Communications in Mathematics and Applications
Vol. 11, No. 3, pp. 325–334, 2020
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v11i3.1395

Research Article

Existence and Uniqueness of Solution of Summation-Difference Equation of Finite Delay in Cone Metric Space

G. C. Done[1], K. L. Bondar*[2] and P. U. Chopade[3]

[1] *P. G. Department of Mathematics, N. E. S. Science College, Nanded, India*
[2] *P. G. Department of Mathematics, Government Vidarbha Institute of Science and Humanities, Amaravati, India*
[3] *Department of Mathematics, D. S. M.'s Arts, Commerce and Science College, Jintur, India*
***Corresponding author:** klbondar_75@rediffmail.com

Abstract. In this research article, we study the existence and uniqueness of solution of Summation-Difference equation of finite delay with nonlocal condition in cone metric space. The result is obtained by using the some extensions of Banach's contraction principle in complete cone metric space and also provide example to demonstrate of main result.

Keywords. Difference equation; Summation equation; Existence of solution; Cone metric space; Banach contraction principle

MSC. 39A05; 54E50; 45N05; 47G20; 34K05; 47H10

Received: April 13, 2020 **Accepted:** June 30, 2020

1. Introduction

In the present years, the study of difference equations and their applications are found to be more useful in the field of numerical and Engineering as well as social sciences. Agrawal [2], Kelley and Peterson [14] had developed theory of difference equation and their inequalities. Some existence and uniqueness results on difference equation and summation equation are obtained by Bondar *et al*. [3–7, 9, 10]. The purpose of this paper is study the existence and uniqueness of solution of Summation Difference equation with nonlocal condition in cone metric space of the form:

$$\Delta x(t) = A(t)x(t) + f(t, x(t), x(t-1)) + \sum_{s=0}^{t} k(s, x(s)), \quad t \in J = [0, b] \tag{1.1}$$

$$x(t-1) = \psi(t), \qquad t \in [0, 1] \tag{1.2}$$

$$x(0) + g(x) = x_0, \tag{1.3}$$

where $A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space $X; f : J \times X \times X \to X$, $k : J \times X \to X$, $g : C(J, X) \to X$ are appropriate continuous functions and x_0 is given element of X.

$\psi(t)$ is a continuous function for $[0,1]$: $\lim_{t\to(1-0)} \psi(t)$ exists, for which we denote by $\psi(1-0) = c_0$. If we observed a function $x(t-1)$ which is unable to define as solution for $[0,1]$. Hence, we have to impose some condition, for example the condition (1.2). We note that, if $[0,1]$, the problem is reduced to Summation Difference equation

$$\Delta x(t) = A(t)x(t) + f(t, x(t), \psi(t)) + \sum_{s=0}^{t} k(s, x(s)), \quad t \in J = [0, b].$$

With initial condition $x(0) + g(x) = x_0$. Here, it is essential to obtain the solutions of (1.1)-(1.3) for $[0, b]$.

In Section 2, we discuss the preliminaries. Section 3, deals with study of existence and uniqueness of solution of Summation-Difference equation with nonlocal condition in cone metric space. Finally, in Section 4, we give example to illustrate the application of our result.

2. Definitions and Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader to [1, 11, 12, 15, 17] for the more details.

Definition 2.1. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq 0$.
2. $a, b \in \Re$, $a, b \geq 0$, $x, y \in P \to ax + by \in P$.
3. $x \in P$ and $-x \in P \to x = 0$.

For a given cone $P \in E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone$P$ is called normal if there is a number $K > 0$ such that $\leq x \leq y$ implies $\|x\| \leq k\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following way, we always suppose E is a real Banach space, P is cone in E with int $P \neq \phi$, and $\leq$ is partial ordering with respect to P.

Note. Throughout this paper, even if it not mention explicitly underlying non empty set $\Re$ denote set of real number.

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \le d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(d_2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(d_3) $d(x,y) \le d(x,z) + d(z,y)$, for all $x,y \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space.

Example 2.1. Let $E = \Re^2$, $p = \{(x,y) \in E : x,y \ge 0\}$, $x = \Re$, and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \ge 0$ is a constat and then (X,d) is cone metric space.

Definition 2.3. Let X be an ordered space. A function $\Phi : X \to X$ is said to a comparison function if every $x,y \in X$, $x \le y$, implies that $\Phi(x) \le \Phi(y)$, $\Phi(x) \le x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2.2. Let $E = \Re^2$, $p = \{(x,y) \in E : x,y \ge 0\}$, it is easy to check that $\Phi : E \to E$ with $\Phi(x,y) = (ax, ay)$, for some $a \in (0,1)$ is a comparison function. Also, if Φ_1, Φ_2 are two comparison function over $\Re$. Then $\Phi(x,y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

3. Main Result

Let X is a Banach space with norm $\|\cdot\|$. $B = C(J,X)$ be the Banach space of all continuous function from J into X endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in [0,b]\}.$$

Let $P = (x,y) : x,y \ge 0 \subset E = \Re^2$, and define

$$d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$$

for every $f,g \in B$, then it is easily seen that (B,d) is a cone metric space.

Definition 3.1. The function $x \in B$ satisfies the summation equation:

Case I: for $t \in [0,1]$

$$x(t) = x_0 - g(x) + \sum_{s=0}^{t} A(s)\left[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\right], \tag{3.1}$$

Case II: for $t \in [1,b]$

$$\begin{aligned} x(t) = x_0 - g(x) &+ \sum_{s=0}^{1} A(s)\left[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\right] \\ &+ \sum_{s=1}^{t} A(s)\left[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\right], \end{aligned} \tag{3.2}$$

is called the mild solution of the equation (1.1)-(1.3).

We need the following theorem for further discussion:

Lemma 3.1. *[16] Let (X,d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that*

$$d(f(x), f(y)) \le \Phi(d(x,y))$$

for very $x, y \in X$. Then f has unique fixed point.

We list the following hypothesis for our convenience:

(H_1): $A(t)$ is a bounded linear operator on X for each $t \in J$, the function$t \to A(t)$ is continuous in the uniform operator topology and hence there exists a constant K such that

$$K = \sup_{t\in J} \|A(t)\|.$$

(H_2): Let $\Phi : \Re^2 \to \Re^2$ be a comparison function

(i): There exist continuous function $p_1, p_2 : J \to \Re^+$ such that

Case I: for $t \in [0,1]$

$$(\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|, \alpha\|f(t,x(t),\psi(t)) - f(t,y(t),\psi(t))\|)$$
$$\le p_1(t)\Phi(d(x,y)).$$

Case II: for $t \in [1,b]$

$$(\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|, \alpha\|f(t,x(t),x(t-1)) - f(t,y(t),y(t-1))\|)$$
$$\le p_2(t)\Phi(d(x,y)),$$

for every $t \in J$ and $x, y \in X$.

(ii): There exist continuous function $q : J \to \Re^+$ such that

$$(\|k(t,x) - k(t,y)\|, \alpha\|k(t,x) - k(t,y)\|) \le q(t)\Phi(d(x,y)),$$

for every $t \in J$ and $x, y \in X$.

(iii): There exists a positive constant G such that

$$(\|g(x) - g(y)\|, \alpha\|g(x) - g(y)\|) \le G\Phi(d(x,y)),$$

for every $x, y \in X$.

(H_3): $\sup_{t\in J} \left\{ G + K \sum_{s=0}^{t} \left[p_1(s) + p_2(s) + \sum_{\tau=0}^{s} q(\tau) \right] \right\} = 1.$

Our main results are given in the following theorem:

Theorem 3.1. *Assume that hypotheses (H_1)-(H_3) hold. Then the evolution equation (1.1)-(1.2) has a unique solution x on J.*

Proof. The operator $F : B \to B$ is defined by:

Case I: for $t \in [0,1]$

$$Fx(t) = x_0 - g(x) + \sum_{s=0}^{t} A(s) \Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau, x(\tau)) \Big], \tag{3.3}$$

Case II: for $t \in [1,b]$

$$F x(t) = x_0 - g(x) + \sum_{s=0}^{1} A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big]$$
$$+ \sum_{s=1}^{t} A(s)\Big[f(s,x(s),x(s-1)) + \sum_{\tau=0}^{s} k(\tau,x(\tau))\Big]. \tag{3.4}$$

By using the hypothesis $(H_1)-(H_3)$, we have

Case I: for $t \in [0,1]$

$$(\|Fx(t)-Fy(t)\|, \alpha\|Fx(t)-Fy(t)\|)$$
$$\leq \Big(\|g(x)-g(y)\| + \sum_{s=0}^{t} \|A(s)\|\Big[\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\| + \sum_{\tau=0}^{s} \|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big],$$
$$\alpha\|g(x)-g(y)\| + \alpha\sum_{s=0}^{t} \|A(s)\|\Big[\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\| + \sum_{\tau=0}^{s} \|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]\Big)$$
$$\leq (\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|)$$
$$+ \sum_{s=0}^{t} K(\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|, \alpha\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\|)$$
$$+ \sum_{s=0}^{t} K \sum_{\tau=0}^{s} (\|k(\tau,x(\tau))-k(\tau,y(\tau))\|, \alpha\|k(\tau,x(\tau))-k(\tau,y(\tau))\|)$$
$$\leq G\Phi(\|x-y\|, \alpha\|x-y\|) + \sum_{s=0}^{t} K p_1(s)\Phi(\|x(s)-y(s)\|, \alpha\|x(s)-y(s)\|)$$
$$+ \sum_{s=0}^{t} K \sum_{\tau=0}^{s} q(\tau)\Phi(\|x(\tau)-y(\tau))\|, \alpha\|x(\tau)-y(\tau))\|)$$
$$\leq G\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) + \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \sum_{s=0}^{t} K\Big[p_1(s) + \sum_{\tau=0}^{s} q(\tau)\Big]$$
$$\leq G\Phi(d(x,y)) + \Phi(d(x,y)) \sum_{s=0}^{t} K\Big[p_1(s) + \sum_{\tau=0}^{s} q(\tau)\Big]$$
$$\leq \Phi(d(x,y))\Big\{G + \sum_{s=0}^{t} K\Big[p_1(s) + \sum_{\tau=0}^{s} q(\tau)\Big]\Big\}$$
$$\leq \Phi(d(x,y))\Big\{G + \sum_{s=0}^{t} K\Big[p_1(s) + p_2(s) + \sum_{\tau=0}^{s} q(\tau)\Big]\Big\}$$
$$\leq \Phi(d(x,y)) \tag{3.5}$$

Case II: for $t \in [1,b]$

$$(\|Fx(t)-Fy(t)\|, \alpha\|Fx(t)-Fy(t)\|)$$
$$\leq \Big(\|g(x)-g(y)\| + \sum_{s=0}^{1} \|A(s)\|\Big[\|f(s,x(s),\psi(s)) - f(s,y(s),\psi(s))\| + \sum_{\tau=0}^{s} \|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]$$
$$+ \sum_{s=1}^{t} \|A(s)\|\Big[\|f(s,x(s),x(s-1)) - f(s,y(s),y(s-1))\| + \sum_{\tau=0}^{s} \|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big],$$

$$
\begin{aligned}
&\alpha\|g(x)-g(y)\|+\alpha\sum_{s=0}^{1}\|A(s)\|\Big[\|f(s,x(s),\psi(s))-f(s,y(s),\psi(s))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]\\
&\quad+\alpha\sum_{s=1}^{t}\|A(s)\|\Big[\|f(s,x(s),x(s-1))-f(s,y(s),y(s-1))\|+\sum_{\tau=0}^{s}\|k(\tau,x(\tau))-k(\tau,y(\tau))\|\Big]\Big)\\
&\le G\Phi(d(x,y))+\sum_{s=0}^{1}K\Big[p_1(s)\Phi(d(x,y))+\sum_{\tau=0}^{s}q(\tau)\Phi(d(x,y))\Big]\\
&\quad+\sum_{s=1}^{t}K\Big[p_2(s)\Phi(d(x,y))+\sum_{\tau=0}^{s}q(\tau)\Phi(d(x,y))\Big]\\
&\le G\Phi(d(x,y))+\sum_{s=0}^{1}K\Big[(p_1(s)+p_2(s))\Phi(d(x,y))+\sum_{\tau=0}^{s}q(\tau)\Phi(d(x,y))\Big]\\
&\quad+\sum_{s=1}^{t}K\Big[(p_1(s)+p_2(s))\Phi(d(x,y))+\sum_{\tau=0}^{s}q(\tau)\Phi(d(x,y))\Big]\\
&\le \Phi(d(x,y))\Big\{G+\sum_{s=0}^{t}K\Big[p_1(s)+p_2(s)+\sum_{\tau=0}^{s}q(\tau)\Big]\Big\}\\
&\le \Phi(d(x,y))
\end{aligned}
\tag{3.6}
$$

for every $x,y\in B$. This implies that $d(Fx,Fy)\le\Phi(d(x,y))$, for every $x,y\in B$. Now, an application of Lemma 3.1, the operator has a unique point in B. This means that the equation (1.1)-(1.2) has unique solution. □

4. Application

In this section, we give an example to illustrate the usefulness of our result discussed in previous section. Let us consider the following evolution equation:

$$\Delta x(t)=\frac{35}{44}e^{-t}x(t)+f(t,x(t),x(t-1))+\sum_{s=0}^{t}\frac{sx(s)}{20},\quad t\in J=[0,2],\ x\in X \tag{4.1}$$

$$x(0)+\frac{x}{8+x}=x_0, \tag{4.2}$$

where

$$
\begin{aligned}
f(t,x(t),x(t-1))&=\frac{te^{-t}x(t)}{(9+e^{t})(1+x(t))}, &&\text{for } t\in[0,1]\\
f(t,x(t),x(t-1))&=\frac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))}, &&\text{for } t\in[1,2].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
A(t)&=\frac{35}{38}e^{-t}, && t\in J\\
f(x,x(t),\psi(t))&=\frac{te^{-t}x(t)}{(9+e^{t})(1+x(t))}, && (t,x)\in J\times X\\
f(t,x(t),x(t-1))&=\frac{2te^{-(t-1)}x(t-1)}{(9+e^{t-1})(1+x(t-1))}, && (t,x)\in J\times X\\
k(t,x(t))&=\frac{tx(t)}{20}, && (t,x)\in J\times X
\end{aligned}
$$

$$g(x)=\frac{x}{8+x}, \qquad x\in X$$

Now for $x,y\in C(J,X)$ and $t\in J$, we have

Case I: for $t\in[0,1]$

$$\begin{aligned}
&(\|f(t,x(t),x(t-1))-f(t,y(t),y(t-1))\|,\alpha\|f(t,x(t),x(t-1))-f(t,y(t),y(t-1))\|)\\
&\quad=\frac{te^{-t}}{9+e^t}\left(\left\|\frac{x(t)}{1+x(t)}-\frac{y(t)}{1+y(t)}\right\|,\alpha\left\|\frac{x(t)}{1+x(t)}-\frac{y(t)}{1+y(t)}\right\|\right)\\
&\quad=\frac{te^{-t}}{9+e^t}\left(\left\|\frac{x(t)-y(t)}{(1+x(t))(1+y(t))}\right\|,\alpha\left\|\frac{x(t)-y(t)}{(1+x(t))(1+y(t))}\right\|\right)\\
&\quad\le\frac{te^{-t}}{9+e^t}(\|x(t)-y(t)\|,\alpha\|x(t)-y(t)\|)\\
&\quad\le\frac{te^{-t}}{9+e^t}(\|x-y\|_\infty,\alpha\|x-y\|_\infty)\\
&\quad\le\frac{te^{-t}}{9+e^t}d(x,y)\\
&\quad\le\frac{t}{10}\Phi(d(x,y)),
\end{aligned}$$

where $p_1(t)=\frac{t}{10}$, which is continuous function of J into $\Re^+$ and a comparison function $\Phi:\Re^2\to\Re^2$ such that $\Phi(d(x,y))=d(x,y)$.

Case II: for $t\in[1,2]$

$$\begin{aligned}
&(\|f(t,x(t),x(t-1))-f(t,y(t),y(t-1))\|,\alpha\|f(t,x(t),x(t-1))-f(t,y(t),y(t-1))\|)\\
&\quad=\frac{2te^{-(t-1)}}{9+e^{t-1}}\left(\left\|\frac{x(t-1)}{1+x(t-1)}-\frac{y(t-1)}{1+y(t-1)}\right\|,\alpha\left\|\frac{x(t-1)}{1+x(t-1)}-\frac{y(t-1)}{1+y(t-1)}\right\|\right)\\
&\quad=\frac{2te^{-(t-1)}}{9+e^{t-1}}\left(\left\|\frac{x(t-1)-y(t-1)}{(1+x(t-1))(1+y(t-1))}\right\|,\alpha\left\|\frac{x(t-1)-y(t-1)}{(1+x(t-1))(1+y(t-1))}\right\|\right)\\
&\quad\le\frac{2te^{-(t-1)}}{9+e^{t-1}}(\|x(t-1)-y(t-1)\|,\alpha\|x(t-1)-y(t-1)\|)\\
&\quad\le\frac{2te^{-(t-1)}}{9+e^{t-1}}(\|x-y\|_\infty,\alpha\|x-y\|_\infty)\\
&\quad\le\frac{2te^{-(t-1)}}{9+e^{t-1}}d(x,y)\\
&\quad\le\frac{t}{5}\Phi(d(x,y)),
\end{aligned}$$

where $p_2(t)=\frac{t}{5}$, which is continuous function of J into $\Re^+$ and a comparison function $\Phi:\Re^2\to\Re^2$ such that $\Phi(d(x,y))=d(x,y)$.

Similarly, we can have

$$\begin{aligned}
(\|k(t,x)-k(t,y)\|,\alpha\|k(t,x)-k(t,y)\|)&=\left(\left\|\frac{tx(t)}{20}-\frac{ty(t)}{20}\right\|,\alpha\left\|\frac{tx(t)}{20}-\frac{ty(t)}{20}\right\|\right)\\
&\le\frac{t}{20}(\|x(t)-y(t)\|,\alpha\|x(t)-y(t)\|)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{t}{20}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)\\
&\le \frac{t}{20}d(x,y)\\
&\le \frac{t}{20}\Phi(d(x,y)),
\end{aligned}
$$

where $q(t) = \frac{t}{20}$, which is continuous function of J into $\Re^+$ and the comparison function Φ defined as above.

Also,

$$
\begin{aligned}
(\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|) &\le 8\left(\frac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}, \alpha\frac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}\right)\\
&\le \frac{8}{64}(\|x-y\|, \alpha\|x-y\|)\\
&\le \frac{1}{8}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)\\
&\le \frac{1}{8}\Phi(d(x,y)),
\end{aligned}
$$

where $G = \frac{1}{8}$, and the comparison function Φ defined as above. Hence the condition (H_1) holds with $K = \frac{35}{44}$.

Moreover,

$$
\begin{aligned}
\sup_{t\in J}\left\{G + K\sum_{s=0}^{t}\left(p_1(s)+p_2(s)+\sum_{\tau=0}^{s}q(\tau)\right)\right\} &= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\sum_{s=0}^{t}\left(\frac{s}{10}+\frac{s}{5}+\sum_{\tau=0}^{s}\frac{\tau}{20}\right)\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\sum_{s=0}^{t}\left(\frac{s}{10}+\frac{s}{5}+\left[\frac{\tau^2}{40}\right]_0^{s+1}\right)\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\sum_{s=0}^{t}\left(\frac{3s}{10}+\frac{(s+1)^2}{40}\right)\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3s^2}{20}+\frac{(s+1)^3}{120}\right]_0^{t+1}\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3(t+1)^2}{20}+\frac{(t+2)^3}{120}\right]\right\}\\
&= \sup_{t\in J}\left\{\frac{1}{8}+\frac{35}{44}\left[\frac{3(t+1)t}{20}+\frac{(t+2)(t+1)t}{120}\right]\right\}\\
&= \left[\frac{1}{8}+\frac{35}{44}\times\left(\frac{18}{20}+\frac{4}{20}\right)\right] = \left[\frac{1}{8}+\frac{35}{44}\times\frac{11}{10}\right]\\
&= \left[\frac{1}{8}+\frac{7}{8}\right] = 1.
\end{aligned}
$$

Since all the conditions of Theorem 3.1 are satisfied, the problem (4.1)-(4.2) has a unique solution x on J.

5. Conclusion

In this paper, we studied the existence for Summation-Difference type equations of finite delay in cone metric spaces and proved that solution of this result is unique. We proved this result by using the some extensions of Banach contraction principle in complete cone metric space. Moreover, we also gave application of above result.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *Journal of Mathematical Analysis and Applications* **341**(1) (2008), 416 – 420, DOI: 10.1016/j.jmaa.2007.09.070.

[2] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel-Dekker, New York (1991), URL `https://doc.lagout.org/science/0_Computer%20Science/3_Theory/Mathematics/Difference%20Equations%20and%20Inequalities%20-%20Theory.pdf`.

[3] K. L. Bondar, Existence and uniqueness of results for first order difference equations, *Journal of Modern Methods in Numerical Mathematical* **2**(1-2) (2011), 16 – 20, URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.682.2335&rep=rep1&type=pdf`.

[4] K. L. Bondar, On some summation-difference inequalities, *International Journal of Mathematical Archive* **2**(9) (2011), 1608 – 1611, URL: `http://ijma.info/index.php/ijma/article/viewFile/533/276`.

[5] K. L. Bondar, Uniqueness results for first order difference equations, *International Journal of Pure and Applied Sciences and Technology* **4**(2) (2011), 57 – 62, URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.419.8444&rep=rep1&type=pdf`.

[6] K. L. Bondar and A. B. Jadhao, Local and global existence of solutions for summation equation, *International Journal of Pure and Applied Sciences and Technology* **13**(1) (2012), 1 – 5, URL: `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.419.9999&rep=rep1&type=pdf`.

[7] P. U. Chopade and K. L. Bondar, Existence of solutions to boundary value problem of functional difference equation, *International Journal of Mathematica Analysis* **5**(13) (2011), 643 – 650, URL: `http://www.m-hikari.com/ijma/ijma-2011/ijma-13-16-2011/chopadeIJMA13-16-2011.pdf`.

[8] S. G. Deo, V. Lakshmikantham and V. Raghavendra, *Text Book of Ordinary Differential Equations*, 2nd Edition, Tata McGraw-Hill Publishing Company Limited, New Delhi (2003), DOI: Missing.

[9] G. C. Done, K. L. Bondar and P. U. Chopade, Existence and uniqueness of solution of first order nonlinear difference equation, *Journal of Mathematical and Computational Science* **10**(5) (2020), 1375 – 1383, DOI: 10.28919/jmcs/4619.

[10] G. C. Done, K. L. Bondar and P. U. Chopade, Existence and uniqueness results on mixed type summation-difference equations in cone metric space, *Journal of Mathematical and Computational Science* **10**(4) (2020), 956 – 963, DOI: 10.28919/jmcs/4530.

[11] L. G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications* **332**(2) (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[12] D. Ilić and V. Rakočević, Common fixed points for maps on cone metric space, *Journal of Mathematical Analysis and Applications* **341**(3) (2008), 876 – 882, DOI: 10.1016/j.jmaa.2007.10.065.

[13] A. Karoui, On the existence of continuous solutions of nonlinear integral equations, *Applied Mathematics Letters* **18**(3) (2005), 299 – 305, DOI: 10.1016/j.aml.2004.09.007.

[14] W. G. Kelley and A. C. Peterson, *Difference Equation*, Academic Press (2001), DOI: Missing.

[15] M. K. Kwong, On Krasnoselskii's cone fixed point theorems, *Fixed Point Theory and Applications* **2008** (2008), Article ID 164537, 18 pages, DOI: /10.1155/2008/164537.

[16] P. Raja and S. M. Vaezpour, Some extensions of Banach's contraction principle in complete cone metric spaces, *Fixed Point Theory and Applications* **2008** (2008), Article ID 768294, 11 pages, DOI: 10.1155/2008/768294.

[17] S. H. Vaezpour and R. Hamlbarani, Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications* **345**(2) (2008), 719 – 724, DOI: 10.1016/j.jmaa.2008.04.049.