Numerical Approach for Differential-Difference Equations with Layer Behaviour

G. Sangeetha¹, G. Mahesh², and K. Phaneendra³,*

¹Department of Mathematics, Bhogi Reddy Engineering College for women, Hyderabad, India
²Department of Mathematics, Geethanjali College of Engineering and Technology, Cheeryal, India
³Department of Mathematics, University College of Engineering, Osmania University, Hyderabad, India

*Corresponding author: kollojuphaneendra@yahoo.co.in

Abstract. A numerical scheme is proposed using a non polynomial spline to solve the differential-difference equations having layer behaviour, with delay as well advanced terms. The retarded terms are handled by using Taylor's series, subsequently the given problem is substituted by an equivalent second order singular perturbation problem. A finite difference scheme using non polynomial spline is derived and it is applied to the singular perturbation problem using non standard differences of the first derivatives. Tridiagonal algorithm is used to solve the resulting system. The method is exemplified on numerical examples with various values of perturbation, delay and advance parameters. Maximum absolute errors are computed and tabulated to support the method. Numerical solutions are pictured in graphs and the effects of small shifts on the boundary layer region has been investigated. Also, the convergence of the proposed method has also been established.

Keywords. Differential-difference equations; Boundary Layer; Non polynomial spline; Maximum absolute error

MSC. 65L10; 65L11; 65L12

Received: October 22, 2019 Accepted: November 9, 2019

1. Introduction

The problems of differential-difference comes about in the modelling of many practical phenomena such as, thermo-elasticity [2], hybrid optical system [3], in population dynamics [10], in models for physiological processes [14], red blood cell system [13], predator-prey models [15] and in the potential in nerve cells by random synaptic inputs in dendrites [18].
For further study of mathematical aspects of the above class of models and singular perturbation problems, one can be found in the collection of books, to name a few, Bellman and Cooke [1], Doolan et al. [5], Driver [6], El’sgol’ts and Norkin [7], Kokotovic [9], Miller et al. [16] and Smith [17].

Lange and Miura [11,12] presented an analysis of differential-difference equations with small shifts, layers having turning points and rapid oscillations. In [4], a fourth order finite difference method with a fitting factor is proposed for the solution of the singularly perturbed differential-difference equations with mixed shifts. The authors in [8], proposed a fitted piecewise-uniform mesh method with analysis for differential difference equation having mixed small shifts having boundary layer. With this motivation, in the next section, we describe the problem and derivation of the of the numerical scheme using non polynomial spline.

2. Numerical Approach

Consider the singularly perturbed differential-difference equation with small delay as well as advance terms of the form:

$$\varepsilon w''(t) + a(t)w'(t) + b(t)w(t - \delta) + c(t)w(t) + d(t)w(t + \eta) = f(t), \quad 0 < t < 1$$ (1)

subject to the interval and boundary conditions

$$w(t) = \phi(t), \quad \text{on} \quad -\delta \leq t \leq 0$$ (2)

$$w(t) = \gamma(t), \quad \text{on} \quad 1 \leq t \leq 1 + \eta$$ (3)

where $a(t), b(t), c(t), d(t), \phi(t)$ and $\gamma(t)$ are bounded and continuously differentiable functions on $(0,1)$, $\varepsilon$ is the small perturbation parameter ($0 < \varepsilon \ll 1$), $\delta$ and $\eta$ are the delay and the advance parameters respectively ($0 < \delta = o(\varepsilon); 0 < \eta = o(\varepsilon)$).

Applying Taylor series on retarded terms in the neighbourhood of the point $t$, we have

$$w(t - \delta) \approx w(t) - \delta w'(t),$$ (4)

$$w(t + \eta) \approx w(t) + \eta w'(t),$$ (5)

Using eqs. (4) and (5) in eq. (1), we get an equivalent second order singular perturbation problem of the form:

$$\varepsilon w''(t) + p(t)w'(t) + q(t)w(t) = f(t)$$ (6)

with boundary conditions

$$w(0) = \phi(0),$$ (7)

$$w(1) = \gamma(1),$$ (8)

where $p(t) = a(t) + d(t)\eta - b(t)\delta$ and $q(t) = b(t) + c(t) + d(t)$.

Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transition from eq. (1) to eq. (6) is admitted. This replacement is significant from the computational point of view (El’sgolt’s and Norkin [3]).

2.1 Non polynomial spline

To construct the difference equation of the problem eqs. (6)-(8), the domain $[0,1]$ is divided into $N$ non overlapping intervals $0 = t_0 < t_1 < t_2 < \ldots < t_N = 1$, each of length $h$. Then, we have
where \( t = t_0 + ih \) for \( i = 0, 1, \ldots, N \). For simplicity, denote \( A(t_i) = A_i \), \( B(t_i) = B_i \), \( f(t_i) = f_i \), \( w(t_0) = w_0 \), \( w(t_i) = w_i \), \( w(t_i + h) = w_{i+1} \), \( w(t_i - h) = w_{i-1} \), \( w'(t_i) = w'_i \), \( w''(t_i) = w''_i \), etc.

For each \( i \)th segment, the cubic non-polynomial spline function \( P_i(t) \) has the form

\[
P_i(t) = \tilde{a}_i + \tilde{b}_i(t - t_i) + \tilde{c}_i \sin \tau(t - t_i) + \tilde{d}_i \cos \tau(t - t_i), \quad i = 0, 1, \ldots, N - 1,
\]

where \( \tilde{a}_i, \tilde{b}_i, \tilde{c}_i \) and \( \tilde{d}_i \) are constants and \( \tau \) is a free parameter.

Let \( w(t) \) be the exact solution and \( w_i \) be an approximation to \( w(t_i) \) obtained by the non-polynomial cubic spline \( P_i(t) \) passing through the points \( (t_i, w_i) \) and \( (t_{i+1}, w_{i+1}) \).

The spline \( P_i(t) \) satisfies interpolatory conditions at \( t_i \) and \( t_{i+1} \), also the continuity of first derivative at the common nodes \( (t_i, w_i) \). The non-polynomial function \( P(t) \) of class \( C^2[A, B] \) interpolates \( w(t) \) at the grid points \( t_i \), for \( i = 0, 1, \ldots, N \), depends on a parameter \( \tau \), and reduces to an ordinary cubic spline \( P(t) \) in \( [A, B] \) as \( \tau \to 0 \).

To derive the expressions for the unknown coefficients of eq. (9) in terms of \( w_i \), \( w_{i+1} \), \( M_i \) and \( M_{i+1} \), define

\[
P_i(t_i) = w_i, \quad P_i(t_{i+1}) = w_{i+1},
\]

\[
P_i'(t_i) = M_i, \quad P_i'(t_{i+1}) = M_{i+1}.
\]

Then by algebraic calculations, we get the following expressions:

\[
\tilde{a}_i = w_i + \frac{M_i}{\tau^2}, \quad \tilde{b}_i = \frac{w_{i+1} - w_i}{h} + \frac{M_{i+1} - M_i}{\tau \theta}, \quad \tilde{c}_i = \frac{M_i \cos \theta - M_{i+1}}{\tau^2 \sin \theta}, \quad \tilde{d}_i = -\frac{M_i}{\tau^2},
\]

where \( \theta = \tau h \), for \( i = 0, 1, 2, \ldots, N - 1 \).

Using the continuity of the first derivative at \( (t_i, w_i) \), that is \( P_i'(t_i) = P_i'(t_i) \), we arrived at the following relation:

\[
aM_{i+1} + 2\beta M_i + aM_{i-1} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \quad \text{for} \quad i = 1, 2, \ldots, N - 1,
\]

where

\[
a = -\frac{1}{\theta^2} + \frac{1}{\theta \sin \theta}, \quad \beta = \frac{1}{\theta^2} - \frac{\cos \theta}{\theta \sin \theta}, \quad M_j = w''(t_j), \quad j = i, i + 1 \quad \text{and} \quad \theta = \tau h.
\]

For our convenience, at the grid points \( x_i \), rearranging eq. (1) we obtain:

\[
\epsilon w''_i = \tilde{A}(t) w'_i + \tilde{B}(t) w_i + f_i,
\]

where \( \tilde{A}(t) = -p(t) \), \( \tilde{B}(t) = -q(t) \).

By using Spline’s second derivatives, we have

\[
\epsilon M_j = \tilde{A}(t) w'_j(t) + \tilde{B}(t) w(t_j) + f(t_j) \quad \text{for} \quad j = i - 1, i, i + 1.
\]

Substituting the above equations in eq. (10) and using the following approximations for the first order derivative of \( y \) at the grid points \( t_1, t_2, \ldots, t_{N-1} \)

\[
w'_{i+1} \approx \frac{w_{i+1} - 4w_i + 3w_{i+1}}{2h},
\]

\[
w'_{i-1} \approx \frac{-3w_{i-1} + 4w_i - w_{i+1}}{2h},
\]

\[
w'_{i} \approx \frac{1 + 2\omega h^2 \tilde{B}_{i+1} + \omega h [3\tilde{A}_{i+1} + \tilde{A}_{i-1}]}{2h} w_{i+1} - 2\omega [\tilde{A}_{i+1} + \tilde{A}_{i-1}] w_i
\]

\[
\text{Communications in Mathematics and Applications, Vol. 10, No. 4, pp. 851–863, 2019}
\]
where
\[
E_i w_{i-1} + F_i w_i + G_i w_{i+1} = H_i \quad \text{for } i = 1, 2, \ldots, N - 1,
\]
(14)

Now we consider the convergence analysis of the non-polynomial spline method, described in this section. Incorporating the boundary conditions we obtain the system of equations in the matrix form as
\[
(D + J)Y + \tilde{R} + T(h) = O,
\]
(15)

where
\[
D = \begin{bmatrix}
-\epsilon & 2\epsilon & 0 & 0 & \cdots & 0 \\
-\epsilon & -\epsilon & 2\epsilon & 0 & \cdots & 0 \\
0 & -\epsilon & -\epsilon & 2\epsilon & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & -\epsilon & 2\epsilon
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
v_1 & u_1 & 0 & 0 & \cdots & 0 \\
v_2 & v_2 & u_2 & 0 & \cdots & 0 \\
v_3 & v_3 & u_3 & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & z_{N-1} & v_{N-1}
\end{bmatrix},
\]

\[
z_i = -\frac{3}{2} \alpha \tilde{A}_{i-1} h + \beta \tilde{A}_i h^2 \omega [ \tilde{A}_{i+1} + 3 \tilde{A}_{i-1} ] - 2 \omega \tilde{A}_i \beta \tilde{B}_{i-1} + \frac{\alpha}{2} \tilde{A}_{i+1} h \alpha \tilde{B}_{i-1} h^2 - h \beta \tilde{A}_i,
\]

\[
v_i = 2 \alpha \tilde{A}_{i-1} h - 4 \beta \tilde{A}_i h^2 \omega [ \tilde{A}_{i+1} + \tilde{A}_{i-1} ] - 2 \alpha \tilde{A}_{i+1} h + 2 \beta \tilde{B}_i h^2,
\]

\[
u_i = -\frac{\alpha}{2} \tilde{A}_{i-1} h + \beta \tilde{A}_i h^2 \omega [ 3 \tilde{A}_{i+1} + \tilde{A}_{i-1} ] + 2 \omega \tilde{A}_i \beta \tilde{B}_{i-1} + \frac{3}{2} \alpha \tilde{A}_{i+1} h + \alpha \tilde{B}_{i+1} h^2 + h \beta \tilde{A}_i,
\]

for \( i = 1, 2, \ldots, N - 1 \),

\[
\tilde{R} = [\tilde{B}_1 + (\epsilon + z_1) y_0, \tilde{B}_2, \tilde{B}_3, \ldots, \tilde{B}_{N-1} + (\epsilon + w_{N-1}) y_1],
\]

\[
\tilde{B}_i = h^2 [(\alpha - 2 \omega \beta \tilde{A}_i h)f_{i-1} + 2 \beta f_i + (\alpha + 2 \omega \beta \tilde{A}_i h) f_{i+1}], \quad i = 1, 2, \ldots, N - 1
\]
and \( W = [W_1, W_2, \ldots, W_{N-1}]^T \), \( T(h) = [T_1, T_2, \ldots, T_{N-1}]^T \), \( O = [0, 0, \ldots, 0]^T \) are the associated vectors with eq. (15).

The local truncation error associated with the scheme developed in eq. (14) is
\[
T(h) = \left[-1 + 2(\alpha + \beta)\right] εh^2 w''(t_i) + \left\{4ωε + \frac{1}{3}\right\} β - \frac{2α}{3} \left[\hat{A}(x_i)w''(t_i) + (-1 + 12α)\frac{ε}{12}w^{(4)}(t_i)\right]\ h^4 + O(h^6)
\]

i.e.,
\[
T(h) = O(h^6), \quad \text{for } \alpha = \frac{1}{12}, \beta = \frac{5}{12}, ω = -\frac{1}{20ε}.
\]

Let \( w = [w_1, w_2, \ldots, w_{N-1}]^T \) \( W \) which satisfies the equation
\[
(D + J)w + \tilde{R} = 0.
\]

Let \( e_i = w_i - W_i, i = 1, 2, \ldots, N-1 \) be the discretization error so that
\[
E = [e_1, e_2, \ldots, e_{N-1}]^T = w - W.
\]

Using eq. (15) and eq. (16), we have the error equation as:
\[
(D + J)E = T(h).
\]

Let \( |\hat{A}(x)| \leq C_1 \) and \( |\hat{B}(x)| \leq C_2 \) where \( C_1, C_2 \) are positive constants. If \( J_{i,j} \) be the \((i,j)\)th element of \( J \), then
\[
|J_{i,i+1}| = |u_i| \leq (h(α + β)C_1 + h^2 αC_2 + 4βωh^2C_1^2 + 2h^3 βωC_1C_2), \quad i = 1, 2, \ldots, N - 2,
\]
\[
|J_{i,i-1}| = |z_i| \leq (h(α + β)C_1 + h^2 αC_2 + 4βωh^2C_1^2 + 2h^3 βωC_1C_2), \quad i = 2, 3, \ldots, N - 1.
\]

Thus for \( h \),
\[
|J_{i,i+1}| < ε, \quad i = 1, 2, \ldots, N - 2
\]

and
\[
|J_{i,i-1}| < ε, \quad i = 2, 3, \ldots, N - 1.
\]

Hence \((D + J)\) is irreducible. Let \( P_i \) be the sum of the elements of the \( i \)th row of the matrix \((D + J)\), then we have
\[
\begin{align*}
\hat{P}_i &= \epsilon + \frac{ah}{2} (3\hat{A}_i - 3\hat{A}_{i-1}) - h β\hat{P}_i + h^2 (α\hat{B}_i + 2β\hat{B}_i) \\
&\quad - h^2 βω\hat{A}_i (3\hat{A}_{i+1} + 3\hat{A}_{i-1}) + 2h^3 βω\hat{A}_i \hat{B}_{i+1}, \quad \text{for } i = 1, \\
\hat{P}_i &= h^2 (α\hat{B}_{i-1} + 2β\hat{B}_{i+1} + α\hat{B}_{i+1}) + 2h^3 β ω (\hat{B}_{i+1} - \hat{B}_{i-1}), \quad \text{for } i = 2, 3, \ldots, N - 2, \\
\hat{P}_i &= \epsilon + \frac{ah}{2} (3\hat{A}_i - 3\hat{A}_{i-1}) - h β\hat{A}_i + h^2 (α\hat{B}_i + 2β\hat{B}_i) \\
&\quad - h^2 βω\hat{A}_i (3\hat{A}_{i+1} + 3\hat{A}_{i-1}) - 2h^3 βω\hat{A}_i \hat{B}_{i-1}, \quad \text{for } i = N - 1.
\end{align*}
\]

Let \( C_1^* = \min_{1 ≤ i ≤ n} |\hat{A}(t)| \) and \( C_2^* = \max_{1 ≤ i ≤ n} |\hat{A}(t)| \), \( C_2^* = \min_{1 ≤ i ≤ n} |\hat{B}(t)| \) and \( C_2^* = \max_{1 ≤ i ≤ n} |\hat{B}(t)| \).

Since \( 0 < \epsilon \ll 1 \) and \( \epsilon \propto O(h) \), it is verify that for \( h, (D + J) \) is monotone.

Hence \((D + J)^{-1}\) exists and \((D + J)^{-1} ≥ 0\).

Thus from eq. (17), we have
\[
\|E\| ≤ \|(D + J)^{-1}\| \cdot \|T\|.
\]
Let \((D + J)^{-1}_{i,k}\) be the \((i,k)\)th element of \((D + J)^{-1}\) and we define

\[
\| (D + J)^{-1} \| = \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (D + J)^{-1}_{i,k} \quad \text{and} \quad \| T(h) \| = \max_{1 \leq i \leq N-1} |T(h)|. \tag{21}
\]

Since \((D + J)^{-1}_{i,k} \geq 0\) and \(\sum_{k=1}^{N-1} (D + J)^{-1}_{i,k} \cdot \bar{P}_k = 1\) for \(i = 1, 2, 3, \ldots, N-1\).

Hence

\[
(D + J)^{-1}_{i,k} \leq \frac{1}{P_i} < \frac{1}{h^2 \left[ (\alpha + 2\beta) C_{2^*} - 4\beta \omega C_{1^*} \right]}, \quad i = 1, \tag{22}
\]

\[
(D + J)^{-1}_{i,k} \leq \frac{1}{P_i} < \frac{1}{h^2 \left[ (\alpha + 2\beta) C_{2^*} - 4\beta \omega C_{1^*} \right]}, \quad i = N-1. \tag{23}
\]

Further

\[
\sum_{k=1}^{N-1} (D + J)^{-1}_{i,k} \leq \frac{1}{\min_{2 \leq i \leq N-2} P_i} < \frac{1}{h^2 \left[ 2(\alpha + \beta) C_{2^*} \right]}. \tag{24}
\]

Using the eqs. (22)-(24), from eq. (20), we get

\[
\| E \| \leq O(h^4).
\]

Hence, the method is fourth order convergent for \(\alpha = \frac{1}{12}, \beta = \frac{5}{12}, \omega = -\frac{1}{20}\varepsilon\) on uniform mesh.

### 4. Numerical Experiments

To demonstrate the proposed method computationally, we consider four boundary value problems. The maximum absolute errors in the solution are estimated by using \(E_{N,\varepsilon} = \max_{0 \leq i \leq N} |w(t_i) - w_i|\), where \(w(t_i)\) is the exact solution and \(w_i\) is the computed solution.

**Example 1.** Consider the differential-difference equation with left end boundary layer

\[
\varepsilon w''(t) + w'(t) + 2w(t - \delta) - 3w(t) = 0 \quad \text{with} \quad w(0) = 1, \quad -\delta \leq t \leq 0 \quad \text{and} \quad w(1) = 1, \quad 1 \leq t \leq 1 + \eta.
\]

**Table 1.** Maximum errors in the solution of Example 1 for \(\varepsilon = 0.1\) with different values of \(\delta\)

<table>
<thead>
<tr>
<th>(N \rightarrow 8)</th>
<th>32</th>
<th>128</th>
<th>512</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta)</td>
<td>(\text{Present method})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.00)</td>
<td>1.0798e-03</td>
<td>4.0303e-06</td>
<td>1.5642e-08</td>
</tr>
<tr>
<td>(0.05)</td>
<td>7.9692e-04</td>
<td>2.9371e-06</td>
<td>1.1431e-08</td>
</tr>
<tr>
<td>(0.09)</td>
<td>6.0862e-04</td>
<td>2.2124e-06</td>
<td>8.6762e-09</td>
</tr>
<tr>
<td>(0.13)</td>
<td>5.0927e-04</td>
<td>1.9751e-06</td>
<td>7.6521e-09</td>
</tr>
</tbody>
</table>

Results in Kadalbajoo and Sharma \[8\]

| \(\delta\)       | \(\text{Present method}\) |
| \(0.00\)          | 0.09907804 | 0.03700736 | 0.00954678 | 0.00214501 |
| \(0.05\)          | 0.09659609 | 0.03640566 | 0.00924661 | 0.00202998 |
| \(0.09\)          | 0.09277401 | 0.03556652 | 0.00895172 | 0.00192488 |

**Example 2.** Consider the equation with left end boundary layer

\[
\varepsilon w''(t) + w'(t) - 3w(t) + 2w(t + \eta) = 0 \quad \text{with} \quad w(0) = 1, \quad -\delta \leq t \leq 0 \quad \text{and} \quad w(1) = 1, \quad 1 \leq t \leq 1 + \eta.
\]
Table 2. Maximum errors in the solution of Example 2 for \( \varepsilon = 0.1 \) with different values of \( \eta \)

<table>
<thead>
<tr>
<th>( N \rightarrow )</th>
<th>8</th>
<th>32</th>
<th>128</th>
<th>512</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta ) ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>1.0798e-03</td>
<td>4.0303e-06</td>
<td>1.5642e-08</td>
<td>6.1089e-11</td>
</tr>
<tr>
<td>0.05</td>
<td>1.4141e-03</td>
<td>5.3400e-06</td>
<td>2.0819e-08</td>
<td>8.1296e-11</td>
</tr>
<tr>
<td>0.09</td>
<td>1.7187e-03</td>
<td>6.5463e-06</td>
<td>2.5726e-08</td>
<td>1.0044e-10</td>
</tr>
</tbody>
</table>

Results in Kadalbajoo and Sharma [8]

<table>
<thead>
<tr>
<th>( \eta ) ↓</th>
<th>0.00</th>
<th>0.05</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon ) 0.00</td>
<td>0.09907804</td>
<td>0.03700736</td>
<td>0.00954678</td>
</tr>
<tr>
<td>( \varepsilon ) 0.05</td>
<td>0.09977501</td>
<td>0.03727087</td>
<td>0.00979659</td>
</tr>
<tr>
<td>( \varepsilon ) 0.09</td>
<td>0.10031348</td>
<td>0.03723863</td>
<td>0.00996284</td>
</tr>
</tbody>
</table>

Example 3. Consider the problem equation with left end boundary layer

\[ \varepsilon w''(t) + w'(t) - 2w(t - \delta) - 5w(t) + w(t + \eta) = 0 \]

with \( w(t) = 1 \), \( -\delta \leq t \leq 0 \) and \( w(t) = 1 \), \( 1 \leq t \leq 1 + \eta \).

Table 3. Maximum errors in Example 3 for \( \varepsilon = 0.1 \) with different values of \( \eta \) and \( \delta \)

<table>
<thead>
<tr>
<th>( \eta ) ( \delta )</th>
<th>( N \rightarrow 8 )</th>
<th>( N \rightarrow 32 )</th>
<th>( N \rightarrow 128 )</th>
<th>( N \rightarrow 512 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta ) ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.00</td>
<td>3.6277e-03</td>
<td>1.4948e-05</td>
<td>5.7829e-08</td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.05</td>
<td>4.6884e-03</td>
<td>1.9939e-05</td>
<td>7.6887e-08</td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.09</td>
<td>5.6439e-03</td>
<td>2.4603e-05</td>
<td>9.4779e-08</td>
</tr>
</tbody>
</table>

Results in Kadalbajoo and Sharma [8]

<table>
<thead>
<tr>
<th>( \eta ) ↓</th>
<th>0.00</th>
<th>0.05</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon ) 0.00</td>
<td>4.1398e-03</td>
<td>1.7332e-05</td>
<td>6.6871e-08</td>
</tr>
<tr>
<td>( \varepsilon ) 0.05</td>
<td>4.6884e-03</td>
<td>1.9939e-05</td>
<td>7.6887e-08</td>
</tr>
<tr>
<td>( \varepsilon ) 0.09</td>
<td>5.1541e-03</td>
<td>2.2193e-05</td>
<td>8.5539e-08</td>
</tr>
</tbody>
</table>

Example 4. Consider the problem with right end boundary layer

\[ \varepsilon w''(t) - w'(t) - 2w(t - \delta) + w(t) = 0 \]

with \( w(0) = 1 \), \( -\delta \leq t \leq 0 \) and \( w(1) = -1 \), \( 1 \leq t \leq 1 + \eta \).

Table 4. Maximum errors in the solution of Example 4 for \( \varepsilon = 0.1 \) with different values of \( \delta \)

<table>
<thead>
<tr>
<th>( \delta ) ↓</th>
<th>( N \rightarrow 8 )</th>
<th>( N \rightarrow 32 )</th>
<th>( N \rightarrow 128 )</th>
<th>( N \rightarrow 512 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta ) ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.00</td>
<td>2.5312e-03</td>
<td>9.4444e-06</td>
<td>3.6657e-08</td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.05</td>
<td>1.7367e-03</td>
<td>6.3970e-06</td>
<td>2.4904e-08</td>
</tr>
<tr>
<td>( \delta ) = 0.5( \varepsilon )</td>
<td>0.09</td>
<td>1.2459e-03</td>
<td>4.5388e-06</td>
<td>1.7815e-08</td>
</tr>
</tbody>
</table>

Results in Kadalbajoo and Sharma [8]

<table>
<thead>
<tr>
<th>( \delta ) ↓</th>
<th>0.00</th>
<th>0.05</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta ) 0.00</td>
<td>0.07847490</td>
<td>0.04678972</td>
<td>0.01727912</td>
</tr>
<tr>
<td>( \eta ) 0.05</td>
<td>0.09222560</td>
<td>0.03828329</td>
<td>0.01487799</td>
</tr>
<tr>
<td>( \eta ) 0.09</td>
<td>0.10509460</td>
<td>0.03149275</td>
<td>0.01299340</td>
</tr>
</tbody>
</table>
Example 5. Consider the differential-difference equation with right end boundary layer:
\[ \varepsilon w''(t) - w'(t) + \omega(t) - 2\omega(t + \eta) = 0 \] with \( \omega(0) = 1, -\delta \leq t \leq 0 \) and \( \omega(1) = -1, 1 \leq t \leq 1 + \eta \).

Table 5. Maximum errors in the solution of Example 5 for \( \varepsilon = 0.1 \) with different values of \( \eta \)

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( N = 8 )</th>
<th>( N = 32 )</th>
<th>( N = 128 )</th>
<th>( N = 512 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>2.5312e-03</td>
<td>9.4444e-06</td>
<td>3.6657e-08</td>
<td>1.4317e-10</td>
</tr>
<tr>
<td>0.05</td>
<td>3.5676e-03</td>
<td>1.3428e-05</td>
<td>5.2345e-08</td>
<td>2.0442e-10</td>
</tr>
<tr>
<td>0.09</td>
<td>4.5639e-03</td>
<td>1.7380e-05</td>
<td>6.8290e-08</td>
<td>2.6661e-10</td>
</tr>
</tbody>
</table>

Results in Kadalbajoo and Sharma [8]

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>( \delta = 0.5 \varepsilon )</th>
<th>( \delta = 0.5 \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.07847490</td>
<td>0.04678972</td>
</tr>
<tr>
<td>0.05</td>
<td>0.06834579</td>
<td>0.05516436</td>
</tr>
<tr>
<td>0.09</td>
<td>0.08328237</td>
<td>0.06168267</td>
</tr>
</tbody>
</table>

Example 6. Consider the equation with right end boundary layer:
\[ \varepsilon w''(t) - w'(t) - 2\omega(t - \delta) + \omega(t) - 2\omega(t + \eta) = 0 \] with \( \omega(0) = 1, -\delta \leq t \leq 0 \) and \( \omega(1) = -1, 1 \leq t \leq 1 + \eta \).

Table 6. Maximum errors in Example 6 for \( \varepsilon = 0.1 \) with different values of \( \eta \) and \( \delta \)

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \eta = 0.5 \varepsilon )</th>
<th>( \eta = 0.5 \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>2.0526e-03</td>
<td>3.7809e-03</td>
</tr>
<tr>
<td>0.05</td>
<td>2.8263e-03</td>
<td>1.4605e-05</td>
</tr>
<tr>
<td>0.09</td>
<td>2.1939e-03</td>
<td>8.2661e-06</td>
</tr>
</tbody>
</table>

\( \delta = 0.5 \varepsilon \)

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \eta = 0.5 \varepsilon )</th>
<th>( \delta = 0.5 \varepsilon )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>3.7809e-03</td>
<td>2.0526e-03</td>
</tr>
<tr>
<td>0.05</td>
<td>2.8263e-03</td>
<td>1.4605e-05</td>
</tr>
<tr>
<td>0.09</td>
<td>2.1939e-03</td>
<td>8.2661e-06</td>
</tr>
</tbody>
</table>

Results of Kadalbajoo and Sharma [8]

5. Discussions and Conclusion

A difference scheme using non-polynomial spline and nonstandard finite differences is implemented to solve differential-difference equation having layer behaviour. Initially, the given problem is minimized by the Taylor’s expansion, to differential equation with layer structure. Then a tridiagonal difference scheme is constructed using the nonstandard finite differences in non-polynomial spline. The method is used with various examples of left layer and right layer, with distinct values of the delay parameter \( \delta \), advanced parameter \( \eta \) and the
perturbation $\varepsilon$. The outcomes of the computations were compared and tabulated. The effect of the delay and the advanced parameters have been examined via graphs on the problem solutions. When the SPDDE solution exhibits the layer on the left-end, the effect of delay or advanced parameters in the layer domain is observed to be negligible, whereas in the outer region it is significant. The variation of the advanced parameter influences the solution in the same way that the change in delay has an influence but reverse effect (see Figures 1-5). In layer region as well as external region, there is an impact when the SPDDEs show right-end layer on the region with respect to the delay or advanced variations. We also observed that the layer thickness decreases as the delay parameter increases while the advanced parameter increases the layer thickness (Figures 6-8). Results show that the proposed scheme is very well suited to the exact solution.

Figure 1. Numerical solution in Example 1 for $\varepsilon = 0.1$ with different values of $\delta$

Figure 2. Numerical solution in Example 2 for $\varepsilon = 0.1$ with different values of $\eta$
Figure 3. Numerical solution in Example 3 for $\varepsilon = 0.1$ with different values of $\delta$

Figure 4. Numerical solution in Example 3 for $\varepsilon = 0.1$ with different values of $\eta$

Figure 5. Numerical solution in Example 4 for $\varepsilon = 0.1$ with different values of $\delta$
Figure 6. Numerical solution in Example 5 for $\varepsilon = 0.1$ with different values of $\eta$.

Figure 7. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of $\eta$.

Figure 8. Numerical solution in Example 6 with $\varepsilon = 0.1$ and for different values of $\delta$. 

Communications in Mathematics and Applications, Vol. 10, No. 4, pp. 851–863, 2019
Competing Interests
The authors declare that they have no competing interests.

Authors’ Contributions
All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

