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IF-dimension of Modules

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Abstract. In this paper, we introduce a dimension, called IF-dimension, for modules. With this new dimension we gives a new characterization of IFD$(-)$ dimension of rings; see [4]. The relations between the IF dimension and other dimensions are discussed.

1. Introduction

Throughout this paper, R denotes a non-trivial associative ring and all modules — if not specified otherwise — are left and unitary.

Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. We use also $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ to denote, respectively, the classical global and weak dimension of R.
In [4], the authors introduced a new global dimension over a ring R, $r.\mathrm{IFD}(R)$, defined as

$$r.\mathrm{IFD}(R) = \sup\{\mathrm{fd}_R(I) \mid I \text{ is a right injective } R\text{-module}\}.$$

For such dimension, Ding and Chen gave a various characterizations (see [4]).

For an R-module M, let $\text{IF-d}_R(M)$ denote the smallest integer $n \geq 0$ such that $\mathrm{Tor}_R^i(I,M) = 0$ for all $i > n$ and all right injective R-module I and call $\text{IF-d}_R(M)$ the IF-dimension of M. If no such n exists, set $\text{IF-d}_R(M) = \infty$. If $\text{IF-d}_R(M) = 0$, then M will be called IF-module.

In this paper, we investigate the IF-dimension and we give the following new characterization $\mathrm{IFD}(R) = \sup\{\text{IF-d}(R/I) \mid I \text{ is a left ideal of } R\}$.

Recall that a subclass $\mathfrak{X}$ of R-modules is called projectively resolving if $\mathfrak{X}$ contain all projective modules, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathfrak{X}$ the conditions $X' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent (see [9, Defnition 1.1]). Also an R-module is called FP-injective if, for all finitely presented N, $\mathrm{Ext}_R^1(N,M) = 0$. The FP-injective dimension of M, denoted by $\text{FP-id}_R(M)$,

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is defined to be the least positive integer n such that $\mathrm{Ext}_R^{n+1}(N,M)=0$ for all finitely presented R-module N. If no such n exists, set $\mathrm{FP}\text{-}\mathrm{id}_R(M)=\infty$ (see [4]). The character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is denoted by M^+ and $E(M)$ denote the envelope injective of the R-module M.

Recall that a short exact sequence of right R-modules

$$0\to A\to B\to C\to 0$$

is pure exact in the sense of Cohn ([2]) if for all (or equivalently for all finitely presented) left R-module K we have the exactness of

$$0\to A\otimes_R K\to B\otimes_R B\to C\otimes_R K\to 0.$$

In this case we say also that A is a pure submodule of B. For instance, a right R-module is FP-injective if, and only if, it is absolutely pure (i.e; it is a pure submodule of every module containing it as a submodule); see [5, 8].

Lemma 1.1 ([3, Theorem 3.1]). *For any ring R, a sequence of right R-modules*

$$0\to A\to B\to C\to 0$$

is pure exact if, and only if, the sequence of character modules

$$0\to C^+\to B^+\to A^+\to 0$$

is split exact.

2. **Main results**

We start with the following proposition:

Proposition 2.1. *The class of all* IF *module is projectively resolving, closed under arbitrary direct sums, and under direct summands.*

Proof. Clearly, every flat (and then every projective) module is IF-module. Now, let $0\to M\to M'\to M''\to 0$ be an exact sequence of R-modules. If M'' is an IF-module. It is trivial that M' is IF-module if, and only if, M is IF-module by using the functor $\mathrm{Tor}_R^i(E,-)$ with E right injective and the long suite exact sequence of homology. Since $\mathrm{Tor}_R^i(E,-)$ commutes with direct sums the other assertions follows immediately. □

We have the following characterization of IF-dimension:

Proposition 2.2. *For any R-module M and any positive integer n, the following are equivalent:*

(1) $\mathrm{IF}\text{-}\mathrm{d}_R(M)\leq n$.
(2) $\mathrm{Tor}_R^i(E,M)=0$ *for all* $i>n$ *and all right injective R-module E.*
(3) $\mathrm{Tor}_R^i(E,M)=0$ *for all* $i>n$ *and all right* FP*-injective R-module E.*
(4) $\mathrm{Tor}_R^i(E,M)=0$ *for all* $i>n$ *and all right R-module E with finite* FP*-injective dimension.*
(5) $\mathrm{Tor}_R^i(F^+,M)=0$ *for all* $i>n$ *and all R-module F with finite flat dimension.*

(6) *If* $0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$ *is exact with* $F_0, .., F_{n-1}$ *are* IF*-modules, then* F_n *is also an* IF*-module.*

Proof. The proof of (1)$\Leftrightarrow$(2)$\Leftrightarrow$(6) is standard homological algebra fare.

(2)$\Rightarrow$(3). Consider an arbitrary right FP-injective R-module K. From [3, Theorem 3.1], the short pure exact sequence of right R-modules $0 \to K \to E(K) \to E(K)/K \to 0$ where $E(K)$ is the injective envelope of K induce the split exact sequence $0 \to (E(K)/K)^+ \to E(K)^+ \to K^+ \to 0$. Thus, K^+ is a direct summand of $E(K)^+$. On the other hand, by adjointness, $\mathrm{Ext}_R^i(M, E(K)^+) \cong (\mathrm{Tor}_R^i(E(K), M))^+ = 0$. Thus, $(\mathrm{Tor}_R^i(K, M))^+ = \mathrm{Ext}_R^i(M, K^+) = 0$, and so $\mathrm{Tor}_R^i(K, M)$ vanishes as desired.

(3)$\Rightarrow$(4). The proof will be by induction on $m = \text{FP-id}_R(K)$. The induction start is clear, from (3). If $m > 0$, pick a short exact sequence $0 \to K \to E(K) \to E(K)/K \to 0$ where $E(K)$ is the envelope injective of K. Then, by a standard homological algebra we can see that $\text{FP-id}_R(E(K)/K) = m - 1$. So, we have, for all $i > n$, the exact sequence

$$\mathrm{Tor}_R^{i+1}(E(K)/K, M) \to \mathrm{Tor}_R^i(K, M) \to \mathrm{Tor}_R^i(E(K), M).$$

Clearly, $\mathrm{Tor}_R^{i+1}(E(K)/K, M)$ and $\mathrm{Tor}_R^i(E(K), M)$ vanishes by the induction hypothesis conditions and since $E(K)$ is injective (and so FP-injective), respectively. Thus, $\mathrm{Tor}_R^i(K, M) = 0$, as desired.

(4)$\Rightarrow$(2). Obvious since every injective module is FP-injective.

(4)$\Rightarrow$(5). Set $m = \mathrm{fd}_R(F) < \infty$. Then, by [6, Theorem 2.1], $m = \mathrm{id}_R(F^+) = \text{FP-id}(F^+)$. Thus, by hypothesis, $\mathrm{Tor}_R^i(F^+, M) = 0$ for all $i > n$.

(5)$\Rightarrow$(2). Let I be a right injective R-module. There exist a flat R-module F such that $F \to I^+ \to 0$ is exact. Then, $0 \to I^{++} \to F^+$ is exact. But $0 \to I \to I^{++}$ is exact (by [7, Proposition 3.52]). Then, $0 \to I \to F^+$ is exact and then I is a direct summand of F^+. So, $\mathrm{Tor}_R^i(I, M) = 0$ as a direct summand of $\mathrm{Tor}_R^i(F^+, M)$. $\square$

Clearly every flat module is IF-module but the inverse implication is not true in the general case as shown by the following example.

Example 2.3. Consider the local quasi-Frobenius ring $R = k[X]/(X^2)$ where k is a field, and denote by $\overline{X}$ the the residue class in R of X. Then, $\overline{X}$ is an IF R-module but not flat.

Proof. Since R is quasi-Frobenius, every right injective module E is projective (and so flat). Then, $\mathrm{Tor}_R^i(E, \overline{X}) = 0$ for all $i > 0$. Thus, $\overline{X}$ is an IF R-module. Now, if we suppose that $\overline{X}$ is flat. Then, it is projective since it is finitely presented. But R is also local. So, $\overline{X}$ must be a free which is absurd by the fact that $\overline{X}^2 = 0$. So, we conclude that $\overline{X}$ is not flat, as desired. $\square$

Proposition 2.4. *Let R be any ring and n a positive integer. The following are equivalents:*

(1) $r.\mathrm{IFD}(R) \leq n$.
(2) $\text{IF-d}_R(M) \leq n$ *for every R-module M.*
(3) $\text{IF-d}_R(M) \leq n$ *for every finitely generated R-module M.*
(4) $\text{IF-d}_R(R/I) \leq n$ *for every ideal I.*

Proof. (1)⇒(2). Let M be an arbitrary R-module and E an injective right R-module. Since $r.\mathrm{IFD}(R) \leq n$, $\mathrm{fd}_R(E) \leq n$. Then, $\mathrm{Tor}_R^i(E,M) = 0$ for all $i > n$. Thus, $\text{IF-d}(M) \leq n$, as desired.

(2)⇒(3)⇒(4). Obvious.

(4)⇒(1). Let E be a right injective R-module. Since $\text{IF-d}_R(R/I) \leq n$ for every ideal I, we have $\mathrm{Tor}_R^i(E,R/I) = 0$ for all $i > n$. Thus, from [11, Lemma 9.18], $\mathrm{fd}_R(E) \leq n$. □

Corollary 2.5 (Theorem 3.5, [4])**.** *For any ring R and any positive integer n, the following are equivalent:*

(1) $\mathrm{IFD}(R) \leq n$.
(2) $\mathrm{fd}_R(E) \leq n$ *for every* FP*-injective right module E.*
(3) $\mathrm{fd}_R(E)$ *for every right R-module with finite* FP*-injective dimension.*
(4) $\mathrm{fd}_R(F^+) \leq n$ *for every R-module with finite flat dimension.*

Proof. Follows immediately from Propositions 2.2 and 2.4. □

For a left coherent ring and a finitely presented modules we get:

Proposition 2.6. *Let R be a left coherent ring and M a finitely presented R-module. Then, the following are equivalent for any positive integer n.*

(1) $r.\text{IF-d}_R(M) \leq n$
(2) $\mathrm{Ext}_R^i(M,R) = 0$ *for all $i > n$.*
(3) $\mathrm{Tor}_R^i(Q,M) = 0$ *for all $i > n$ where Q is any injective cogenerator in the class of the right R-module.*
(4) $\mathrm{Tor}_R^i(E(S),M) = 0$ *for all $i > n$ and for all right simple module S.*

Proof. (1)⇒(3). Trivial.

(3)⇒(1). Let I be a right injective R-module. Since Q is an injective cogenerator in the class of the right R-module, I can be embedded in some product of Q; $\prod Q$. Thus, I is a direct summand of $\prod Q$. Using [10, Lemma 1], $\mathrm{Tor}_R^i\left(\prod Q, M\right) = \prod \mathrm{Tor}_R^i(Q,M) = 0$ for all $i > n$. Thus, $\mathrm{Tor}_R^i(I,M) = 0$ for all $i > n$, as desired.

(1)⇒(2). Since R is left coherent, $(\mathrm{Ext}_R^i(M,R))^+ = \mathrm{Tor}_R^i(R^+,M) = 0$ for all $i > 0$. Thus, $\mathrm{Ext}_R^i(M,R) = 0$.

(2)⇒(1). Follows from the implication (3)⇒(1) since R^+ is an injective cogenerator in the class of right R-module and since $(\mathrm{Ext}^i_R(M,R))^+ = \mathrm{Tor}^i_R(R^+, M)$ (recall that R is left coherent).

(1)⇒(4). Trivial.

(4)⇒(1). Using [1, Proposition 18.15], $Q = \prod E(S)$ (direct product of the envelope injective of simple right modules) is an injective cogenerator in the class of right R-modules. On the other hand, from [10, Lemma 1], $\mathrm{Tor}^i_R\Big(\prod E(S), M\Big) = \prod \mathrm{Tor}^i_R(E(S), M\Big) = 0$. Thus, this implication follows from (3)⇒(1). □

Proposition 2.7. *For any R-module M, we have* $\text{IF-d}_R(M) \leq \mathrm{fd}_R(M)$ *with equality if* $\mathrm{fd}_R(M)$ *is finite.*

Proof. The first inequality follows from the fact that every flat module is an IF-module. Now, set $\mathrm{fd}_R(M) = n < \infty$ and suppose that $\text{IF-d}_R(M) = m < n$. Thus, there is a right R-module N such that $\mathrm{Tor}^n_R(N, M) \neq 0$. Clearly, N can not be injective. Thus, we can consider the exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$ where $E(N)$ is the envelope injective of N. Then, we have

$$0 = \mathrm{Tor}^{n+1}_R(E(N), M) \to \mathrm{Tor}^{n+1}_R(E(N)/N, M) \to \mathrm{Tor}^n_R(N, M) \to \mathrm{Tor}^n_R(E(N), M) = 0$$

Therefore, $\mathrm{Tor}^{n+1}_R(E(N)/N, M) \neq 0$. Absurd, since $\mathrm{fd}_R(M) = n$. □

The proof of the next proposition is standard homological algebra.

Proposition 2.8. *Let $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If two of* $\text{IF-d}_R(A)$, $\text{IF-d}_R(B)$, *and* $\text{IF-d}_R(C)$ *are finite, so is the third. Moreover,*

(1) $\text{IF-d}_R(B) \leq \sup\{\text{IF-d}_R(A), \text{IF-d}_R(C)\}$.
(2) $\text{IF-d}_R(A) \leq \sup\{\text{IF-d}_R(B), \text{IF-d}_R(C) - 1\}$.
(3) $\text{IF-d}_R(C) \leq \sup\{\text{IF-d}_R(B), \text{IF-d}_R(A) + 1\}$.

The next corollary is an immediate consequence of Proposition 2.8

Corollary 2.9. *Let R be a ring, $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If B is an* IF*-module and* $\text{IF-d}(C) > 0$ *then,* $\text{IF-d}(C) = \text{IF-d}(A) + 1$.

Proposition 2.10 (Flat base change). *Consider a flat homomorphism of commutative rings $R \to S$ (that is, S is flat as an R-module). Then for any R-module M we have the inequality,* $\text{IF-d}_S(M \otimes_R S) \leq \text{IF-d}_R(M)$.

Proof. Suppose that $\text{IF-d}_R(M) \leq n$. If I is an injective S-module, then since S is R-flat, I is also an injective R-module. On the other hand, from [11, Theorem 11.64], we have $\mathrm{Tor}^i_S(I, M \otimes_R S) \cong \mathrm{Tor}^i_R(I, M) = 0$ for all $i > n$. Thus, $\text{IF-d}_S(M \otimes_R S) \leq n$, as desired. □

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