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Research Article

Stability Analysis of a Fractional Order Discrete Anti-Periodic Boundary Value Problem

A. George Maria Selvam*, Mary Jacintha and R. Dhineshbabu

Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur 635601, Tamil Nadu, India
***Corresponding author:** agmshc@gmail.com

Abstract. This article aims at investigating stability properties for a class of discrete fractional equations with anti-periodic boundary conditions of fractional order $\delta \in (3,4]$. Utilizing contraction mapping principle and fixed point theorem due to Brouwer [2], new criteria for the uniqueness and existence of the solutions are developed and two types of Ulam stability are analyzed. The theoretical outcomes are corroborated with examples.

Keywords. Existence; Ulam stability; Boundary value problem; Caputo fractional difference operator

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1. Introduction

Fractional calculus whose origin dates back to seventeenth century has gained much momentum during the past three decades among mathematicians and researchers as fractional derivatives have greater ability to consider hereditary and memory properties of various processes and materials in relation to integer order derivatives. Derivatives of fractional order, with their non-local property can be applied both in technical and applied sciences. Mathematical modelling has entered an exciting phase primarily due to fractional calculus which is effective in modelling various phenomena often arising in physics, engineering, biology and scientific fields namely synchronization of chaotic systems [19, 36], anomalous diffusion [33], models to analyse the spread and control of diseases [10, 30, 31], models to study the interaction of species in ecology [27], control theory [17], non-linear oscillation of earthquake [29], blood flow problems [34], etc.

Fractional difference equations (FDEs) with fractional order difference and sum operators as primary notions are the discrete counter part of fractional differential equations. Ever since, Kuttner [25] mentioned the fractional order differences in 1956, the theory of difference equations of fractional order has systematically evolved over the years (see [14–16, 23, 26]) and the references therein. The investigation of the qualitative properties of these equations has hit bull's eye recently with scores of publications analyzing the behavior of their solutions (see [7–9, 18, 22]). The research on FDEs extends from the theoretical features of uniqueness and existence of solutions to the analytical and numerical procedures of obtaining solutions. Nevertheless, the area of *boundary value problems* (BVPs) for nonlinear FDEs is open to be explored. The area of interest for researchers in BVP is the existence and uniqueness of solutions. By employing different fixed-point theorems, many interesting results have been developed for the existence of solution to boundary value problems for FDEs (see [3, 6, 24, 28, 32]).

In recent times, researchers have focused their attention on BVPs with anti-periodic boundary conditions which makes up a significant category of BVP. The anti-periodic boundary conditions are quite common in mathematical models of various physical processes namely anti-periodic trigonometric polynomials in the study of interpolation problems, anti-periodic wavelets, ordinary, abstract, partial, and impulsive differential equations and difference equations (see [1, 4, 5, 14]).

In [35], Wang considered an anti-periodic fractional BVP with the Caputo fractional derivative of order $\delta \in (1,2]$. The study was carried on with the aid of fixed point theorem due to Schauder and contraction mapping principle.

In [11], Cernea investigated an anti-periodic fractional BVP with the Caputo fractional derivative of order $\delta \in (2,3]$. New results were obtained by applying suitable fixed point theorems.

Motivated by [11, 20, 35], existence and uniqueness criteria are established for the discrete fractional BVP given below

$$\begin{cases} {}^{C}\Delta^{\delta} x(\ell) = \Psi(\ell+\delta-1, x(\ell+\delta-1)), \ \ell \in [0, e+3]_{\mathbb{N}_0}, \\ x(\delta-4) = -x(\delta+e), \qquad \Delta x(\delta-4) = -\Delta x(\delta+e), \\ \Delta^2 x(\delta-4) = -\Delta^2 x(\delta+e), \ \Delta^3 x(\delta-4) = -\Delta^3 x(\delta+e), \end{cases} \tag{1.1}$$

where $\Psi : [\delta-3, \delta+e]_{\mathbb{N}_{\delta-3}} \times \mathbb{R} \to \mathbb{R}$ is a function which is continuous and ${}^{C}\Delta^{\delta}$ is the Caputo fractional difference operator of order $3 < \delta \leq 4$.

2. Preliminaries

The subsequent definitions, notations and properties of fractional order sum and difference operators are essential to prove the main results.

The falling factorial for any $\ell \geq 0$ is

$$r^{(\ell)} = \frac{\Gamma(r+1)}{\Gamma(r+1-\ell)}. \tag{2.1}$$

Definition 2.1 (See [13, 21])**.** The δth fractional sum of a function Ψ is defined by

$$\Delta^{-\delta}\Psi(\ell) = \frac{1}{\Gamma(\delta)} \sum_{r=a}^{\ell-\delta} (\ell-r-1)^{(\delta-1)} \Psi(r), \tag{2.2}$$

for $\ell \in \{a+\delta, a+\delta+1, \cdots\} := \mathbb{N}_{a+\delta}$.

Definition 2.2 (Ssee [13,20]). Let $\delta > 0$ and $n-1 < \delta \le n$. Set $\sigma = n-\delta$. The δ^{th} Caputo fractional difference operator is defined as

$$^{C}\Delta^{\delta}\Phi(\ell) = \Delta^{-\sigma}\left(\Delta^{n}\Phi(\ell)\right) = \frac{1}{\Gamma(\sigma)}\sum_{r=a}^{\ell-\sigma}(\ell-r-1)^{(\sigma-1)}\Delta^{n}\Phi(r), \tag{2.3}$$

for $\ell \in \mathbb{N}_{a+\sigma}$, where $n = \lceil\delta\rceil$, $\lceil . \rceil$ ceiling of number.

Lemma 2.3 ([13, 20]). *Suppose that $\delta > 0$ and Φ is defined on $\mathbb{N}_a$. Then*

$$\begin{aligned}\Delta^{-\delta}\,{}^{C}\Delta^{\delta}\Phi(\ell) &= \Phi(\ell) - \sum_{j=0}^{n-1}\frac{(\ell-a)^{(j)}}{j!}\Delta^{j}\Phi(a)\\ &= \Phi(\ell) + C_0 + C_1\ell + \cdots + C_{n-1}\ell^{(n-1)},\end{aligned} \tag{2.4}$$

for all $C_i \in \mathbb{R}$, where $0 \le i \le n-1$.

Lemma 2.4. *Let $\delta > 0$. Then the following identities hold* [20].

(1) $\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)} = \frac{1}{\delta}\frac{\Gamma(\ell+1)}{\Gamma(\ell-\delta+1)}$.

(2) $\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)} = \frac{1}{\delta}\frac{\Gamma(\delta+e+1)}{\Gamma(e+1)}$.

(3) $\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)} = \frac{1}{(\delta-1)}\frac{\Gamma(\delta+e+1)}{\Gamma(e+2)}$.

(4) $\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)} = \frac{1}{(\delta-2)}\frac{\Gamma(\delta+e+1)}{\Gamma(e+3)}$.

(5) $\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)} = \frac{1}{(\delta-3)}\frac{\Gamma(\delta+e+1)}{\Gamma(e+4)}$.

3. Existence and Uniqueness of Solutions

The criteria for the existence and uniqueness of solutions of a discrete fractional anti-periodic boundary value problem (FABVP) (1.1) is established in this section, with the condition that the solution exists.

Theorem 3.1. *Let $\Psi : [\delta-3, \delta+e]_{\mathbb{N}_{\delta-3}} \to \mathbb{R}$ and $3 < \delta \le 4$ be given. A function $x(\ell)$ is a solution of the discrete FABVP of the form*

$$\begin{cases} ^{C}\Delta^{\delta}x(\ell) = \Psi(\ell+\delta-1), \ \ell \in [0, e+3]_{\mathbb{N}_0}, \\ x(\delta-4) = -x(\delta+e), \qquad \Delta x(\delta-4) = -\Delta x(\delta+e), \\ \Delta^2 x(\delta-4) = -\Delta^2 x(\delta+e), \ \Delta^3 x(\delta-4) = -\Delta^3 x(\delta+e), \end{cases} \tag{3.1}$$

if and only if $x(\ell)$, for $\ell \in [\delta-4, \delta+e]_{\mathbb{N}_{\delta-4}}$ has the form

$$\begin{aligned} x(\ell) = {}& \frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1) - \frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1) \\ & + \frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1) - \frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1) \\ & - \frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
P(\ell) &= 2(\delta-\ell)+e-4,\\
Q(\ell) &= \left[2(\delta^2+\ell^2)+\delta(2e-6-4\ell)-e(2\ell+7)+(6\ell-4)\right],\\
R(\ell) &= \delta^2[12(1+\ell)-6e]+\delta[12e(\ell+3)-12\ell(\ell+2)+40]\\
&\quad -e[6\ell^2+4(1+9\ell)]+e^2(e+12)+4(\ell^3-\delta^3)+\ell(12\ell-40)-48.
\end{aligned} \tag{3.3}$$

Proof. Suppose that $x(\ell)$ defined on $[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$ is a solution of (3.1). In line with Lemma 2.3, a general solution for (3.1) is as given below

$$x(\ell)=\Delta^{-\delta}\Psi(\ell+\delta-1)-C_0-C_1\ell-C_2\ell^{(2)}-C_3\ell^{(3)},\quad \ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$$

where $C_i\in\mathbb{R}$ for $0\le i\le 3$. Then by Definition 2.1, we obtain

$$x(\ell)=\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1)-C_0-C_1\ell-C_2\ell^{(2)}-C_3\ell^{(3)}. \tag{3.4}$$

Now taking Δ, Δ^2 and Δ^3 operator on either sides of (3.4), leads to

$$\Delta x(\ell)=\frac{1}{\Gamma(\delta-1)}\sum_{r=0}^{\ell-\delta+1}(\ell-r-1)^{(\delta-2)}\Psi(r+\delta-1)-C_1-2C_2\ell-3C_3\ell(\ell-1),$$

$$\Delta^2 x(\ell)=\frac{1}{\Gamma(\delta-2)}\sum_{r=0}^{\ell-\delta+2}(\ell-r-1)^{(\delta-3)}\Psi(r+\delta-1)-2C_2-6C_3\ell,$$

$$\Delta^3 x(\ell)=\frac{1}{\Gamma(\delta-3)}\sum_{r=0}^{\ell-\delta+3}(\ell-r-1)^{(\delta-4)}\Psi(r+\delta-1)-6C_3.$$

The values of C_i for $0\le i\le 3$ are obtained by considering the boundary conditions as given in (1.1). So in view of boundary conditions $x(\delta-4)=-x(\delta+e)$, $\Delta x(\delta-4)=-\Delta x(\delta+e)$, $\Delta^2 x(\delta-4)=-\Delta^2 x(\delta+e)$ and $\Delta^3 x(\delta-4)=-\Delta^3 x(\delta+e)$, we obtain

$$\begin{aligned}
C_0 &= \frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1)-\frac{U}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1)\\
&\quad +\frac{V}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1)+\frac{W}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1),\\
C_1 &= \frac{1}{2\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1)-\frac{[2\delta-4+e]}{4\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1)\\
&\quad -\frac{[2\delta(3-e-\delta)+7e+4]}{8\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1),\\
C_2 &= \frac{1}{4\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1)-\frac{[2\delta-4+e]}{8\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1),\\
C_3 &= \frac{1}{12\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1),
\end{aligned}$$

where $U=[2\delta+e-4], V=[2\delta(\delta+e-3)-(7e+4)]$ and $W=[6\delta^2(2-e)+4\delta(10+9e)-4\delta^3+e(e^2+12e-4)-48]$. Now considering C_i for $0\le i\le 3$ in $x(\ell)$ brings us to,

$$x(\ell)=\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1)-\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1)$$

$$+\frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1)-\frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1)$$
$$-\frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1),$$

where $P(\ell)$, $Q(\ell)$ and $R(\ell)$ are defined in (3.3). Conversely, if (3.2) is a solution, it is clear that the solution satisfies the discrete FABVP (3.1). The proof is completed. Let $\mathbb{E}$ be the set of all real sequences $x=\{x(\ell)\}_{\ell=\delta-4}^{\delta+e}$ with norm $\|x\|=\sup|x(\ell)|$ for $\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$. Then $\mathbb{E}$ is a Banach space. □

Now we define the following operator,

$$\begin{aligned}(Tx)(\ell)=&\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1,x(r+\delta-1))\\
&-\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1,x(r+\delta-1))\\
&+\frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1,x(r+\delta-1))\\
&-\frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1,x(r+\delta-1))\\
&-\frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1,x(r+\delta-1)),\end{aligned}\tag{3.5}$$

for $\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$. Obviously, $x(\ell)$ is a solution of (1.1) if it is a fixed point of the operator T.

We consider the following hypotheses:

(H_1) There exists a constant $\Omega>0$ such that $|\Psi(\ell,x)-\Psi(\ell,y)|\le\Omega|x-y|$ for each $\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$ and all $x,y\in\mathbb{E}$.

(H_2) There exists a bounded function $\Phi:[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}\to\mathbb{R}$ such that $|\Psi(\ell,x)|\le\Phi(\ell)|x|$ for all $x\in\mathbb{E}$.

(H_3) For a non decreasing function $\varphi\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}\to\mathbb{R}$, there exists a constant $\lambda>0$ such that

$$\frac{\epsilon}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{\delta-1}\varphi(r+\delta-1)\le\lambda\epsilon\varphi(\ell+\delta-1),\quad \ell\in[0,e+3]_{\mathbb{N}_0}.$$

Theorem 3.2. *If the hypothesis* (H_1) *holds, then the discrete FABVP* (1.1) *has a unique solution in* $\mathbb{E}$ *provided that*

$$\Omega\beta\frac{\Gamma(\delta+e+1)}{2\Gamma(\delta-2)\Gamma(e+1)}<1,\tag{3.6}$$

where $\beta=\left[\frac{3}{\delta^{(3)}}+\frac{e+4}{2(\delta-1)^{(2)}(e+1)}+\frac{e+4}{4(\delta-2)(e+1)(e+2)}+\frac{\xi}{24(e+1)(e+2)(e+3)}\right]$ *such that* $\xi=e\left(e^2+12e+44\right)+48$.

Proof. Let $x, y \in \mathbb{E}$; then for each $\ell \in [\delta-4, \delta+e]_{\mathbb{N}_{\delta-4}}$, we have

$$\begin{aligned}
&|(Tx)(\ell)-(Ty)(\ell)| \\
&\le \frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}|\Psi(r+\delta-1,x(r+\delta-1))-\Psi(r+\delta-1,y(r+\delta-1))| \\
&\quad+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}|\Psi(r+\delta-1,x(r+\delta-1))-\Psi(r+\delta-1,y(r+\delta-1))| \\
&\quad+\frac{|P(\ell)|}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}|\Psi(r+\delta-1,x(r+\delta-1))-\Psi(r+\delta-1,y(r+\delta-1))| \\
&\quad+\frac{|Q(\ell)|}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}|\Psi(r+\delta-1,x(r+\delta-1))-\Psi(r+\delta-1,y(r+\delta-1))| \\
&\quad+\frac{|R(\ell)|}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}|\Psi(r+\delta-1,x(r+\delta-1))-\Psi(r+\delta-1,y(r+\delta-1))|.
\end{aligned}$$

By assumption (H_1) and with the help of Lemma 2.4, the above inequality becomes

$$\begin{aligned}
\|Tx-Ty\| &\le \Omega\left[\frac{\Gamma(\ell+1)}{\Gamma(\delta+1)\Gamma(\ell+1-\delta)}+\frac{\Gamma(\delta+e+1)}{2\Gamma(\delta+1)\Gamma(e+1)}+\frac{|P(\ell)|\Gamma(\delta+e+1)}{4\Gamma(\delta)\Gamma(e+2)}\right.\\
&\qquad\left.+\frac{|Q(\ell)|\Gamma(\delta+e+1)}{8\Gamma(\delta-1)\Gamma(e+3)}+\frac{|R(\ell)|\Gamma(\delta+e+1)}{48\Gamma(\delta-2)\Gamma(e+4)}\right]\|x-y\| \\
&\le \Omega\left[\frac{3\Gamma(\delta+e+1)}{2\Gamma(\delta+1)\Gamma(e+1)}+\frac{(e+4)\Gamma(\delta+e+1)}{4\Gamma(\delta)\Gamma(e+2)}+\frac{(e+4)\Gamma(\delta+e+1)}{8\Gamma(\delta-1)\Gamma(e+3)}\right.\\
&\qquad\left.+\frac{\xi\Gamma(\delta+e+1)}{48\Gamma(\delta-2)\Gamma(e+4)}\right]\|x-y\| \\
\|Tx-Ty\| &\le \Omega\left[\beta\frac{\Gamma(\delta+e+1)}{2\Gamma(\delta-2)\Gamma(e+1)}\right]\|x-y\|,
\end{aligned}$$

which implies that T is a contraction. Hence from Banach fixed point theorem T has a unique fixed point which is the unique solution of the discrete FABVP (1.1). □

Theorem 3.3. *The discrete FABVP* (1.1) *has at least one solution under the hypothesis* (H_2) *and the inequality*

$$\Gamma(\delta+e+1)\le\frac{2}{\omega\Phi^*}, \tag{3.7}$$

where $\omega=\left[\frac{3}{\Gamma(\delta+1)\Gamma(e+1)}+\frac{e+4}{2\Gamma(\delta)\Gamma(e+2)}+\frac{e+4}{4\Gamma(\delta-1)\Gamma(e+3)}+\frac{\xi}{24\Gamma(\delta-2)\Gamma(e+4)}\right]$
such that $\xi=e\left(e^2+12e+44\right)+48$ *and* $\Phi^*=\max\left\{\Phi(\ell):\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}\right\}$.

Proof. Let $M>0$ and define the set $S=\left\{x(\ell):[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}\to\mathbb{R},\ \|x\|\le M\right\}$. To prove this theorem, we only need to show that T maps S into S. For $x(\ell)\in S$, we have

$$\begin{aligned}
|(Tx)(\ell)| &\le \frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}|\Psi(r+\delta-1,x(r+\delta-1))| \\
&\quad+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}|\Psi(r+\delta-1,x(r+\delta-1))|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|P(\ell)|}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\,|\Psi(r+\delta-1,x(r+\delta-1))|\\
&+\frac{|Q(\ell)|}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\,|\Psi(r+\delta-1,x(r+\delta-1))|\\
&+\frac{|R(\ell)|}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\,|\Psi(r+\delta-1,x(r+\delta-1))|
\end{aligned}
$$

$$
\begin{aligned}
|(Tx)(\ell)|\le{}&\frac{\Phi(r)}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\,|x(r+\delta-1)|\\
&+\frac{\Phi(r)}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\,|x(r+\delta-1)|\\
&+\frac{\Phi(r)\,|P(\ell)|}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\,|x(r+\delta-1)|\\
&+\frac{\Phi(r)\,|Q(\ell)|}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\,|x(r+\delta-1)|\\
&+\frac{\Phi(r)\,|R(\ell)|}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\,|x(r+\delta-1)|.
\end{aligned}
\tag{3.8}
$$

From Lemma 2.4, the inequality (3.8) becomes

$$
\begin{aligned}
\|Tx\|\le{}&\frac{\Phi(r)\,\|x\|}{\Gamma(\delta)}\,\frac{\Gamma(\ell+1)}{\delta\Gamma(\ell+1-\delta)}+\frac{\Phi(r)\,\|x\|}{2\Gamma(\delta)}\,\frac{\Gamma(\delta+e+1)}{\delta\Gamma(e+1)}+\frac{|P(\ell)|\,\Phi(r)\,\|x\|}{4\Gamma(\delta-1)}\,\frac{\Gamma(\delta+e+1)}{(\delta-1)\Gamma(e+2)}\\
&+\frac{|Q(\ell)|\,\Phi(r)\,\|x\|}{8\Gamma(\delta-2)}\,\frac{\Gamma(\delta+e+1)}{(\delta-2)\Gamma(e+3)}+\frac{|R(\ell)|\,\Phi(r)\,\|x\|}{48\Gamma(\delta-3)}\,\frac{\Gamma(\delta+e+1)}{(\delta-3)\Gamma(e+4)}
\end{aligned}
$$

$$
\|Tx\|\le\left[\frac{3\Gamma(\delta+e+1)}{2\Gamma(\delta+1)\Gamma(e+1)}+\frac{(e+4)\Gamma(\delta+e+1)}{4\Gamma(\delta)\Gamma(e+2)}+\frac{(e+4)\Gamma(\delta+e+1)}{8\Gamma(\delta-1)\Gamma(e+3)}+\frac{\xi\Gamma(\delta+e+1)}{48\Gamma(\delta-2)\Gamma(e+4)}\right]\Phi^*\,\|x\|,
$$

$$
\|Tx\|\le\left[\omega\frac{\Gamma(\delta+e+1)}{2}\right]\Phi^*\,\|x\|.
$$

From (3.7), we have $\|Tx\|\le M$ implying that T maps S in S. Thus T has at least one fixed point, which is a solution of the FABVP (1.1) according to Brouwer fixed-point theorem [2] which completes the proof. □

4. The Ulam Stability

Stability analysis is discussed for the discrete FABVP (1.1) in this section. The following definitions for FDE are given on the basis of ([13, 20]).

Definition 4.1 ([13, 20])**.** The nonlinear discrete FABVP (1.1) is Hyers-Ulam stable if for every $\epsilon>0$, there is a constant $\mathcal{K}>0$ and for every solution $y\in\mathbb{E}$ of

$$
\left|{}^{C}\Delta^{\delta}y(\ell)-\Psi(\ell+\delta-1,y(\ell+\delta-1))\right|\le\epsilon,\quad \ell\in[0,e+3]_{\mathbb{N}_0},
\tag{4.1}
$$

there exists a solution $x\in\mathbb{E}$ of (1.1) such that

$$
|y(\ell)-x(\ell)|\le\mathcal{K}\epsilon,\qquad \ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}.
\tag{4.2}
$$

Definition 4.2 ([13,20])**.** The nonlinear discrete FABVP (1.1) is Hyers-Ulam Rasias stable if for every $\epsilon>0$, there is a constant $\mathcal{K}_1>0$ and for every solution $y\in\mathbb{E}$ of

$$\left|{}^{C}\Delta^{\delta}y(\ell)-\Psi(\ell+\delta-1,y(\ell+\delta-1))\right|\le\varphi(\ell+\delta-1)\epsilon,\quad \ell\in[0,e+3]_{\mathbb{N}_0},\tag{4.3}$$

there exists a solution $x\in\mathbb{E}$ of (1.1) such that

$$|y(\ell)-x(\ell)|\le\mathcal{K}_1\epsilon\varphi(\ell+\delta-1),\quad \ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}.\tag{4.4}$$

Remark 4.3. A function $y\in\mathbb{E}$ is a solution of (4.1) if and only if there exists a function $g:[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}\to\mathbb{R}$ such that

(i) $|g(\ell+\delta-1)|\le\epsilon$, for $\ell\in[0,e+3]_{\mathbb{N}_0}$,

(ii) ${}^{C}\Delta^{\delta}y(\ell)=\Psi(\ell+\delta-1,y(\ell+\delta-1))+g(\ell+\delta-1)$, for $\ell\in[0,e+3]_{\mathbb{N}_0}$.

A similar remark is true for inequality (4.3).

Theorem 4.4. *Assume that* (H_1) *holds. Let $y\in\mathbb{E}$ be a solution of inequality* (4.1) *and let $x\in\mathbb{E}$ be a solution of the discrete FABVP* (1.1)*. Then, BVP* (1.1) *is the Ulam-Hyers stable provided that*

$$\Omega<\frac{2\Gamma(\delta-2)\Gamma(e+1)}{\beta\Gamma(\delta+e+1)},\tag{4.5}$$

where β is defined in Theorem 3.2.

Proof. From inequality (4.1) and Remark 4.3, for $\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$, it follows that

$$\begin{aligned}&\Bigg|y(\ell)-\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1,y(r+\delta-1))\\&+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1,y(r+\delta-1))\\&-\frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1,y(r+\delta-1))\\&+\frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1,y(r+\delta-1))\\&+\frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1,y(r+\delta-1))\Bigg|\\&\le\frac{\epsilon\Gamma(\delta+e+1)}{\Gamma(\delta+1)\Gamma(e+1)}.\end{aligned}\tag{4.6}$$

Combining (3.2) and (4.6), for $\ell\in[\delta-4,\delta+e]_{\mathbb{N}_{\delta-4}}$, we have

$$\begin{aligned}|y(\ell)-x(\ell)|\le&\Bigg|y(\ell)-\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1,x(r+\delta-1))\\&+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{b}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1,x(r+\delta-1))\\&-\frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1,x(r+\delta-1))\end{aligned}$$

$$
\begin{aligned}
&+\frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1,x(r+\delta-1))\\
&+\frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1,x(r+\delta-1))\Bigg|\\
\le&\Bigg|y(\ell)-\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\Psi(r+\delta-1,y(r+\delta-1))\\
&+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\Psi(r+\delta-1,y(r+\delta-1))\\
&-\frac{P(\ell)}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\Psi(r+\delta-1,y(r+\delta-1))\\
&+\frac{Q(\ell)}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\Psi(r+\delta-1,y(r+\delta-1))\\
&+\frac{R(\ell)}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\Psi(r+\delta-1,y(r+\delta-1))\Bigg|\\
&+\frac{1}{\Gamma(\delta)}\sum_{r=0}^{\ell-\delta}(\ell-r-1)^{(\delta-1)}\big|\Psi(r+\delta-1,y(r+\delta-1))-\Psi(r+\delta-1,x(r+\delta-1))\big|\\
&+\frac{1}{2\Gamma(\delta)}\sum_{r=0}^{e}(\delta+e-r-1)^{(\delta-1)}\big|\Psi(r+\delta-1,y(r+\delta-1))-\Psi(r+\delta-1,x(r+\delta-1))\big|\\
&+\frac{|P(\ell)|}{4\Gamma(\delta-1)}\sum_{r=0}^{e+1}(\delta+e-r-1)^{(\delta-2)}\big|\Psi(r+\delta-1,y(r+\delta-1))-\Psi(r+\delta-1,x(r+\delta-1))\big|\\
&+\frac{|Q(\ell)|}{8\Gamma(\delta-2)}\sum_{r=0}^{e+2}(\delta+e-r-1)^{(\delta-3)}\big|\Psi(r+\delta-1,y(r+\delta-1))-\Psi(r+\delta-1,x(r+\delta-1))\big|\\
&+\frac{|R(\ell)|}{48\Gamma(\delta-3)}\sum_{r=0}^{e+3}(\delta+e-r-1)^{(\delta-4)}\big|\Psi(r+\delta-1,y(r+\delta-1))-\Psi(r+\delta-1,x(r+\delta-1))\big|.
\end{aligned}
$$

Solving the above inequality with the aid of Lemma 2.4 and assumption (H_1), we obtain

$$\|y-x\|\le\mathcal{K}\epsilon,$$

where $\mathcal{K}=\dfrac{\Gamma(\delta+e+1)}{\delta^{(3)}\left[\Gamma(\delta-2)\Gamma(e+1)-\Omega\beta\frac{\Gamma(\delta+e+1)}{2}\right]}>0$. Thus, a discrete FABVP (1.1) is Ulam-Hyers stable. □

Theorem 4.5. *Assume that* (H_1) *and* (H_3) *hold. Let* $y\in\mathbb{E}$ *be a solution of inequality* (4.3) *and let* $x\in\mathbb{E}$ *be a solution of* (1.1)*. Then a discrete FABVP* (1.1) *is Hyers-Ulam-Rasias stable provided that* (4.5) *holds.*

Proof. Using the argument as in the proof of Theorem 4.4, we get

$$\|y-x\|\le\mathcal{K}_1\epsilon\varphi(\ell+\delta-1),$$

where $\mathcal{K}_1=\dfrac{\lambda\Gamma(\delta-2)\Gamma(e+1)}{\Gamma(\delta-2)\Gamma(e+1)-\Omega\beta\frac{\Gamma(\delta+e+1)}{2}}>0$ such that β is defined in Theorem 3.2. Thus a discrete FABVP (1.1) is Ulam-Hyers-Rasias stable. □

5. Examples

We consider the following examples to demonstrate our theoretical findings.

Example 5.1. Suppose that $\delta = \frac{7}{2}$ and $e = 3$. Let $\Psi(\ell, x) = \frac{\cos(x(\ell))}{1000+\ell^2}$ and $\Omega = \frac{1}{1000}$. Then discrete FABVP (1.1) becomes

$$\begin{cases} {}^{C}\Delta^{\frac{7}{2}} x(\ell) = \frac{\cos\left(x\left(\ell+\frac{5}{2}\right)\right)}{1000+\left(\ell+\frac{5}{2}\right)^2}, \quad \ell \in [0,6]_{\mathbb{N}_0}, \\ x\left(-\frac{1}{2}\right) = -x\left(\frac{13}{2}\right), \qquad \Delta x\left(-\frac{1}{2}\right) = -\Delta x\left(\frac{13}{2}\right), \\ \Delta^2 x\left(-\frac{1}{2}\right) = -\Delta^2 x\left(\frac{13}{2}\right), \ \Delta^3 x\left(-\frac{1}{2}\right) = -\Delta^3 x\left(\frac{13}{2}\right). \end{cases} \tag{5.1}$$

In this case, inequality (3.6) is

$$\Omega\beta \frac{\Gamma(\delta+e+1)}{2\Gamma(\delta-2)\Gamma(e+1)} \le 0.1108 < 1.$$

Therefore, from Theorem 3.2, we conclude that boundary value problem (5.1) has a unique solution.

Example 5.2. Suppose that $\delta = \frac{18}{5}$, $e = 2$ and $M = 1000$ with $\Psi(\ell, x) = \frac{1}{27}\ell e^{-\frac{x^2(\ell)}{10}}$. Then discrete FABVP (1.1) takes the form

$$\begin{cases} {}^{C}\Delta^{\frac{18}{5}} x(\ell) = \frac{1}{27}\left(\ell+\frac{13}{5}\right) e^{-\frac{x^2\left(\ell+\frac{13}{5}\right)}{10}}, \quad \ell \in [0,5]_{\mathbb{N}_0}, \\ x\left(-\frac{2}{5}\right) = -x\left(\frac{28}{5}\right), \qquad \Delta x\left(-\frac{2}{5}\right) = -\Delta x\left(\frac{28}{5}\right), \\ \Delta^2 x\left(-\frac{2}{5}\right) = -\Delta^2 x\left(\frac{28}{5}\right), \ \Delta^3 x\left(-\frac{2}{5}\right) = -\Delta^3 x\left(\frac{28}{5}\right). \end{cases} \tag{5.2}$$

The Banach space is $\mathbb{E} := \left\{ x(\ell) | \left[-\frac{2}{5}, \frac{28}{5}\right]_{\mathbb{N}_{-\frac{2}{5}}} \to \mathbb{R}, \|x\| \le 1000 \right\}$. We note that

$$\frac{2M}{\omega\Gamma(\delta+e+1)} \approx 15.8987.$$

It is clear that $|\Psi(\ell, x)| \le \frac{28}{135} < 15.8987$, whenever $x \in [-1000, 1000]$. Therefore by Theorem 3.3, we conclude that the boundary value problem (1.1) has at least one solution.

Example 5.3. Suppose that $\delta = \frac{16}{5}$, $e = 1$. Let $\Psi(\ell, x) = \frac{x(\ell)}{39+\ell^2}$ and $\Omega = \frac{1}{39}$. Then discrete FABVP (1.1) becomes

$$\begin{cases} {}^{C}\Delta^{\frac{16}{5}} x(\ell) = \frac{x\left(\ell+\frac{11}{5}\right)}{39+\left(\ell+\frac{11}{5}\right)^2}, \quad \ell \in [0,4]_{\mathbb{N}_0}, \\ x\left(-\frac{4}{5}\right) = -x\left(\frac{21}{5}\right), \qquad \Delta x\left(-\frac{4}{5}\right) = -\Delta x\left(\frac{21}{5}\right), \\ \Delta^2 x\left(-\frac{4}{5}\right) = -\Delta^2 x\left(\frac{21}{5}\right), \ \Delta^3 x\left(-\frac{4}{5}\right) = -\Delta^3 x\left(\frac{21}{5}\right). \end{cases} \tag{5.3}$$

Since

$$\frac{2\Gamma(\delta-2)\Gamma(e+1)}{\beta\Gamma(\delta+e+1)} \approx 0.0476.$$

If $\Omega = 0.0256 < 0.0476$ and the inequality

$$\left| {}^{C}\Delta^{\frac{16}{5}} y(\ell) - \Psi\left(\ell + \frac{16}{5}, y\left(\ell + \frac{16}{5}\right)\right)\right| \le \epsilon, \quad \ell \in [0,4]_{\mathbb{N}_0},$$

hold, then (5.3) is the Ulam-Hyers stable by Theorem 4.4.

6. Conclusion

The authors, in this article, considered a class of discrete fractional equations with anti-periodic boundary conditions with fractional order $3 < \delta \leq 4$. Models involving design and manufacturing process in modern technology need construction and analysis of boundary value problems. By means of contraction mapping principle and fixed-point theorem due to Brouwer [2], new criteria for the existence and uniqueness of the solutions are established, followed by the analysis of Hyers-Ulam stability and Hyers-Ulam Rasias stability of the problem. Appropriate examples are chosen to authenticate the main theoretical findings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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