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Research Article

$\mathcal{D}$-squares and E-squares

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Abstract. The concept of E-squares introduced by Prof. K.S.S. Nambooripad plays an important role in the study of structure of Semigroups. Multiplicative semigroups of rings form an important class of semigroups and one theme in the study of semigroups is how the structure of this semigroup affects the structure of the ring. An important tool in analyzing the structure of a semigroup are the Green's relations. In this paper, we study some properties of these relations on the multiplicative semigroup of a regular ring [5] and hence the properties of E-squares and $\mathcal{D}$-squares.

Keywords. $\mathcal{D}$-square; Green's relations; E-squares

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1. Introduction

A semigroup [3] is a set with an associative binary operation. In particular every ring is a semigroup, considering its multiplication alone. The concept of regularity for elements of a ring was introduced by von Neumann: an element x of a ring R is said to be *regular* [10], if there exists an element x' in R such that $xx'x = x$. The ring itself is said to be regular [6], if all its elements are regular [1]. An element e in a ring R is an idempotent if $e^2 = e$. If e and f are idempotents in a ring R, then $e \le f$ if and only if $ef = e = fe$ [9].

Green's $\mathcal{D}$-relation [7] is the motivation for the definition of $\mathcal{D}$-squares. The set of elements $\{e, f, x, x'\}$ in a semigroup can be written in the form of a square called a $\mathcal{D}$-square as $\begin{Bmatrix} e & x' \\ x & f \end{Bmatrix}$ if $e\mathcal{R}x'\mathcal{L}f$ and $e\mathcal{L}x\mathcal{R}f$ where e and f are idempotents and x' is the generalized inverse of x. That is $\begin{Bmatrix} e & x' \\ x & f \end{Bmatrix}$ is a $\mathcal{D}$-square [8] if $xx' = f$, $x'x = e$, $xx'x = x$ and $x'xx' = x$. $\begin{Bmatrix} e & f \\ h & g \end{Bmatrix}$ is an E-square [9] if $e\mathcal{R}f\mathcal{L}g\mathcal{R}h$ is an E-chain.

Not all $\mathcal{D}$-squares are E-squares as the elements of the $\mathcal{D}$-square need not be idempotents. Also not all E-squares are $\mathcal{D}$-squares as the off diagonal entries need not be generalized inverses of each other.

2. Main Results

Theorem 1. *Let $\begin{Bmatrix} e & x' \\ x & f \end{Bmatrix}$ and $\begin{Bmatrix} g & y' \\ y & h \end{Bmatrix}$ be $\mathcal{D}$-squares in a ring R satisfying $e \le 1-g$ and $f \le 1-h$. Then $\begin{Bmatrix} e+g & x'+y' \\ x+y & f+h \end{Bmatrix}$ is a $\mathcal{D}$-square.*

Proof. Given $g \le 1-e$. So $g(1-e) = g = (1-e)g$. That is $g - ge = g = g - eg$. Thus $ge = eg = 0$. Similarly $fh = hf = 0$. Thus $(e+g)$ and $(f+h)$ are idempotents. Also, $xx' = f$, $x'x = e$, $yy' = h$, and $y'y = g$. Now,

$$xy' = (xe)(gy') = x(eg)y' = x.0.y' = 0$$

and

$$x'y = (x'f)(hy) = x'(fh)y = x'.0.y = 0.$$

Similarly

$$xy' = y'x = 0.$$

Thus

$$(x+y)(x'+y') = xx' + xy' + yx' + yy' = f + h.$$

Similarly,

$$(x'+y')(x+y) = e + g.$$

Also

$$fy = f(hy) = (fh)y = 0$$

and

$$hx = h(fx) = 0$$

and so

$$(x+y)(x'+y')(x+y) = (f+h)(x+y) = fx + fy + hx + hy = x + y.$$

Similarly,

$$(x'+y')(x+y)(x'+y') = x' + y'.$$ □

Theorem 2. *If $\begin{Bmatrix} e & x' \\ x & f \end{Bmatrix}$ is a $\mathcal{D}$-square in a ring and g is any idempotent satisfying $g \le 1-e$ and $g \le 1-f$, then $\begin{Bmatrix} e+g & x'+g \\ x+g & f+g \end{Bmatrix}$ is also a $\mathcal{D}$-square.*

Theorem 3. *If* $\begin{Bmatrix} e & f \\ h & g \end{Bmatrix}$ *is and E-square and w is an idempotent satisfying $w \le 1-e$ and $w \le 1-g$, then $w \le 1-f$ and $w \le 1-h$. Moreover,* $\begin{Bmatrix} e+w & f+w \\ h+w & g+w \end{Bmatrix}$ *is an E-square.*

Proof. As $w \le 1-e$ and $w \le 1-g$, we get $ew = we = 0$ and $gw = wg = 0$. Then

$$fw = (fg)w = f(gw) = 0$$

and

$$wf = w(ef) = (we)f = 0.$$

So,

$$w(1-f) = w - wf = w$$

and

$$(1-f)w = w - fw = w.$$

Thus $w \le 1-f$. Similarly, $w \le 1-h$. Thus

$$ew = we = fw = wf = gw = wg = hw = wh = 0$$

and so $(e+w)$, $(f+w)$, $(g+w)$ and $(h+w)$ are idempotents. Now

$$(e+w)(f+w) = ef + ew + wf + w^2 = f + w$$

and $(f+w)(e+w) = e+w$. Thus $(e+w)\mathcal{R}(f+w)$.
Similarly we can prove that $(f+w)\mathcal{L}(g+w)\mathcal{R}(h+w)\mathcal{L}(e+w)$. □

Theorem 4. *If* $\begin{Bmatrix} e & x' \\ x & f \end{Bmatrix}$ *is a $\mathcal{D}$-square and u is a unit in a ring R, then* $\begin{Bmatrix} u^{-1}eu & u^{-1}x'u \\ u^{-1}xu & u^{-1}fu \end{Bmatrix}$ *is again a $\mathcal{D}$-square.*

Proof. Clearly $u^{-1}eu$ and $u^{-1}fu$ are idempotents[8]. Again $u^{-1}xu$ and $u^{-1}x'u$ are generalized inverses of each other[4] and

$$(u^{-1}xu)(u^{-1}x'u) = u^{-1}(xx')u = u^{-1}fu$$

and similarly

$$(u^{-1}x'u)(u^{-1}xu) = u^{-1}(x'x)u = u^{-1}eu.$$ □

Theorem 5. *If e and f are idempotents with $e\mathcal{L}f$ then $ef = e$ and $fe = f$. Hence $efe = e$ and $fef = f$. Thus* $\begin{Bmatrix} e & e \\ f & f \end{Bmatrix}$ *is a degenerate $\mathcal{D}$-square as well as an E-square. Similarly if $e\mathcal{R}f$ then* $\begin{Bmatrix} e & f \\ e & f \end{Bmatrix}$ *is a $\mathcal{D}$-square as well as an E-square. Also* $\begin{Bmatrix} e & e \\ e & e \end{Bmatrix}$ *is a $\mathcal{D}$-square as well as an E-square. If $x, x' \in H_e$, then* $\begin{Bmatrix} e & x' \\ x & e \end{Bmatrix}$ *is a $\mathcal{D}$-square. In particular if* $\begin{Bmatrix} 1 & x' \\ x & 1 \end{Bmatrix}$ *is a $\mathcal{D}$-square, then $xx' = x'x = 1$ and so x and x' are inverses of each other.*

Theorem 6. *Let* $\begin{Bmatrix} e & f \\ h & g \end{Bmatrix}$ *and* $\begin{Bmatrix} k & l \\ n & m \end{Bmatrix}$ *be* E*-squares in a ring* R *satisfying* $e \le 1-k$, $f \le 1-l$, $g \le 1-m$, $h \le 1-n$. *Then* $\begin{Bmatrix} e+k & f+l \\ h+n & g+m \end{Bmatrix}$ *is an* E*-square.*

Proof. By assumption we get

$$ek = ke = fl = lf = gm = mg = hn = nh = 0.$$

So $e+l$, $f+m$, $g+n$, $h+k$ are idempotents [2]. Also,

$$el = e(kl) = (ek)l = 0.l = 0.$$

Similarly,

$$kf = k(ef) = (ke)f = 0.f = 0.$$

Then

$$(e+k)(f+l) = ef + el + kf + kl = f + l$$

and similarly $(f+l)(e+k) = e+k$. Thus $(e+k)\mathcal{R}(f+l)$. Similarly, we can prove that $(f+l)\mathcal{L}(g+m)\mathcal{R}(h+n)\mathcal{L}(e+k)$. □

Theorem 7. *Let* R *be a ring. Then each module isomorphism between left modules* Re *and* Rf *corresponds to a* $\mathcal{D}$*-square and conversely.*

Proof. Let $Re \cong Rf$ and let $\phi : Re \to Rf$ be a module isomorphism. So ϕ^{-1} is also a module isomorphism. For any $x \in R$, $\phi(xe) = x\phi(e)$. Thus $\phi(e)$ determines ϕ. Let $\phi(e) = a \in Rf$. Then $\phi(e) = a = af$. So $e = \phi^{-1}(af) = a\phi^{-1}(f)$. Similarly, let $\phi^{-1}(f) = b \in Re$. Then $\phi^{-1}(f) = b = be$. Now

$$\phi^{-1}(f)\phi(e) = be\phi(e) = b\phi(e) = \phi(be) = f$$

as $\phi^{-1}(f) = be$ and

$$\phi(e)\phi^{-1}(f) = af\phi^{-1}(f) = a\phi^{-1}(f) = \phi^{-1}(af) = e.$$

Also,

$$\phi^{-1}(f)\phi(e)\phi^{-1}(f) = f\phi^{-1}(f) = \phi^{-1}(f)$$

and

$$\phi(e)\phi^{-1}(f)\phi(e) = e\phi(e) = \phi(e).$$

Thus $\begin{Bmatrix} e & \phi(e) \\ \phi^{-1}(f) & f \end{Bmatrix}$ is a $\mathcal{D}$-square.

Conversely, if $\begin{Bmatrix} e & a' \\ a & f \end{Bmatrix}$ is a $\mathcal{D}$-square then $aa' = f$, $a'a = e$. Define $\phi : Re \to Rf$ by $\phi(x) = xa'$. Then for $x, y \in Re$,

$$\phi(xy) = (xy)a' = x(ya') = x\phi(y).$$

Also,

$$\phi(x+y) = (x+y)a' = xa' + ya' = \phi(x) + \phi(y).$$

So ϕ is a module homomorphism. Let $\phi(x) = \phi(y)$ where $x, y \in Re$. Then $xa' = ya'$ and so

$xa'a = ya'a$. That is $xe = ye$. Since $x, y \in Re$, this implies that $x = y$ as $xe = x$ and $ye = y$. Thus ϕ is one to one. Now, let $z \in Rf$. Then $z = zf = zaa' = \phi(za)$ where $za = zae \in Re$. Thus ϕ is onto. Thus ϕ is a module isomorphism. □

Theorem 8. *Let* $\begin{Bmatrix} e & f \\ h & g \end{Bmatrix}$ *be an E-square then* $e\mathcal{R}f\mathcal{L}g\mathcal{R}h\mathcal{L}e$. *So,*

$$(1-e)\mathcal{L}(1-f)\mathcal{R}(1-g)\mathcal{L}(1-h)\mathcal{R}(1-e)$$

and hence $\begin{Bmatrix} 1-e & 1-h \\ 1-f & 1-g \end{Bmatrix}$ *is an E-square.*

Theorem 9. *Let* $\begin{Bmatrix} e & f \\ h & g \end{Bmatrix}$ *and* $\begin{Bmatrix} k & l \\ n & m \end{Bmatrix}$ *be E-squares in a ring R satisfying $k \le e$, $n \le f$, $m \le g$, $l \le h$. Then* $\begin{Bmatrix} e-k & f-n \\ h-l & g-m \end{Bmatrix}$ *is an E-square.*

Proof. By previous note $\begin{Bmatrix} 1-e & 1-h \\ 1-f & 1-g \end{Bmatrix}$ is an E-square. As $k \le e$ we get $1-e \le 1-k$. Similarly, $1-f \le 1-n$, $1-g \le 1-m$, $1-h \le 1-l$. So, by Theorem 6, $\begin{Bmatrix} 1-e+k & 1-h+l \\ 1-f+n & 1-g+m \end{Bmatrix}$ is an E-square. That is $\begin{Bmatrix} 1-(e-k) & 1-(h-l) \\ 1-(f-n) & 1-(g-m) \end{Bmatrix}$. Thus by previous note we get $\begin{Bmatrix} (e-k) & (f-n) \\ (h-l) & (g-m) \end{Bmatrix}$ is an E-square. □

Definition 10. An E- array is a matrix $A = (e_{i_\lambda}), i \in I, \lambda \in \Lambda$ such that e_{i_λ} are idempotents with $e_{i_\lambda}\mathcal{R}e_{i_\mu}$ and $e_{i_\lambda}\mathcal{L}e_{j_\lambda}$ for all $i, j \in I$ and $\lambda, \mu \in \Lambda$

Theorem 11. *Let e, f, k, l be idempotents in a ring R.*

(i) *If $e\mathcal{L}f$, $k\mathcal{L}l$ with $e \le (1-k)$ and $f \le (1-l)$ then $(e+k)\mathcal{L}(f+l)$.*

(ii) *If $e\mathcal{R}f$, $k\mathcal{R}l$ with $e \le (1-k)$ and $f \le (1-l)$ then $(e+k)\mathcal{R}(f+l)$.*

(iii) *If $e\mathcal{L}f$, $k\mathcal{R}l$ with $k \le e$ and $l \le f$ then $(e-k)\mathcal{L}(f-l)$.*

(iv) *If $e\mathcal{R}f$, $k\mathcal{L}l$ with $k \le e$ and $l \le f$ then $(e-k)\mathcal{R}(f-l)$.*

Proof. (1) Given that $e \le (1-k)$ and $f \le (1-l)$. So $ek = ke = 0$ and $fl = lf = 0$. Hence $(e+k)$ and $(f+l)$ are idempotents. Also, given that $e\mathcal{L}f$, $k\mathcal{L}l$. So $ef = e$, $fe = f$, $kl = k$, $lk = l$. Hence

$$el = (ef)l = e(fl) = 0.$$

Similarly,

$$le = (lk)e = l(ke) = 0.$$

Also,

$$kf = (kl)f = k(lf) = 0$$

and

$$fk = (fe)k = f(ek) = 0.$$

Thus

$$(e+k)(f+l) = ef + el + kf + kl = e+k$$

and

$$(f+l)(e+k) = fe + fk + le + lk = f+l.$$

Hence $(e+k)\mathcal{L}(f+l)$.

The proof of (2) is similar.

(3) Given that $e\mathcal{L}f$ and so $(1-e)\mathcal{R}(1-f)$. Since $k \le e$ and $l \le f$ we get $(1-e) \le 1-k$ and $(1-f) \le 1-l$. So by (i) $(1-e+k)\mathcal{R}(1-f+l)$. That is $1-(e-k)\mathcal{R}1-(f-l)$. Hence $(e-k)\mathcal{L}(f-l)$. Similarly we can prove (iv). □

Theorem 12. (i) *If $A = (e_{i_\lambda})$ and $B = (f_{i_\lambda})$ be two E-array of the same order satisfying $e_{i_\lambda} \le 1 - f_{i_\lambda}$, then $A + B = (e_{i_\lambda} + f_{i_\lambda})$ is also an E-array.*

(ii) *If $A = (e_{i_\lambda})$ and $B = (f_{i_\lambda})$ be two square E-array of the same order satisfying $f_{i_\lambda} \le e_{\lambda_i}$, then $A - B^T = (e_{i_\lambda} - f_{\lambda_i})$ is also an E-array.*

3. Conclusion

The concepts of Green's relations were primarily introduced for Semigroups. In rings these relations possess additional properties that are proved by defining $\mathcal{D}$-squares and $\mathcal{E}$-squares. The $\mathcal{D}$-relation among the elements of rings helps to derive some ring theoretic results. Many new results in ring theory can be proved with the help of $\mathcal{D}$-squares.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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