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Spectral Conditions for Composition Operators on Algebras of Functions

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Abstract. We establish general sufficient conditions for maps between function algebras to be composition or weighted composition operators, which extend previous results in [2, 4, 6, 7]. Let X be a locally compact Hausdorff space and $A \subset C(X)$ a dense subalgebra of a function algebra, not necessarily with unit, such that $X = \partial A$ and $p(A) = \delta A$, where ∂A is the Shilov boundary, δA – the Choquet boundary, and $p(A)$ – the set of p-points of A. If $T : A \to B$ is a surjective map onto a function algebra $B \subset C(Y)$ such that either $\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg)$ for all $f, g \in A$, or, alternatively, $\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg)$ for all $f, g \in A$, then there is a homeomorphism $\psi\colon \delta B \to \delta A$ and a function α on δB so that $(Tf)(y) = \alpha(y) f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. If, instead, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and either $\sigma_\pi(f) \subset \sigma_\pi(Tf)$ for all $f \in A$, or, alternatively, $\sigma_\pi(Tf) \subset \sigma_\pi(f)$ for all $f \in A$, then $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. In particular, if A and B are uniform algebras and $T : A \to B$ is a surjective map with $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, that has a limit, say b, at some $a \in A$ with $a^2 = 1$, then $(Tf)(y) = b(y)\, a(\psi(y))\, f(\psi(y))$ for every $f \in A$ and $y \in \delta B$.

1. **Introduction**

Let A be a *function algebra* on a locally compact Hausdorff space, that is, A is an algebra of bounded continuous functions on X, which is closed under the *sup*-norm $\|f\| = \sup_{x \in X} |f(x)|$ and strongly separates the points of X, namely, for every $x, y \in X$, $x \neq y$, there is a function $f \in A$ so that $f(x) \neq f(y)$, and for every $x \in X$ there is a function $f \in A$ so that $f(x) \neq 0$. If A is unital, then its maximal ideal space, $\mathcal{M}_A$, and its Shilov boundary, ∂A, are compact spaces. If A is not unital, then $\mathcal{M}_A$ is a locally compact space and the Gelfand transform $\widehat{A}$ of A is a subset of $C_0(\mathcal{M}_A)$, the space of continuous functions on $\mathcal{M}_A$ that vanish at infinity.

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Denote by $\sigma(f)$ the spectrum, by $\sigma_\pi(f) = \{z \in \sigma(f) \colon |z| = \max_{u\in\sigma(f)} |u|\}$ the *peripheral spectrum*, and by $E(f) = \{x \in X \colon |f(x)| = \|f\|\}$ the *maximum modulus set* of $f \in A$. An $h \in A$ with $\|h\| = 1$ and $|h(x)| < 1$ whenever $h(x) \neq 1$ is said to be a *peaking function* of A. The set of all peaking functions of A is denoted by $\mathscr{P}(A)$. The set of all peaking functions h of A such that $h(x_0) = 1$ for a fixed $x_0 \in X$ will be denoted by $\mathscr{P}_{x_0}(A)$. A point $x \in X$ is called a *p-point*, or a *strong boundary point* for A if for every neighborhood V of x there is a peaking function h of B so that $h(x) = 1$ and $E(h) \subset V$. The set of all p-points for A is denoted by $p(A)$. If A is a function algebra then $p(A)$ is a boundary, namely, the Choquet boundary δA of A. If A is not a function algebra, then the set $p(A)$ is, in general, a proper subset of δA. Peaking functions can be utilized to find the values of algebra functions on Choquet boundaries. Namely,

Lemma 1.1 (Strong Multiplicative Bishop's Lemma [7]). *Let $A \subset C(X)$ be a function algebra on $X = \partial A$, not necessarily with unit. If $f \in A$ and $x_0 \in X$ is a p-point of A with $f(x_0) \neq 0$, then there exists a peaking function $h_0 \in \mathscr{P}_{x_0}(A)$ such that*

$$\sigma_\pi(fh_0) = \{f(x_0)\}. \tag{1.1}$$

If U is an open set of X containing x_0, then h_0 can be chosen so that $E(fh_0) = E(h_0) \subset U$.

Let $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, and let $\psi\colon Y \to X$ be a continuous mapping. A map $T : A \to B$ is called

(i) a *ψ-composition operator on Y* if $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in Y$, and
(ii) a *weighted ψ-composition operator on Y* if there is a continuous function α on Y so that $(Tf)(y) = \alpha(y)f(\psi(y))$ for all $f \in A$ and $y \in Y$.

Clearly, composition operators are algebra isomorphisms. Any weighted composition operator $T = \alpha(f \circ \psi)$ is linear, while the operator T/α is linear and multiplicative.

There is considerable interest in finding conditions for maps between algebras of functions to be composition type operators (e.g. [1, 3, 4, 6, 7]). In particular, sufficient conditions for maps between two algebras of functions to be weighted composition operators "in modulus" are given in [7]. Namely,

Theorem 1.2 ([7]). *Let $A \subset C(X)$ and $B \subset C(Y)$ be dense subalgebras of function algebras on $X = \partial A$ and $Y = \partial B$ with $p(A) = \delta A$ and $p(B) = \delta B$. If $T : A \to B$ is a surjection such that $\|Tf \cdot Tg\| = \|fg\|$ for all $f, g \in A$, then there is a homeomorphism $\psi\colon p(B) \to p(A)$ such that*

$$|(Tf)(y)| = |f(\psi(y))| \tag{1.2}$$

for all $f \in A$ and $y \in p(B)$.

In [6] it is shown that if A and B are uniform algebras on compact Hausdorff spaces X and Y respectively, then a necessary and sufficient condition for a surjective unital operator $T : A \to B$ to be a composition operator on δB is for T to be *peripherally-multiplicative*, i.e. to satisfy the equality $\sigma_\pi(Tf \cdot Tg) = \sigma_\pi(fg)$ for every $f, g \in A$.

In this paper we establish general sufficient conditions for maps between function algebras to be composition or weighted composition operator, which extend the main results in [2, 4, 6, 7].

2. **The main theorems**

In [4, Corollary 3] it is shown that if $T : A \to B$ is a surjective mapping between two uniform algebras such that

$$\sigma_\pi(Tf) = \sigma_\pi(f) \tag{2.1}$$

for every $f \in A$, and either $\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg)$ for all $f, g \in A$, or, alternatively, $\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg)$ for all $f, g \in A$, then T is a composition operator on δB. We show that condition (2.1) is not necessary for arbitrary function algebras.

Theorem 2.1. *Let $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, not necessarily with units, where X and Y are locally compact Hausdorff spaces. If $T : A \to B$ is a surjection such that*

$$\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg) \tag{2.2}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $y \in \delta B$.

Proof. First we show that $(Tf)(y)^2 = f(\psi(y))^2$ for every $f \in A$ and $y \in \delta B$. Let $f \in A$ and $y_0 \in \delta B$. Equality (2.2) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi : \delta B \to \delta A$ be the homeomorphism from Theorem 1.2, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $f(\psi(y_0)) = 0$.

Suppose $f(\psi(y_0)) \neq 0$. If $V \subset \delta B$ is an arbitrary open neighborhood of y_0 in δA, then, clearly, $U = \psi(V)$ is an open neighborhood of $\psi(y_0)$. By Lemma 1.1 there exists a peaking function $h \in \mathscr{P}_{\psi(y_0)}(A)$ with $\sigma_\pi(fh) = \{f(\psi(y_0))\}$ such that $E(fh) = E(h) \subset U$. Denote $k = Th$. Note that $\sigma_\pi(Tf \cdot k) = \{f(\psi(y_0))\}$ since, by (2.2), $\sigma_\pi(Tf \cdot k) \subset \sigma_\pi(fh) = \{f(\psi(y_0))\}$. Therefore, there is a point $y_1 \in \delta B$ so that $(Tf \cdot k)(y_1) = f(\psi(y_0))$, i.e. $(Tf)(y_1)k(y_1) = f(\psi(y_0))$. Since, by (1.2), $|f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_1)||(Th)(y_1)| = |f(\psi(y_0))|$ and $\sigma_\pi(fh) = \{f(\psi(y_0)\}$, we deduce that the function fh attains the maximum of its modulus at $\psi(y_1)$. Hence $\psi(y_1) \in E(fh) = E(h) \subset U = \psi(V)$, thus $y_1 \in V$.

Since V is an arbitrary neighborhood of y_0, the continuity of Tf and k implies that $(Tf)(y_0)\,k(y_0) = f(\psi(y_0))$, and consequently,

$$(Tf)(y_0)^2\,k(y_0)^2 = f(\psi(y_0))^2. \tag{2.3}$$

Equality (2.2) implies that $\sigma_\pi(k^2) = \sigma_\pi((Th)^2) \subset \sigma_\pi(h^2) = \{1\}$, and therefore, $\sigma_\pi(k^2) = \{1\}$. Since, by (1.2), $|(Tf)(y_0)| = |f(\psi(y_0))|$ we deduce that $|k^2(y_0)| = 1$, thus $k^2(y_0) \in \sigma_\pi(k^2) = \{1\}$, and therefore, $k(y_0)^2 = 1$. Hence (2.3) becomes $(Tf)(y_0)^2 = f(\psi(y_0))^2$, as claimed.

Consequently, there exists a number $\alpha_f(y_0) = \pm 1$, possibly depending on f, such that

$$(Tf)(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{2.4}$$

We claim that, in fact, the number $\alpha_f(y_0)$ does not depend on $f \in A$. First we show that $\alpha_h(y_0)$ has the same value for all peaking functions h in $\mathscr{P}_{\psi(y_0)}(A)$. Indeed, if $h_1, h_2 \in \mathscr{P}_{\psi(y_0)}(A)$, then, by (2.2), $\sigma_\pi(Th_1 \cdot Th_2) \subset \sigma_\pi(h_1 h_2) = \{1\}$, and therefore, $\sigma_\pi(Th_1 \cdot Th_2) = \{1\}$. Since $|(Th_1)(y_0)(Th_2)(y_0)| = |h_1(\psi(y_0))||h_2(\psi(y_0))| = 1$, the function $Th_1 \cdot Th_2$ attains its maximum modulus at y_0. Hence $(Th_1)(y_0)(Th_2)(y_0) \in \sigma_\pi(Th_1 \cdot Th_2) = \{1\}$, and therefore, $(Th_1)(y_0)(Th_2)(y_0) = 1$. Consequently, by (2.4), the numbers $\alpha_{h_i}(y_0) = \alpha_{h_i}(y_0) h_i(\psi(y_0)) = (Th_i)(y_0)$, $i = 1, 2$, have the same sign, thus $\alpha_{h_1}(y_0) = \alpha_{h_2}(y_0)$.

By Lemma 1.1 there is an $h \in \mathscr{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$. Since, by (2.2), $\sigma_\pi(Tf \cdot Th) \subset \sigma_\pi(fh) = \{f(\psi(y_0))\}$, we have $\sigma_\pi(Tf \cdot Th) = \{f(\psi(y_0)\}$. Hence $|(Tf)(y_0)(Th)(y_0)| = |(Tf)(y_0)||(Th)(y_0)| = |f(\psi(y_0))||h(\psi(y_0))| = |f(\psi(y_0))|$. Consequently, the function $Tf \cdot Th$ attains the maximum of its modulus at y_0, so we must have $(Tf)(y_0)(Th)(y_0) \in \sigma_\pi(Tf \cdot Th) = \{f(\psi(y_0))\}$, thus $(Tf)(y_0)(Th)(y_0) = f(\psi(y_0))$. Therefore,

$$\alpha_f(y_0)\,\alpha_h(y_0) = \frac{(Tf)(y_0)}{f(\psi(y_0))}\,\frac{(Th)(y_0)}{h(\psi(y_0))} = \frac{1}{h(\psi(y_0))} = 1.$$

Hence $\alpha_f(y_0) = \alpha_h(y_0)$, thus the number $\alpha_f(y_0)$ has the same value for all $f \in A$ with $f(\psi(y_0)) \neq 0$. Consequently, the function $\alpha(y) = \alpha_f(y)$, $y \in \delta B$, $f \in A$, $f(\psi(y)) \neq 0$, is well defined, and $\alpha^2 = \alpha_h^2 = 1$. Now (2.4) becomes $(Tf)(y_0) = \alpha(y_0) f(\psi(y_0))$, as desired.

To show that α is continuous at any $y \in \delta B$, let $f \in A$ with $f(\psi(y)) \neq 0$ and let $V \subset \delta B$ be a neighborhood of y such that $f \circ \psi \neq 0$ on V. Since Tf, f and ψ are continuous on V, so is the function $\alpha = Tf/(f \circ \psi)$. In particular, α is continuous at $y \in V$. □

For surjections $T : A^{-1} \to B^{-1}$ between the sets of invertible elements of uniform algebras a similar result is proven in [3]. Alternatively, we have:

Theorem 2.2. *Let $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, not necessarily with units, where X and Y are locally compact Hausdorff spaces. If $T : A \to B$ is a* *surjection such that*

$$\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg) \tag{2.5}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi\colon \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $y \in \delta B$.

Proof. As before we show first that $(Tf)(y)^2 = f(\psi(y))^2$ for every $f \in A$ and $y \in \delta B$. Let $f \in A$ and $y_0 \in \delta B$. The equality (2.5) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$, and therefore, Theorem 1.2 applies. Let $\psi\colon \delta B \to \delta A$ be the homeomorphism from Theorem 1.2, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Suppose $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0. By Lemma 1.1 there exists a peaking function $k \in \mathscr{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$ and $E(Tf \cdot k) = E(k) \subset V$. Hence for every $h \in T^{-1}(k)$ we have $\sigma_\pi(fh) \subset \sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$, i.e. $\sigma_\pi(fh) = \{(Tf)(y_0)\}$. Therefore, there is a point $x_1 \in \delta A$ so that $(fh)(x_1) = (Tf \cdot k)(y_0) = (Tf)(y_0)$. The surjectivity of ψ implies that there is an $y_1 \in \delta B$ so that $x_1 = \psi(y_1)$. Hence $f(\psi(y_1))h(\psi(y_1)) = (Tf)(y_0)$. Since $\sigma_\pi(fh) = \{(Tf)(y_0)\}$ and $|(Tf)(y_1)||k(y_1)| = |f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_0)|$ by (1.2), the function $Tf \cdot k$ attains the maximum of its modulus at y_1. Consequently, $y_1 \in E(Tf \cdot k) = E(k) \subset V$. Since V is an arbitrary neighborhood of y_0, the continuity of f, ψ and h imply $f(\psi(y_0))h(\psi(y_0)) = (Tf)(y_0)$ and therefore

$$(Tf)(y_0)^2 = f(\psi(y_0))^2 h(\psi(y_0))^2. \tag{2.6}$$

Since, by (1.2), $|(Tf)(y_0)| = |f(\psi(y_0))|$ we have $|h(\psi(y_0))| = 1$. The condition (2.5) implies that $\sigma_\pi(h^2) \subset \sigma_\pi(k^2) = \{1\}$, thus $\sigma_\pi(h^2) = \{1\}$, hence $h(\psi(y_0))^2 \in \sigma_\pi(h^2) = \{1\}$, and, therefore, $h(\psi(y_0))^2 = 1$. Hence (2.6) becomes $(Tf)(y_0)^2 = f(\psi(y_0))^2$, as claimed.

Consequently, there is a number $\alpha_f(y_0) = \pm 1$, possibly dependent on f, such that

$$(Tf)(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{2.7}$$

We claim that $\alpha_f(y_0)$ does not depend on $f \in A$. First we show that $\alpha_h(y_0)$ has the same value for any $h \in T^{-1}(k)$ such that $k \in \mathscr{P}_{y_0}(B)$. If $k_1, k_2 \in \mathscr{P}_{y_0}(B)$ and $h_i \in T^{-1}(k_i)$, $i = 1, 2$, then $\sigma_\pi(h_1 h_2) \subset \sigma_\pi(Th_1 \cdot Th_2) = \sigma_\pi(k_1 k_2) = \{1\}$, thus $\sigma_\pi(h_1 h_2) = \{1\}$. Since $|h_1(\psi(y_0))||h_2(\psi(y_0))| = |(Th_1)(y_0)(Th_2)(y_0)| = 1$ we deduce that $h_1(\psi(y_0))h_2(\psi(y_0)) \in \sigma_\pi(h_1 h_2) = \{1\}$, hence $h_1(\psi(y_0))h_2(\psi(y_0)) = 1$. By (2.7), $\alpha_{h_i}(y_0)h_i(\psi(y_0)) = (Th_i)(y_0) = k_i(y_0) = 1$. Consequently, the numbers $\alpha_{h_i}(y_0) = 1/h_i(\psi(y_0))$, $i = 1, 2$, have the same sign and therefore, $\alpha_{h_1}(y_0) = \alpha_{h_2}(y_0)$.

Now let $f \in A$ be arbitrary. According to Lemma 1.1 there exists a $k \in \mathscr{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$. Let $h \in T^{-1}(k)$. Equality (2.5) implies that $\sigma_\pi(fh) \subset \sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$, hence $\sigma_\pi(fh) = \{(Tf)(y_0)\}$. Therefore,

$$|f(\psi(y_0))\,h(\psi(y_0))| = |f(\psi(y_0))||h(\psi(y_0))| = |(Tf)(y_0)\,(Th)(y_0)| = |(Tf)(y_0)|.$$

It follows that the function fh attains the maximum of its modulus at $\psi(y_0)$, so we must have $f(\psi(y_0))\,h(\psi(y_0)) \in \sigma_\pi(fh)$, thus, $f(\psi(y_0))\,h(\psi(y_0)) = (Tf)(y_0)$. Therefore,

$$\alpha_f(y_0)\,\alpha_h(y_0) = \frac{(Tf)(y_0)}{f(\psi(y_0))}\,\frac{(Th)(y_0)}{h(\psi(y_0))} = (Th)(y_0) = k(y_0) = 1.$$

Hence $\alpha_f(y_0) = \alpha_h(y_0)$, thus the number $\alpha_f(y_0)$ has the same value for all $f \in A$ with $(Tf)(y_0) \neq 0$.

Consequently, the function $\alpha(y) = \alpha_f(y)$, $y \in \delta B$, $f \in A$, $(Tf)(y) \neq 0$, is well defined. Now (2.7) becomes $(Tf)(y_0) = \alpha(y_0)\,f(\psi(y_0))$. The proof completes as in Theorem 2.1. □

More generally, we have the following

Theorem 2.3. *Let X be a locally compact Hausdorff space and $A \subset C(X)$ a dense subalgebra of a function algebra, not necessarily with unit, such that $X = \partial A$ and $p(A) = \delta A$. If $T : A \to B$ is a surjection onto a function algebra $B \subset C(Y)$ such that either*

(a) $\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg)$ *for all $f, g \in A$, or,*
(b) $\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg)$ *for all $f, g \in A$,*

then T is a weighted composition operator on δB. That is, there is a homeomorphism $\psi : \delta B \to \delta A$ and a function α on δB with $\alpha^2 = 1$ so that $(Tf)(y) = \alpha(y)\,f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. In particular, A is necessarily a function algebra and T/α is linear and multiplicative operator, i.e. an algebra isomorphism.

More general, *weakly peripherally-multiplicative* operators, that satisfy the condition $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, are considered in [4]. It is not known, though, whether every weakly peripherally-multiplicative operator $T : A \to B$ is a weighted composition operator. However, if T preserves, in addition, the peripheral spectra of all algebra elements, then T is necessarily a composition operator [4, Proposition 2]. Namely,

Proposition 2.4 ([4])**.** *If a weakly peripherally-multiplicative surjective map $T : A \to B$ between uniform algebras preserves the peripheral spectra of algebra elements, i.e.*

$$\sigma_\pi(Tf) = \sigma_\pi(f) \tag{2.8}$$

for all $f \in A$, then it is a composition operator on δB, i.e. an isometric algebra isomorphism.

Below we expand this result for algebras of functions and simultaneously relax the condition (2.8).

Theorem 2.5. *Let X be a locally compact Hausdorff space and $A \subset C(X)$ is a dense subalgebra of a function algebra, not necessarily with unit, such that $X = \partial A$ and $p(A) = \delta A$. If $T: A \to B$ is a surjection onto a function algebra $B \subset C(Y)$ such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \emptyset \text{ for all } f, g \in A \tag{2.9}$$

and either

(a) $\sigma_\pi(f) \subset \sigma_\pi(Tf)$ *for all $f \in A$, or,*
(b) $\sigma_\pi(Tf) \subset \sigma_\pi(f)$ *for all $f \in A$,*

then T is a bijective composition operator on δB with respect to a homeomorphism $\psi: \delta B \to \delta A$. That is,

$$(Tf)(y) = f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, A is necessarily a function algebra and T is an algebra isomorphism.

Proof. Let $y_0 \in p(B) = \delta B$. Condition (2.9) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi: \delta B \to \delta A$ be the homeomorphism from Theorem 1.2, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Let $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0.

Case (a): According to Lemma 1.1, there exists a peaking function $k \in \mathscr{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$ and $E(Tf \cdot k) = E(k) \subset V$. Note that if $h \in T^{-1}(k)$ then $(Tf)(y_0) \in \sigma_\pi(fh)$ since, by (a), $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \emptyset$. Therefore, there is a point $x_1 \in \delta A$ so that $(Tf \cdot k)(y_0) = (fh)(x_1)$. Since ψ is surjective, there is an $y_1 \in \delta B$ so that $x_1 = \psi(y_1)$. Hence

$$(Tf)(y_0) = (Tf)(y_0)\, k(y_0) = f(\psi(y_1))\, h(\psi(y_1)). \tag{2.10}$$

By (1.2), $|(Tf)(y_0)| = |(Tf)(y_0)||k(y_0)| = |f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_1)|$ $|k(y_1)|$. Hence $y_1 \in E(Tf \cdot k) = E(k) \subset V$. Therefore, $|h(\psi(y_1))| = |k(y_1)| = 1$. Condition (a) implies that $\sigma_\pi(h) \subset \sigma_\pi(k) = \{1\}$, thus $h(\psi(y_1)) \in \sigma_\pi(h)$, hence $h(\psi(y_1)) = 1$. Now the equality (2.10) becomes $(Tf)(y_0) = f(\psi(y_1))$. Since V was an arbitrary neighborhood of y_0, the continuity of f and ψ yield $(Tf)(y_0) = f(\psi(y_0))$ as desired.

Case (b): Note that $U = \psi(V)$ is an open neighborhood of $\psi(y_0)$ in δA. By Lemma 1.1 there exists a peaking function $h \in \mathscr{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$ and $E(f \cdot h) = E(h) \subset U$. If $Th = k$, then $f(\psi(y_0)) \in \sigma_\pi(Tf \cdot k)$ since, by (2.9), $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \emptyset$. Therefore, there is a point $y_1 \in p(B)$ so that $(Tf \cdot k)(y_1) = f(\psi(y_0))$. Hence

$$f(\psi(y_0)) = f(\psi(y_0))\, h(\psi(y_0)) = (Tf)(y_1)\, k(y_1). \tag{2.11}$$

By (1.2), $|f(\psi(y_0))| = |(Tf)(y_1)||k(y_1)| = |f(\psi(y_1))||(Th)(y_1)| = |f(\psi(y_1))|$ $|h(\psi(y_1))|$. Hence $\psi(y_1) \in E(f \cdot h) = E(h) \subset U = \psi(V)$, thus $y_1 \in V$. We have that $|h(\psi(y_1))| = |(Th)(y_1)| = |k(y_1)| = 1$. Since, by condition (b), $\sigma_\pi(k) \subset \sigma_\pi(h) = \{1\}$, we deduce that $k(y_1) \in \sigma_\pi(k) = \{1\}$, hence $k(y_1) = 1$. Then the equality (2.11) becomes $f(\psi(y_0)) = (Tf)(y_1)$. Since V was an arbitrary neighborhood of y_0, the continuity of Tf yields $(Tf)(y_0) = f(\psi(y_0))$ as claimed. $\square$

In [2] it is shown that a surjective weakly peripherally-multiplicative map T between uniform algebras is a composition operator if conditions (a) or (b) in Theorem 2.5 are replaced by the single condition T to be continuous at the unity. Below we generalize this result.

Recall that a set $E \subset X$ is called a *peak set* for a function algebra $A \subset C(X)$ if E is the maximum modulus set of a peaking function, i.e. if $E = E(h)$, $h \in \mathscr{P}(A)$. It is known that if $\lambda \in \sigma_\pi(f)$ for some $f \in A$, then $f^{-1}(\lambda)$ is a peak set for A (e.g. [5]).

Theorem 2.6. *Let A and B be uniform algebras on compact Hausdorff spaces X and Y. If $T : A \to B$ is a surjective map such that*

(i) *$\sigma_\pi(Tf \cdot T) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$ and*

(ii) *There exist an $a \in A$ with $a^2 = 1$ such that T has a limit, say b, at a,*

then $b^2 = 1$ and $(Tf)(y) = b(y)\,a(\psi(y))\,f(\psi(y))$ for every $f \in A$ and $y \in \delta B$, i.e. the map $f \mapsto b\,T(af)$ is an isometric algebra isomorphism.

Proof. Condition (i) implies that $\|Tf \cdot Tg\| = \|fg\|$ for all $f, g \in A$. In particular, $\|(Tf)^2\| = \|f^2\|$ and therefore, $\|Tf\| = \|f\|$ for every $f \in A$.

We claim that $\sigma_\pi(f) \subset \sigma_\pi(b\,T(af))$ for every $f \in A$. Let $f \in A$ and $\lambda \in \sigma_\pi(f)$. If $\lambda = 0$, then $\|f\| = 0$ and so $f = 0$, thus $af = 0$ and hence $\|T(af)\| = 0$. Consequently, $b\,T(af) = 0$ and, therefore, $\lambda \in \sigma_\pi(b\,T(af))$.

If $\lambda \neq 0$, then $f^{-1}(\lambda)$ is a peak set in X, so there exists a peaking function $h \in \mathscr{P}(A)$ such that $E(h) = f^{-1}(\lambda)$. Define $h_n = a\frac{n+h}{n+1}$. Clearly, $ah_n \in \mathscr{P}(A)$ for every n. Note that since $(ah_n)^{-1}(1) = f^{-1}(\lambda)$ for every n, we have that $\sigma_\pi(ah_n f) = \{\lambda\}$. Condition (i) implies that $\lambda \in \sigma_\pi(Th_n \cdot T(af))$ for every n. Since h_n converges uniformly to a, we must have $Th_n \to b$ and therefore $\lambda \in \sigma_\pi(b\,T(af))$. Consequently,

$$\sigma_\pi(f) \subset \sigma_\pi(b\,T(af)) \tag{2.12}$$

as claimed. Theorem 2.5 implies that the map $f \mapsto b\,T(af)$ is a ψ-composition operator on δB. Hence $b(y)(T(af)(y) = f(\psi(y))$, and therefore, $T(f)(y) = b(y)\,a(\psi(y))\,f(\psi(y))$ for all $y \in \delta B$ and $f \in A$.

To show that $b^2 = 1$, let $y \in \delta B$ and consider the set $K = b^{-1}(b(y))$. Since $y \in \delta B$, K is a peak set and, therefore, there exists a peaking function $k \in \mathscr{P}(B)$ with $E(k) = K$. Let $h \in A$ be such that $T(ah) = k$. According to (2.12), $\sigma_\pi(h) \subset \sigma_\pi(b\,T(ah)) = \sigma_\pi(bk) = \{b(y)\}$. Hence $\sigma_\pi(h^2) = \sigma_\pi((ah)^2) = \{b(y)^2\}$, so by (a), $\{b(y)^2\} \in \sigma_\pi(T(ah)^2) = \sigma_\pi(k^2) = \{1\}$ since $k \in \mathscr{P}(B)$. Therefore $b(y)^2 = 1$ for every $y \in \delta B$. $\square$

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