, & x \in (0,a), \\
\varphi'(0) = 0, & \varphi'(a) = 1.
\end{cases}
$$

Then there exists constants $a_1 > 0$ such that the solution $u(x,t)$ of (1.1) exists globally when $u_0(x) \leq a_1\varphi(x)$.

Proof. Let $w_2(x) = a_1\varphi(x)$ where $a_1 > 0$ is chosen so that

$$
-(|w_{2x}|^{p-2}w_{2x})_x = a_1 > \overline{\lambda} a_1^m \left(\int_0^a \varphi^\alpha(\xi)d\xi + k\varphi^\alpha(x) \right) = \overline{\lambda} \left(\int_0^a w_2^\alpha(\xi)d\xi + kw_2^\alpha(x) \right),
$$

$$
w_{2x}|_{x=a} = a_1 \geq \overline{\lambda} w_2^q.
$$

By Lemma 3.1 it follows that $u(x,t)$ exists globally provided that $u_0(x) \leq w_2(x) = a_1\varphi(x)$. □

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