Characterization of Joined Graphs

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Abstract. The join of simple graphs $G_1$ and $G_2$, written by $G_1 \vee G_2$, is the graph obtained from the disjoint union between $G_1$ and $G_2$ by adding the edges \{xy : x \in V(G_1), y \in V(G_2)\}. We call a simple graph $G$ as a joined graph if there are $G_1$ and $G_2$ that $G = G_1 \vee G_2$. In this paper, we give conditions to determine that which graphs are joined graphs and use its properties to investigate the chromatic number of joined graphs.

1. Introduction and Preliminaries

In this paper, graphs must be simple graphs which can be trivial graphs but not empty graphs. We follow West [2] for terminologies and notations not defined here. Let $G_1$ and $G_2$ be any two graphs. The join of graphs $G_1$ and $G_2$, written by $G_1 \vee G_2$, is the graph obtained from the disjoint union between $G_1$ and $G_2$ by adding the edges \{xy : x \in V(G_1), y \in V(G_2)\).

We call a simple graph $G$ as a joined graph if there are $G_1$ and $G_2$ that $G = G_1 \vee G_2$. Clearly that $G_1$ and $G_2$ are subgraphs of $G_1 \vee G_2$. If a graph $G$ is a joined graph of $G_1$ and $G_2$, $G = G_1 \vee G_2$, we refer $G_1$ and $G_2$ as factors of $G$.

In generally, we may define $G_1 \vee G_2 \vee G_3$ as $G_1 \vee (G_2 \vee G_3)$. We note here that $G_1 \vee (G_2 \vee G_3) = G_1 \vee G_2 \vee G_3 = (G_1 \vee G_2) \vee G_3$ where $G_1$, $G_2$ and $G_3$ are graphs.

Figure 1. $K_4 \vee K_3 = K_2 \vee K_2 \vee K_2$

Theorem 1.1. Let $G_1$ and $G_2$ be graphs. If $H_1$ and $H_2$ are subgraphs of $G_1$ and $G_2$, respectively, then $H_1 \vee H_2 \subseteq G_1 \vee G_2$.

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Proof. Let \( G_1 \) and \( G_2 \) be graphs. Assume that \( H_1 \) and \( H_2 \) are subgraphs of \( G_1 \) and \( G_2 \), respectively. Clearly that \( V(H_1 \lor H_2) \subseteq V(G_1 \lor G_2) \). Next, let \( e \) be an edge in \( H_1 \lor H_2 \) with endpoints \( u \) and \( v \). If \( u, v \in V(H_1) \), then \( e \in E(H_1) \subseteq E(G_1) \subseteq E(G_1 \lor G_2) \). Similarly, if \( u, v \in V(H_2) \), then \( e \in E(H_2) \subseteq E(G_2) \subseteq E(G_1 \lor G_2) \). Suppose that \( u \in V(H_1) \) and \( v \in V(H_2) \). So \( e \in \{uv : u \in V(G_1), v \in V(G_2)\} \subseteq E(G_1 \lor G_2) \).

Therefore \( H_1 \lor H_2 \subseteq G_1 \lor G_2 \).

In [3], there are Theorems about property of joined graphs as follow.

**Theorem 1.2.** Any joined graphs are always connected.

**Theorem 1.3.** Any joined graphs are bipartite graphs or contain \( K_3 \).

By applying Theorem 1.2 and Theorem 1.3, we have necessary conditions to be a joined graph as Theorem 1.4.

**Theorem 1.4.** Let \( G \) be a graph. If \( G \) has properties that

(i) \( G \) is not connected or

(ii) \( G \) is not a bipartite graph and have no \( K_3 \) or

(iii) girth of \( G \) are not \( \infty, 3 \) or \( 4 \),

then \( G \) is not a joined graph.

Because \( K_n \) where \( n \in \mathbb{N} \) is not connected, so \( \overline{K_n} \) is not a joined graph. Since girth of \( C_{2n} \) where \( n \in \mathbb{N} \) and \( n > 2 \) is \( 2n \), we have that \( C_{2n} \) is not a joined graph. We can conclude that \( C_4 \) is the only one bipartite graph that is a joined graph where \( C_4 = \overline{K_2} \lor \overline{K_2} \).

We end this section by giving the Theorem about complement of graphs to use in the next section.

**Theorem 1.5.** Let \( G \) be a graph and let \( H \) be a spanning subgraph of \( G \). We have \( G \subseteq H \).

Proof. Let \( G \) be a graph and let \( H \) be a spanning subgraph of \( G \). Clearly that \( n(\overline{G}) = n(\overline{H}) \). Let \( e \) be an edge in \( \overline{G} \) with endpoints \( u \) and \( v \). Then \( u, v \in V(G) = V(H) \) and \( u \) is not adjacent to \( v \) in \( G \). Since \( H \) is a subgraph of \( G \), we have \( u \) is not adjacent to \( v \) in \( H \). So \( e \in E(H) \). Hence \( \overline{G} \subseteq H \). 

### 2. Necessary and Sufficient Conditions

We begin this section by giving the definition of operator \( + \) and a relation between \( + \) and \( \lor \). Let \( G_1 \) and \( G_2 \) be distinct two graphs. The sum of \( G_1 \) and \( G_2 \), denoted by \( G_1 \lor G_2 \), is the graph that \( V(G_1 \lor G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \lor G_2) = E(G_1) \cup E(G_2) \). Clearly that \( G_1, G_2 \subseteq G_1 \lor G_2 \subseteq G_1 \lor G_2 \).

**Theorem 2.1.** For any graphs \( G_1 \) and \( G_2 \), \( G_1 \lor G_2 \) is the only one bipartite graph that is a joined graph.
Proof. Let $G_1$ and $G_2$ be graphs. By the definition of the sum of a graph, we have $G_1 + G_2 \subseteq G_1 \lor G_2$. Next, let $e \in E(G_1 \lor G_2)$ with endpoints $u$ and $v$. So $u$ and $v$ are not adjacent in $G_1 \lor G_2$. Hence $u, v \in V(G_1)$ or $u, v \in V(G_2)$. Without loss of generality, we may assume that $u, v \in V(G_1)$. Since $G_1 \subseteq G_1 \lor G_2$, we have $u$ and $v$ are not adjacent in $G_1$. Thus $e \in G_1 \subseteq G_1 \lor G_2$. Therefore $G_1 \lor G_2 = G_1 + G_2$. \hfill \Box

In the previous section, we have only necessary conditions to be a joined graph. We next show the sufficient conditions.

Theorem 2.2. For any graph $G$, the following are equivalent (and characterize the joined graph).

(i) $G$ is a joined graph.

(ii) $G$ have a spanning complete bipartite as a subgraph.

(iii) $\overline{G}$ is a disconnected graph.

Proof. Let $G$ be a graph.

(i)→(ii) Assume that $G$ is a joined graph. Let $G_1$ and $G_2$ be graphs that $G = G_1 \lor G_2$. So $n(G_1) + n(G_2) = n(G)$. Let $G_i'$ be a graph obtained by deleting all edges in $G_i$ for all $i = 1, 2$. Then $G_i' \subseteq G_1$ and $G_i' \subseteq G_2$. By Theorem 1.1, we have $G_1' \lor G_2' \subseteq G_1 \lor G_2 = G$ and $n(G_1') + n(G_2') = n(G_1) + n(G_2) = n(G)$. Therefore $G$ have a spanning complete bipartite, $G_1' \lor G_2'$, as subgraph.

(ii)→(iii) Assume that $G$ have a spanning complete bipartite as a subgraphs, called $H \cong K_{m,n}$ where $m + n = n(G)$. By Theorem 1.5, we have $G \subseteq H \cong \overline{K_{m,n}}$. Clearly that $H \cong \overline{K_{m,n}}$ is disconnected. Therefore $\overline{G}$ is a disconnected graph.

(iii)→(i) We assume that $\overline{G}$ is a disconnected graph. Let $H$ be a connected induce subgraph of $\overline{G}$. So $\overline{G} = H + \overline{G} \not\subseteq H$. By Theorem 2.1, we have that $G = H \lor \overline{G} \not\subseteq H$. Hence $G = H \lor \overline{G} \not\subseteq H$. Therefore $G$ is a joined graph. \hfill \Box

Corollary 2.3. Let $G$ be a graph. If $n(G) + e(G) > \frac{n(n-1)}{2} + 1$, then $G$ is a joined graph.

Proof. Let $G$ be a graph. We assume that $n(G) + e(G) > \frac{n(n-1)}{2} + 1$. We know that $e(\overline{G}) = \frac{n(n-1)}{2} - e(G)$. So $e(\overline{G}) = \frac{n(n-1)}{2} - e(G) < n(G) - 1 = n(\overline{G}) - 1$. Hence $\overline{G}$ is not a connected graph. Therefore $G$ is a joined graph by Theorem 2.2. \hfill \Box

The converse of Corollary 2.3 is not true. For example, $K_2$ is a joined graph but $n(K_2) + e(K_2) = 3 = \frac{2(1)}{2} + 1$.

Because the complement of the Petersen graph is a connected graph, so we can conclude that the Petersen graph is not a joined graph (see Figure 2).

3. Joined Graphs and It's Chromatic Number

To find the chromatic number of a graph, we use clique number to be a lower bound and find a proper coloring to get a upper bound. Sometime, it's not easy to
find a clique number for a graph with many edges as Example 3.2, but if we know that a graph is a joined graph, we can find the chromatic number of that graph easier by the next Theorem.

**Theorem 3.1.** Let $G_1$ and $G_2$ be graphs. Then $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.

**Proof.** Let $G_1$ and $G_2$ be graphs. Let $f$ and $g$ be proper colorings of $G_1$ and $G_2$, respectively. Define $\alpha : V(G_1) \cup V(G_2) \rightarrow \{1, 2, \ldots, \chi(G_1) + \chi(G_2)\}$ by for all $v \in G_1 \cup V(G_2)$

$$\alpha(v) = \begin{cases} f(v), & \text{if } v \in V(G_1), \\ \chi(G_1) + g(v), & \text{if } v \in V(G_2). \end{cases}$$

It is easy to see that $\alpha$ is proper. So $\chi(G_1 \vee G_2) \leq \chi(G_1) + \chi(G_2)$. Suppose that $\chi(G_1 \vee G_2) < \chi(G_1) + \chi(G_2)$. There exist $u \in V(G_1)$ and $v \in V(G_2)$ such that $\alpha(u) = \alpha(v)$. So $u$ and $v$ are not adjacent in $G_1 \vee G_2$. This contradicts to the definition of the join graphs. Hence $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$. \qed

We know that Wheel with $n$ vertices, denote by $W_n$, is a joined graph where $W_n = C_{n-1} \vee K_1$. Since $\chi(C_{n-1}) = 2$ or 3, then by Theorem 3.1 we have that

$$\chi(W_n) = \begin{cases} 3, & \text{if } n \text{ is an even integer}, \\ 4, & \text{if } n \text{ is an odd integer}. \end{cases}$$

**Example 3.2.** Let $G$ be a graph as Figure 3. We can see that $\overline{G}$ is disconnected. By Theorem 2.2, we have $G$ is a joined graph.

Next, we find factors of $G$. By following the proof of Theorem 2.2, we get that factors of $G$ are the complement of it’s component. So we have factors of $G$ as Figure 4. So $G = H_1 \vee H_2 \vee H_3$ where $H_1$, $H_2$ and $H_3$ are factors of $G$. Hence $\chi(G) = \chi(H_1) + \chi(H_2) + \chi(H_3) = 2 + 3 + 2 = 7$.

We conclude the results here that a jointed graph is a graph that its complement is disconnected graph and chromatic number of jointed graph is equal to the sum of chromatic number of their factors.
Characterization of Joined Graphs

Figure 3. A graph $G$ and its complement

Figure 4. Factors of $G$

References


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