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Research Article

On Initial Chebyshev Polynomial Coefficient Problem for Certain Subclass of Bi-Univalent Functions

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Abstract. In this paper, we firstly, introduced the subclass $R_\Sigma(\tau,\alpha,\gamma;t)$ satisfying subordinate conditions. Subsequently, considering this defined subclass, initial coefficient estimations are established using by Chebyshev polynomials expansions, and Fekete-Szegö inequalities are also derived for functions belonging to the said subclass. Furthermore, Some relevant consequences of these results are also discussed.

Keywords. Initial coefficients problem; Bi-univalent function; Chebyshev polinomials; Fekete-Szegö problem

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. In addition, we indicate by $\mathcal{S}$ the class of all univalent functions in Δ.

For f and g, analytic functions in Δ, the function $f(z)$ is subordinate to $g(z)$ in Δ, and it can be represented by

$$f(z) \prec g(z) \quad (z \in \Delta)$$

if there exists an analytic Schwarz function $w(z)$ being as

$$w(0) = 0 \ \text{ and } \ |w(z)| < 1 \quad (z \in \Delta)$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

Especially, if the function g is univalent in Δ, then the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Every function $f \in \mathcal{S}$ is known to have an inverse f^{-1}, given by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f^{-1}(f(w)) = w \left(|w| < r_0(f);\, r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \tag{2}$$

In order to have bi-univalent $f \in \mathcal{A}$ in Δ, both $f(z)$ and $f^{-1}(z)$ must be univalent in Δ.

Let represent the class of bi-univalent functions in Δ, given by the Taylor-Maclaurin series expansion (1) by Σ.

Some well-known subclasses of the $\mathcal{S}$, which are denoted by class $S^*(\alpha)$ of starlike functions of order α in Δ and the class $K(\alpha)$ of convex functions of order α in Δ, are respectively shown as follows:

$$S^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha;\ z \in \Delta;\ 0 \leq \alpha < 1 \right\} \tag{3}$$

and

$$K(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha; z \in \Delta;\ 0 \leq \alpha < 1 \right\}. \tag{4}$$

For $0 \leq \alpha < 1$, if both f and f^{-1} are respectively starlike or convex functions of order α, $f \in \Sigma$ is in the class $S_\Sigma^*(\alpha)$ of bi-starlike function of order α, or $K_\Sigma(\alpha)$ of bi-convex function of order α.

The significance of Chebyshev polynomial in numerical analysis is increased in terms of both theoretical and practical points of view. On the other hand, many researchers have also been dealing with orthogonal Chebyshev polynomials. One can see the details of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications in the references [4, 5, 7]. The well known first and second kinds Chebyshev polynomials are defined as follows:

$$T_n(t) = \cos(n\theta) \text{ and } U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1)$$

where the subscript n shows the polynomial degree and t is equal to $\cos\theta$.

In geometric function theory, the Fekete-Szegö functional $|a_3 - \eta a_2^2|$ for normalized univalent functions of the form given by (1) is well known in this field. Historically, its origin is based on Fekete and Szegö of the 1933 conjecture of Littlewood and Paley. In their study the coefficients of odd univalent functions are bounded by unity (see [6]). Since Fekete-Szegö functional has received great attention, especially in many subclasses of the family of univalent functions, this topic had become of interest among researchers (see, e.g., [1,2,8,9,11]).

Definition 1.1. A function $f \in \Sigma$ given by the equation (1) is said to be in the class $R_\Sigma(\tau,\alpha,\gamma;t)$, for $\alpha \ge 1$, $\gamma \ge 0$, $\tau \in \mathbb{C}\setminus\{0\}$, $t \in \left(\frac{1}{2},1\right]$ and all $z,w \in \Delta$ if the following subordination conditions hold:

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\gamma z f''(z)-1\right] \prec H(z,t) := \frac{1}{1-2tz+z^2} \tag{5}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{g(w)}{w}+\alpha g'(w)+\gamma z g''(w)-1\right] \prec H(w,t) := \frac{1}{1-2wt+w^2}, \tag{6}$$

where the function $g = f^{-1}$ is given by (2).

It is remarkable that if $t = \cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1-2\cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in \Delta).$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots. \quad (z \in \mathbb{U}).$$

$H(z,t)$ can be written as from reference [10],

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 \dots \quad (z \in \Delta),\ t \in (-1,1)$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N})$$

indicates the second kind of Chebyshev polynomials and we have some initial coefficients as follows:

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t,\ \ U_2(t) = 4t^2 - 1,\ \ U_3(t) = 8t^3 - 4t,\ \ U_4(t) = 16t^4 - 12t^3 + 1 \dots. \tag{7}$$

The first kind of Chebyshev polynomial $T_n(t)$, $t \in [-1,1]$, is indicated by

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \Delta).$$

The first kind of Chebyshev polynomial $T_n(t)$ is related to second kind of Chebyshev polynomial $U_n(t)$ which is given below:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Remark 1.2. (i) For $\tau = 1$, $\alpha = \lambda$ and $\gamma = 0$, we get the class $R_\Sigma(1,\lambda,0;t) = B_\Sigma(\lambda,t)$ consists of functions $f \in \Sigma$ satisfying the condition

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (2) (Bulut et al. [3]).

(ii) For $\tau = 1$, $\alpha = 1$ and $\gamma = 0$, we have the class $R_\Sigma(1,1,0;t) = B_\Sigma(t)$ consists of functions f satisfying the condition

$$f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$g'(w) \prec H(w,t) := \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is defined by (2).

In the current investigation, we present a subclass $R_\Sigma(\tau,\alpha,\gamma;t)$ of analytic and bi-univalent functions. Additionally, we derived the initial coefficient bounds and Fekete-Szegö inequality by means of Chebyshev polynomials expansions. Furthermore, we introduce some corollaries associated with our main results.

2. Main Results

In this section, we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö inequality for functions in the class $R_\Sigma(\tau,\alpha,\gamma;t)$, which is introduced by Definition 1.1. These inequalities are asserted by Theorem 2.1.

Theorem 2.1. *For $\alpha \geq 1$, $\gamma \geq 0$, $\tau \in \mathbb{C}\setminus\{0\}$ and $t \in \left(\frac{1}{2},1\right]$, let $f \in R_\Sigma(\tau,\alpha,\gamma;t)$. Then*

$$|a_2| \leq \frac{2|\tau|t\sqrt{2t}}{\sqrt{|4[\tau(1+2\alpha+6\gamma)-(1+\alpha+2\gamma)^2]t^2+(1+\alpha+2\gamma)^2|}}, \tag{8}$$

$$|a_3| \leq \frac{4|\tau|^2t^2}{(1+\alpha+2\gamma)^2} + \frac{2|\tau|t}{1+2\alpha+6\gamma} \tag{9}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{2|\tau|t}{1+2\alpha+6\gamma}, & |\eta-1| \leq \dfrac{|(1+\alpha+2\gamma)^2-4[(1+\alpha+2\gamma)^2-\tau(1+2\alpha+6\gamma)]t^2|}{4(1+2\alpha+6\gamma)t^2} \\ \dfrac{8|\tau|^2|\eta-1|t^3}{|(1+\alpha+2\gamma)^2-4[(1+\alpha+2\gamma)^2-\tau(1+2\alpha+6\gamma)]t^2|}, & \\ & |\eta-1| \geq \dfrac{|(1+\alpha+2\gamma)^2-4[(1+\alpha+2\gamma)^2-\tau(1+2\alpha+6\gamma)]t^2|}{4(1+2\alpha+6\gamma)t^2}. \end{cases} \tag{10}$$

Proof. Let $f \in R_\Sigma(\tau,\alpha,\gamma;\beta)$. Then from the equations (5) and (6), we obtain

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\gamma z f''(z)-1\right]=1+U_1(t)p(z)+U_2(t)p^2(z)+\dots \tag{11}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{g(w)}{w}+\alpha g'(w)+\gamma w g''(w)-1\right]=1+U_1(t)q(w)+U_2(t)q^2(w)+\dots \tag{12}$$

for some analytic functions

$$p(z)=c_1z+c_2z^2+c_3z^3+\dots,\qquad (z\in\Delta) \tag{13}$$

$$q(w)=d_1w+d_2w^2+d_3w^3+\dots,\quad (w\in\Delta) \tag{14}$$

such that $p(0)=q(0)=0$, $|p(z)|<1$ $(z\in\Delta)$ and $|q(w)|<1$ $(w\in\Delta)$.

Obviously, if $|p(z)|<1$ and $|q(w)|<1$, then

$$|c_j|\le 1 \text{ and } |d_j|\le 1 \quad \text{for } j\in N. \tag{15}$$

From (11), (12), (13) and (14), we have

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\gamma z f''(z)-1\right]=1+U_1(t)c_1z+[U_1(t)c_2+U_2(t)c_1^2]z^2+\dots \tag{16}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{g(w)}{w}+\alpha g'(w)+\gamma w g''(w)-1\right]=1+U_1(t)d_1w+[U_1(t)d_2+U_2(t)d_1^2]w^2+\dots. \tag{17}$$

When we equate the coefficients (16) and (17), we have

$$\frac{1}{\tau}(1+\alpha+2\gamma)a_2=U_1(t)c_1, \tag{18}$$

$$\frac{1}{\tau}(1+2\alpha+6\gamma)a_3=U_1(t)c_2+U_2(t)c_1^2, \tag{19}$$

$$-\frac{1}{\tau}(1+\alpha+2\gamma)a_2=U_1(t)d_1, \tag{20}$$

$$\frac{1}{\tau}(1+2\alpha+6\gamma)(2a_2^2-a_3)=U_1(t)d_2+U_2(t)d_1^2. \tag{21}$$

From (18) and (20), we obtain

$$c_1=-d_1 \tag{22}$$

and

$$\frac{2}{\tau^2}(1+\alpha+2\gamma)^2a_2^2=U_1^2(t)(c_1^2+d_1^2). \tag{23}$$

Also, by using (19) and (21), we obtain

$$\frac{2}{\tau}(1+2\alpha+6\gamma)a_2^2=U_1(t)(c_2+d_2)+U_2(t)(c_1^2+d_1^2). \tag{24}$$

By using (23) in (24), we get

$$\left[\frac{2}{\tau}(1+2\alpha+6\gamma)-\frac{2U_2(t)}{\tau^2U_1^2(t)}(1+\alpha+2\gamma)^2\right]a_2^2=U_1(t)(c_2+d_2). \tag{25}$$

From the eqs. (7), (15) and (25), we obtain the inequality (8).

Thereafter, we get the following equality, by subtracting (21) from (19)

$$\frac{2}{\tau}(1+2\alpha+6\gamma)(a_3-a_2^2)=U_1(t)(c_2-d_2)+U_2(c_1^2-d_1^2). \tag{26}$$

In addition, in the light of (22), we get

$$a_3=a_2^2+\frac{\tau U_1(t)(c_2-d_2)}{2(1+2\alpha+6\gamma)}. \tag{27}$$

Thus, we get convenient inequality (9) by using (23) and applying (7).

Now, we get the following equalities by using (25) and (27) for some $\eta\in\mathbb{R}$

$$(a_3-\eta a_2^2)=(1-\eta)\left[\frac{\tau^2U_1^3(t)(c_2+d_2)}{2\tau U_1^2(t)(1+2\alpha+6\gamma)-2U_2(t)(1+\alpha+2\gamma)^2}\right]+\frac{\tau U_1(t)(c_2-d_2)}{2(1+2\alpha+6\gamma)} \tag{28}$$

and

$$a_3-\eta a_2^2=\tau U_1(t)\left[\left(h(\eta)+\frac{1}{2(1+2\alpha+6\gamma)}\right)c_2+\left(h(\eta)-\frac{1}{2(1+2\alpha+6\gamma)}\right)d_2\right],$$

where

$$h(\eta)=\frac{\tau U_1^2(t)(1-\eta)}{(2\tau U_1^2(t)(1+2\alpha+6\gamma)-2U_2(t)(1+\alpha+2\gamma)^2)}.$$

So, we conclude that

$$|a_3-\eta a_2^2|\le\begin{cases}\dfrac{2|\tau|t}{1+2\alpha+6\gamma}, & 0\le|h(\eta)|\le\dfrac{1}{2(1+2\alpha+6\gamma)}\\[2ex] 4|\tau|\,|h(\eta)|t, & |h(\eta)|\ge\dfrac{1}{2(1+2\alpha+6\gamma)}.\end{cases}$$

So, this is the proof which confirms Theorem 2.1. In the special case, we can obtain the Corollary 2.2 for the parameters $\tau=1$, $\alpha=\lambda$ and $\gamma=0$ in Theorem 2.1. □

Corollary 2.2 ([3]). *Let the function $f\in\mathcal{B}_\Sigma(\lambda,t)$, for $\lambda\ge1$ and $t\in(1/2,1]$. Then*

$$|a_2|\le\frac{2t\sqrt{2t}}{\sqrt{|(1+\lambda)^2-4\lambda^2t^2|}},$$

$$|a_3|\le\frac{4t^2}{(1+\lambda)^2}+\frac{2t}{(1+2\lambda)}$$

and for some $\eta\in\mathbb{R}$,

$$|a_2-\eta a_3|\le\begin{cases}\dfrac{2t}{1+2\lambda}, & |\eta-1|\le\dfrac{|(1+\lambda)^2-4\lambda^2t^2|}{4(1+2\lambda)t^2}\\[2ex] \dfrac{8|\eta-1|t^3}{|(1+\lambda)^2-4\lambda^2t^2|}, & |\eta-1|\ge\dfrac{|(1+\lambda)^2-4\lambda^2t^2|}{4(1+2\lambda)t^2}.\end{cases}$$

In the special case, we can obtain the following Corollary 2.3 for the parameters $\tau=1$, $\alpha=1$ and $\gamma=0$ in Theorem 2.1.

Corollary 2.3. *Let the function $f\in B_\Sigma(t)$, for $t\in\left(\frac{1}{2},1\right]$. Then*

$$|a_2|\le\frac{t\sqrt{2t}}{\sqrt{|1-t^2|}},$$

$$|a_3| \le t^2 + \frac{2t}{3},$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{2t}{3}, & |\eta - 1| \le \frac{1-t^2}{3t^2} \\ \frac{2|\eta-1|t^3}{1-t^2}, & |\eta - 1| \ge \frac{1-t^2}{3t^2}. \end{cases}$$

3. Conclusion

Using the concept of Chebyshev polynomials, we have introduced a new subclass in the unit disc associated with subordination. We have then derived the initial coefficient estimations using by Chebyshev polynomials expansions, and also Fekete-Szegö inequalities for functions belonging to this subclass. Our main result is stated and proved as Theorem 2.1. Additionally, by specializing the some parameters, some relevant interesting consequences of these results which were studied in previous works, are obtained. So, these general results presented in this paper, are motivated essentially by the earlier works which are pointed out.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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