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Research Article

Bounds for Toader Mean in Terms of Arithmetic and Second Seiffert Means

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Abstract. In the article, we prove that the double inequalities

$$\alpha_1 T(a,b)+(1-\alpha_1)A(a,b)<TD(a,b)<\beta_1 T(a,b)+(1-\beta_1)A(a,b),$$
$$T^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b)<TD(a,b)<T^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 3/4$, $\beta_1\geq 1$, $\alpha_2\leq 3/4$ and $\beta_2\geq 1$, where $A(a,b)$, $TD(a,b)$ and $T(a,b)$ are the arithmetic, Toader and second Seiffert means of a and b, respectively.

Keywords. Toader mean; Second Seiffert mean; Arithmetic mean

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1. Introduction

"A real-valued function $M:(0,\infty)\times(0,\infty)\to(0,\infty)$" is said to be a symmetric and homogeneous bivariate mean of degree one if $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$, $M(x,y)=M(y,x)$ and $M(\lambda x,\lambda y)=\lambda M(x,y)$ for all $x,y\in(0,\infty)$ and $\lambda>0$. The study of bivariate mean has a history of several hundred years, it has widely applications in mathematics, physics, and many other many other natural and human social sciences. Recently, the relations and comparison for different bivariate means have attracted the attention of many researchers.

Let $p\in\mathbb{R}$, $r\in(0,1)$ and $a,b>0$ with $a\neq b$. Then the complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ [1–32] of the first and second kinds, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, contra-harmonic mean $C(a,b)$, pth Lehmer mean $L_p(a,b)$ [33], Toader mean $TD(a,b)$ [34–37], pth power mean $M_p(a,b)$ [38–43] and second Seiffert mean $T(a,b)$ [44] of a and b are given by

$$\begin{aligned}
\mathscr{K}(r) &= \int_0^{\pi/2} \left[1-r^2\sin^2(t)\right]^{-1/2} dt,\\
\mathscr{E}(r) &= \int_0^{\pi/2} \left[1-r^2\sin^2(t)\right]^{1/2} dt,\\
A(a,b) &= \frac{a+b}{2},\\
Q(a,b) &= \sqrt{\frac{a^2+b^2}{2}},\\
C(a,b) &= \frac{a^2+b^2}{a+b},\\
L_p(a,b) &= \frac{a^{p+1}+b^{p+1}}{a^p+b^p},\\
TD(a,b) &= \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2(t)+b^2\sin^2(t)}dt \qquad (1.1)\\
&= \begin{cases} \frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), & a>b,\\ \frac{2b}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right), & a<b, \end{cases}\\
M_p(a,b) &= \left(\frac{a^p+b^p}{2}\right)^{1/p}
\end{aligned}$$

and

$$T(a,b) = \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}, \qquad (1.2)$$

respectively.

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b>0$ with $a \neq b$ and the inequalities

$$A(a,b) = M_1(a,b) < T(a,b) < Q(a,b) = M_2(a,b) < C(a,b) \qquad (1.3)$$

hold for all $a,b>0$ with $a \neq b$.

Recently, the bounds for the second Seiffert mean $T(a,b)$ and Toader mean $TD(a,b)$ have attracted the attention of many researchers.

Seiffert [45] established that

$$M_1(a,b) < T(a,b) < M_2(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [46, 47], the authors proved that the double inequalities

$$\begin{aligned}
&L_0(a,b) < T(a,b) < L_{1/3}(a,b),\\
&M_{\log 2/\log(\pi/2)}(a,b) < T(a,b) < M_{5/3}(a,b) \qquad (1.4)
\end{aligned}$$

hold all $a,b>0$ with $a \neq b$.

Chu and Wang [48] proved that the double inequality

$$L_p(a,b) < TD(a,b) < L_q(a,b)$$

holds all $a,b>0$ with $a \neq b$ if and only if $p \leq 0$ and $q \geq 1/4$.

In [49–52], the authors proved that the two-sided inequalities

$$\lambda_1 Q(a,b)+(1-\lambda_1)A(a,b) < TD(a,b) < \mu_1 Q(a,b)+(1-\mu_1)A(a,b),$$
$$Q^{\lambda_2}(a,b)A^{1-\lambda_2}(a,b) < TD(a,b) < Q^{\mu_2}(a,b)A^{1-\mu_2}(a,b),$$
$$\lambda_3 C(a,b)+(1-\lambda_3)A(a,b) < TD(a,b) < \mu_3 C(a,b)+(1-\mu_3)A(a,b),$$
$$\frac{\lambda_4}{A(a,b)}+\frac{1-\lambda_4}{C(a,b)} < \frac{1}{TD(a,b)} < \frac{\mu_4}{A(a,b)}+\frac{1-\mu_4}{C(a,b)},$$
$$M_{\lambda_5}(a,b) < TD(a,b) < M_{\mu_5}(a,b) \tag{1.5}$$

are valid for all $a,b>0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/2$, $\mu_1 \geq (4-\pi)/[(\sqrt{2}-1)\pi] = 0.6597\cdots$, $\lambda_2 \leq 1/2$, $\mu_2 \geq 4-2\log\pi/\log 2 = 0.6970\cdots$, $\lambda_3 \leq 1/4$, $\mu_3 \geq 4/\pi - 1 = 0.2732\cdots$, $\lambda_4 \leq \pi/2-1 = 0.5707\cdots$, $\mu_4 \geq 3/4$, $\lambda_5 \leq 3/2$ and $\mu_5 \geq \log 2/\log(\pi/2) = 1.5349\cdots$.

From (1.3)-(1.5) we clearly see that the double inequality

$$A(a,b) < TD(a,b) < T(a,b) \tag{1.6}$$

holds for all $a,b>0$ with $a \neq b$.

Motivated by inequality (1.6), the main purpose of the article is to present the best possible parameters α_1, α_2, β_1 and β_2 such that the double inequalities

$$\alpha_1 T(a,b)+(1-\alpha_1)A(a,b) < TD(a,b) < \beta_1 T(a,b)+(1-\beta_1)A(a,b),$$
$$T^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < TD(a,b) < T^{\beta_2}(a,b)A^{1-\beta_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$.

2. Lemmas

In order to prove our main results, we need several formulas and lemmas which we present in this section.

The following formulas for $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be found in the literature [53]:

$$\frac{d\mathscr{K}(r)}{dr} = \frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{rr'^2},$$
$$\frac{d\mathscr{E}(r)}{dr} = \frac{\mathscr{E}(r)-\mathscr{K}(r)}{r},$$
$$\frac{d[\mathscr{E}(r)-r'^2\mathscr{K}(r)]}{dr} = r\mathscr{K}(r),$$
$$\frac{d[\mathscr{K}(r)-\mathscr{E}(r)]}{dr} = \frac{r\mathscr{E}(r)}{r'^2},$$
$$\mathscr{K}(0^+) = \mathscr{E}(0^+) = \frac{\pi}{2},$$
$$\mathscr{K}(1^-) = \infty,$$
$$\mathscr{E}(1^-) = 1,$$

where and in what follows $r' = \sqrt{1-r^2}$ for $r \in (0,1)$.

Lemma 2.1 (See [53, Theorem 1.25])**.** *Let* $-\infty < a < b < \infty$, $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},$$

$$\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *The following statements are true:*

(i) *The function* $r \mapsto \left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$*;*

(ii) *The function* $r \mapsto [\mathscr{K}(r) - \mathscr{E}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, \infty)$*;*

(iii) *The function* $r \mapsto \phi(r) = \left[2\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]/\sqrt{1+r^2}$ *is strictly decreasing from* $(0,1)$ *onto* $(\sqrt{2}, \pi/2)$*;*

(iv) *The function* $r \mapsto \varphi(r) = r(1+r^2)[\arctan(r)]^2/\left[(1+r^2)\arctan(r) - r\right]$ *is strictly increasing from* $(0,1)$ *onto* $(3/2, \pi^2/[4(\pi-2)])$*;*

(v) *The function* $r \mapsto \psi(r) = r^2\sqrt{1+r^2}\arctan(r)/\left[(1+r^2)\arctan(r) - r\right]$ *is strictly increasing from* $(0,1)$ *onto* $(3/2, \sqrt{2}\pi/[2(\pi-2)])$.

Proof. Parts (i) and (ii) can be found in [53, Theorem 3.21(1) and Exercise 3.43(11)]. For part (iii), it is not difficult to verify that

$$\phi(0^+) = \frac{\pi}{2}, \quad \phi(1^-) = \sqrt{2}. \tag{2.1}$$

Making use of part (ii) and differentiating $\phi(r)$ lead to

$$\phi'(r) = -\frac{rr'^2}{(1+r^2)^{3/2}}\left[\frac{\mathscr{K}(r) - \mathscr{E}(r)}{r^2}\right] < 0 \tag{2.2}$$

for $r \in (0,1)$.

Therefore, part (iii) follows from (2.1) and (2.2).

For part (iv), let $t = \arctan(r) \in (0, \pi/4)$, then simple computations lead to

$$\varphi(r) = \frac{t^2\sin(t)}{\cos(t)[t - \sin(t)\cos(t)]} = \frac{\sin(t)/[t\cos(t)]}{[t-\sin(t)\cos(t)]/t^3}. \tag{2.3}$$

It is easy to verify that the function $t \mapsto \sin(t)/[t\cos(t)]$ is strict increasing and positive on $(0, \pi/4)$ and the function $t \mapsto [t - \sin(t)\cos(t)]/t^3$ is strictly decreasing and positive on $(0, \pi/4)$. Then from (2.3) we know that the function $\varphi(r)$ is strictly increasing on $(0,1)$.

Note that

$$\varphi(0^+) = \frac{3}{2}, \quad \varphi(1^-) = \frac{\pi^2}{4(\pi-2)} = 2.1613\cdots. \tag{2.4}$$

Therefore, part (iv) follows from (2.4) and the monotonicity of $\varphi(r)$.

For part (v), let $t = \arctan(r) \in (0, \pi/4)$, then $\psi(r)$ can be rewritten as

$$\psi(r) = \frac{t\sin^2(t)}{\cos(t)[t - \sin(t)\cos(t)]} = \frac{\sin^2(t)/[t^2\cos(t)]}{[t-\sin(t)\cos(t)]/t^3}. \tag{2.5}$$

It is not difficult to verify that the function $t \mapsto \sin^2(t)/[t^2\cos(t)]$ is strictly increasing and positive on $(0,\pi/4)$ and the function $t \mapsto [t-\sin(t)\cos(t)]/t^3$ is strictly decreasing and positive on $(0,\pi/4)$. Then (2.5) leads to the conclusion that $\psi(r)$ is strictly increasing on $(0,1)$.

Note that

$$\psi(0^+) = \frac{3}{2}, \quad \psi(1^-) = \frac{\sqrt{2}\pi}{2(\pi-2)} = 1.9459\cdots. \tag{2.6}$$

Therefore, part (v) follows from (2.6) and the monotonicity of $\psi(r)$. □

3. Main Results

Theorem 3.1. *The double inequality*

$$\alpha_1 T(a,b) + (1-\alpha_1)A(a,b) < TD(a,b) < \beta_1 T(a,b) + (1-\beta_1)A(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \le 3/4$ and $\beta_1 \ge 1$.

Proof. Since $A(a,b)$, $T(a,b)$ and $TD(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0,1)$. Then from (1.1) and (1.2) we get

$$TD(a,b) = \frac{2}{\pi}A(a,b)\left[2\mathscr{E}(r) - r'^2\mathscr{K}(r)\right], \tag{3.1}$$

$$T(a,b) = A(a,b)\frac{r}{\arctan(r)}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{TD(a,b)-A(a,b)}{T(a,b)-A(a,b)} = \frac{\frac{2}{\pi}\left[2\mathscr{E}(r)-r'^2\mathscr{K}(r)\right]-1}{\frac{r}{\arctan(r)}-1} := F(r). \tag{3.3}$$

Let $f_1(r) = 2\left[2\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]/\pi - 1$ and $g_1(r) = r/\arctan(r) - 1$. Then elaborated computations lead to

$$f_1(0^+) = g_1(0^+) = 0, \quad F(r) = \frac{f_1(r)}{g_1(r)}, \tag{3.4}$$

$$\frac{f_1'(r)}{g_1'(r)} = \frac{2}{\pi}\left[\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r^2}\right]\varphi(r), \tag{3.5}$$

where $\varphi(r)$ is defined as in Lemma 2.2(iv).

It follows from Lemma 2.2(i) and (iv) together with (3.5) that $f_1'(r)/g_1'(r)$ is strictly increasing on $(0,1)$. Then Lemma 2.1 and (3.4) lead to the conclusion that $F(r)$ is strictly increasing on $(0,1)$. Moreover,

$$F(0^+) = \frac{3}{4}, \quad F(1^-) = 1. \tag{3.6}$$

Therefore, Theorem 3.1 follows from (3.3) and (3.6) together with the monotonicity of $F(r)$. □

Theorem 3.2. *The double inequality*

$$T^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < TD(a,b) < T^{\beta_2}(a,b)A^{1-\beta_2}(a,b) \tag{3.7}$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha_2 \le 3/4$ and $\beta_2 \ge 1$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0,1)$. Then from (1.1) and (1.2) we clearly see that inequality (3.7) is equivalent to

$$\alpha_2 < \frac{\log[TD(a,b)] - \log[A(a,b)]}{\log[T(a,b)] - \log[A(a,b)]} = \frac{\log\left[\frac{2}{\pi}\left(2\mathscr{E}(r) - r'^2\mathscr{K}(r)\right)\right]}{\log[\frac{r}{\arctan(r)}]} := G(r) < \beta_2. \tag{3.8}$$

Let $f_2(r) = \log\left[2\left(2\mathscr{E}(r) - r'^2\mathscr{K}(r)\right)/\pi\right]$ and $g_2(r) = \log[r/\arctan(r)]$. Then elaborated computations lead to

$$f_2(0^+) = g_2(0^+) = 0, \quad G(r) = \frac{f_2(r)}{g_2(r)}, \tag{3.9}$$

$$\begin{aligned}\frac{f_2'(r)}{g_2'(r)} &= \frac{(1+r^2)[\arctan(r)][\mathscr{E}(r) - r'^2\mathscr{K}(r)]}{[(1+r^2)\arctan(r) - r][2\mathscr{E}(r) - r'^2\mathscr{K}(r)]} \\ &= \frac{[\mathscr{E}(r) - r'^2\mathscr{K}(r)]/r^2}{[2\mathscr{E}(r) - r'^2\mathscr{K}(r)]/\sqrt{1+r^2}}\psi(r),\end{aligned} \tag{3.10}$$

where $\psi(r)$ is defined as in Lemma 2.2(v).

It follows from Lemma 2.2(i), (iii) and (v) together with (3.10) that $f_2'(r)/g_2'(r)$ is strictly increasing on $(0,1)$. Then Lemma 2.1 and (3.9) lead to the conclusion that $G(r)$ is strictly increasing on $(0,1)$. Moreover,

$$G(0^+) = \frac{3}{4}, \quad G(1^-) = 1. \tag{3.11}$$

Therefore, Theorem 3.2 follows from (3.8) and (3.11) together with the monotonicity of $G(r)$. □

Let $a = 1$ and $b = r'$. Then (1.1), (1.2) and Theorems 3.1 lead to Corollary 3.3 immediately.

Corollary 3.3. *The double inequality*

$$\frac{\pi}{16}\left[\frac{3(1-r')}{\arctan\left(\frac{1-r'}{1+r'}\right)} + (1+r')\right] < \mathscr{E}(r) < \frac{\pi}{4}\left[\frac{1-r'}{\arctan\left(\frac{1-r'}{1+r'}\right)}\right]$$

holds for all $r \in (0,1)$.

4. Conclusion

In the article, we find the best possible upper and lower bounds for the Toader mean

$$TD(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2(t) + b^2\sin^2(t)}dt$$

in terms of the convex combination of the arithmetic mean $A(a,b) = (a+b)/2$ and second Seiffert mean $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, and provide new bounds for the complete elliptic of the second kind

$$\mathscr{E}(r) = \int_0^{\pi/2}\left[1 - r^2\sin^2(t)\right]^{1/2}dt.$$

The given idea is novel and interesting, it may stimulate further research in the theory of bivariate means and special functions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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