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A Note on Sharp Embedding Theorems for Holomorphic Classes Based on Lorentz Spaces on A Unit Circle

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Abstract. We provide new sharp embedding theorems for holomorphic classes in the unit disk based on Lorentz classes on the unit circle.

1. Introduction

Let μ be a positive Borel measure on a unit disk $\mathbb{D} = \{z : |z| < 1\}$, $T = \{z : |z| = 1\}$ be as usual unit circle. Let also $L^{p,q}(T)$, $0 < p, q \le \infty$ be the Lorentz space on T (see [3]), and $dm(z)$ the normalized Lebesgue measure on T and $dm_2(z)$ be normalized Lebesgue measure on $\mathbb{D}$. Let also Δ_{jk} be the dyadic decomposition of $\mathbb{D}$, that is (see [1])

$$\Delta_{jk} = \{z = a\xi : \xi \in I_{j,k}, 2^{-k-1} \le 1 - a < 2^{-k}\}$$

and

$$I_{jk} = \{\xi \in T : \xi = e^{i\theta}, j\pi 2^{-k} < \theta \le (j+1)\pi 2^{-k}\}$$

$k = 0, 1, 2, \dots$, $j = 0, 1, \dots, 2^{k+1} - 1$. $|\Delta_{jk}| = m_2(\Delta_{jk}) \approx 2^{-2k}$, $\mu(\Delta_{jk}) = \int_{\Delta_{jk}} d\mu(z)$ and $|I_{jk}| \approx 2^{-k}$, $k = 0, 1, 2, \dots$, $j = 0, 1, \dots, 2^{k+1} - 1$.

Let further $H(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$. For $0 < p$, $q \le \infty$, $\alpha > -1$, $0 < s < \infty$,

$$H^{p,q}_{\alpha,s} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \|f_\rho\|^s_{L^{p,q}(T)} (1-\rho)^\alpha d\rho < \infty \right\},$$

where $f_\rho(\xi) = f(\rho\xi)$, $\rho \in (0,1)$, $\xi \in T$.

Throughout the paper we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in chain of inequalities) but is independent of the functions or variables being discussed. We say A is equivalent to B, denoted by $A \approx B$, if there exist two constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. The goal of this note is to find some new sharp embedding theorems for $H^{p,q}_{\alpha,s}$ space for $s = p$, $q = \infty$ case in unit disk. Note

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such and other classes of holomorphic functions on the unit disk based on Lorentz spaces on the unit circle were studied in recent papers of Marc Lengfield (see [4, 5, 6]). Sharp embedding theorems for different holomorphic classes in $\mathbb{D}$ are well-known in literature see for example [1, 7].

2. **Main Results**

Theorem 2.1. *Let μ be a positive Borel measure on $\mathbb{D}$, $0 < p < \infty$, $\tau > 0$, $\alpha > 0$, $\tau > 2\alpha + 1$. Then for $f \in H(\mathbb{D})$*

$$\sup_{j,k} \frac{1}{|\Delta_{jk}|^{\alpha}} \int_{\Delta_{jk}} |f(z)|^p d\mu(z) \le C \int_0^1 \|f_\rho\|^p_{L^{p,\infty}(T)} (1-\rho)^{\tau-2-2\alpha} d\rho$$

if and only if $\mu(\Delta_{jk}) 2^{k\tau} <$ const.

Theorem 2.2. *Let $f \in H(\mathbb{D})$, μ be a positive Borel measure on $\mathbb{D}$, $0 < p < \infty$, $\tau > 0$, $\alpha > 0$, $\tau > \alpha + 1$ and $V = \tau - 2$. Then the following conditions are equivalent:*

(a) $\mu(\Delta_{jk}) 2^{k\tau} \le C$;

(b) $\displaystyle\int_0^1 \sup_{I_{jk} \subset T} \frac{1}{|I_{jk}|^{\alpha}} \int_{I_{jk}} |f_\rho(\xi)|^p d\mu(\rho\xi) \le C \int_0^1 \|f_\rho\|^p_{L^{p,\infty}} (1-\rho)^{V-\alpha} d\rho$;

(c) $\displaystyle\sup_{I_{jk} \subset T} \frac{1}{|I_{jk}|^{\alpha}} \int_{I_{jk}} \int_0^1 |f_\rho(\xi)|^p d\mu(\rho\xi) \le C \int_0^1 \|f_\rho\|^p_{L^{p,\infty}} (1-\rho)^{V-\alpha} d\rho$.

Theorem 2.3. *Let $0 < p < \infty$, $\alpha \ge 0$, $\tau > 0$, μ be a positive Borel measure on $\mathbb{D}$ and $f \in H(\mathbb{D})$. Then*

$$\sup_{\Delta_{jk}} \left(\frac{1}{|\Delta_{jk}|^{\tau}} \int_{\Delta_{jk}} |f(z)|^p d\mu(z) \right) \le C \sup_{\rho} (\|f_\rho\|^p_{L^{p,\infty}(T)} (1-\rho)^{\alpha})$$

if and only if

$$\sup_{j,k} \frac{\mu(\Delta_{jk}) 2^{k(1+\alpha)}}{|\Delta_{jk}|^{\tau}} \le C.$$

***Proof of Theorem* 2.1.** Note that the following estimates are true

$$\begin{aligned}
\int_{\Delta_{jk}} |f(z)|^p d\mu(z) \; &\le \; \big(\max_{\Delta_{jk}} |f(z)|^p\big) \mu(\Delta_{jk}) \\
&\le \; C 2^{k(p/q+1)} \mu(\Delta_{jk}) \int_{1-2^{-k+1}}^{1-2^{-k-2}} \left(\int_{I^*_{jk}} |f(\rho\xi)|^q dm(\xi) \right)^{p/q} \rho d\rho.
\end{aligned}$$

By subharmonicity of $|f(z)|^p$ (see [1], Lemma 2.5), we used the estimates

$$\max_{z \in \Delta_{jk}} |f(z)|^p \le C 2^{k(p/q+1)} \int_{1-2^{-k+1}}^{1-2^{-k-2}} \left(\int_{I^*_{jk}} |f(\rho\xi)|^q dm(\xi) \right)^{p/q} d\rho$$

for $q \in (0, \infty)$, $p > q$, $f \in H(\mathbb{D})$, where I^*_{jk} is an enlarged arc (see [1], Lemma 2.5).

Using the fact that $p > q$ and that $V = \tau - 2$,

$$\|f\|^q_{L^{p,\infty}(T)} \approx \sup_{I \subset T} \frac{1}{|I|^{1-q/p}} \int_I |f(\xi)|^q dm(\xi),$$

(see [3]), we have finally from above

$$\begin{aligned} \frac{1}{|\Delta_{jk}|^\alpha} \int_{\Delta_{jk}} |f(z)|^p d\mu(z) \; &\leq \; C \sup_{j,k} \mu(\Delta_{jk}) 2^{k\tau} \\ &\quad \cdot \int_{1-2^{-k+1}}^{1-2^{-k-2}} \|f_\rho\|^p_{L^{p,\infty}(T)} (1-\rho)^{V-2\alpha} d\rho \end{aligned}$$

if $\sup\limits_{j,k} \mu(\Delta_{jk}) 2^{k\tau} < \infty$.

Let us prove the reverse. We will use the fact that for $\alpha > 0$, $|(1-|w|z)|^\alpha \approx (1-|w|)^\alpha$ if w is a center of Δ_{jk} and $z \in \Delta_{Jk}$, which can be checked by direct calculation. Let β be big enough positive number.

$$f(z) = \frac{(1-|w|)^{\beta - \tau/p + 1/p}}{(1-|w||z|\xi)^{1/p+\beta}}, \quad z = |z|\xi, \xi \in T, |z| \in (0,1),$$

where w is a center of Δ_{j_0,k_0}. Using the fact that $(1-z)^{-1/p} \in H^{p,\infty} = H(\mathbb{D}) \cap L^{p,\infty}(T)$ (see [4, 5]), we get for fixed j_0, k_0

$$\sup_{j,k} \frac{1}{|\Delta_{jk}|^\alpha} \int_{\Delta_{jk}} |f(z)|^p d\mu(z) \geq 2^{2k_0\alpha + k_0\tau} \mu(\Delta_{j_0k_0})$$

and also

$$\int_0^1 \|f_\rho\|^p_{H^{p,\infty}(T)} (1-\rho)^{V-2\alpha} d\rho \leq C(1-|w|)^{-2\alpha} = 2^{2k_0\alpha}.$$

So for any fixed j_0, k_0 we will have if our embedding holds then

$$\mu(\Delta_{j_0k_0}) \cdot 2^{k_0\tau} < C.$$

So our theorem is proved. □

The proof of Theorem 2.2 is very similar to the proof of Theorem 2.1 and we omit it. The third assertion can be obtained from the following chain of estimates

$$\begin{aligned} \int_{\Delta_{jk}} |f(z)|^p d\mu(z) \; &\leq \; \big(\max_{\Delta_{jk}} |f(z)|^p\big) \mu(\Delta_{jk}) \\ &\leq \; C 2^{k(p/q+1)} \mu(\Delta_{jk}) \int_{1-2^{-k+1}}^{1-2^{-k-2}} \left(\int_{I^*_{jk}} |f(\rho\xi)|^q dm(\xi) \right)^{p/q} \rho d\rho. \end{aligned}$$

$$\begin{aligned}\frac{1}{|\Delta_{jk}|^{\tau}}\int_{\Delta_{jk}}|f(z)|^{p}d\mu(z) \;\leq\; & C\sup_{I^{*}_{jk}}\frac{2^{-k\alpha}}{|I^{*}_{jk}|^{(1/q-1/p)p}}\\ & \times\Big(\int_{I^{*}_{jk}}|f(\rho\xi)|^{q}dm(\xi)\Big)^{p/q}\cdot 2^{k(p/q+1)}\\ & \cdot\mu(\Delta_{jk})\cdot|I^{*}_{jk}|^{(1/q-1/p)p}2^{-k}\\ \leq\; & C\sup_{\rho}\left[\|f_{\rho}\|^{p}_{H^{p,\infty}}(1-\rho)^{\alpha}\right]\cdot\sup_{j,k}\frac{\mu(\Delta_{jk}2^{k(1+\alpha)})}{|\Delta_{jk}|^{\tau}}\end{aligned}$$

for $\alpha > 0$, $0 < p < \infty$ and $\tau > 0$. This condition on measure is sufficient. The reverse implication can be obtained as above using the following standard test function

$$f_{\rho}(\xi)=\frac{1}{(1-\rho\xi w)^{1/p+\alpha/p}}\,,$$

where w is the center of Δ_{jk}.

Remark 2.4. We note that the following similar assertion was given in [7] for $p < 2$ and in [6] for all $p < q$. Let μ be a positive measure, $f \in H(\mathbb{D})$, $0 < p < q$, $\alpha > 0$. Then

$$\sup_{I\subset T}\frac{1}{|I|^{1-p/q}}\int_{DI}|f(z)|^{p}d\mu(z)\leq C\|f\|^{p}_{A^{q}_{\alpha}},$$

$$\|f\|_{A^{q}_{\alpha}}=\Big(\int_{\mathbb{D}}|f(z)|^{q}(1-|z|)^{q\alpha-1}dm_{2}(z)\Big)^{1/q}$$

if and only if

$$\sup_{I\subset T}\frac{1}{|I|}\int_{DI}\left(\frac{\mu(\Delta_{z})}{(1-|z|)^{1+\alpha p}}\right)^{\frac{q}{q-p}}\frac{dm_{2}(z)}{1-|z|}\leq C\,,$$

where DI is well-known Carleson box and

$$\Delta_{z}=\left\{z'=ze^{i\theta'}:|\theta-\theta'|<1-|z|,\frac{1-|z|}{2}<1-|z'|<2(1-|z|)\right\}.$$

Note that the proof of mentioned assertions in [6, 7] is based on completely different ideas.

Remark 2.5. Let μ be a positive measure, $f \in H(\mathbb{D})$. Using Standard covering lemmas (see [1], Lemma 2.5), we can show that

$$\sup_{j,k}\frac{1}{|\Delta_{jk}|}\int_{\Delta_{jk}}|f(z)|^{p}d\mu(z)<\infty$$

if

$$\sup_{B(w,a)\subset\mathbb{C}}\frac{1}{|B(w,a)|}\int_{B(w,a)\bigcap\mathbb{D}}|f(z)|^{p}d\mu(z)<\infty,\quad 0<p<\infty,$$

for $B(w,a)=\{z:|z-w|<a\}$, $a>0$, where $|B(w,a)|$ is a Lebesgue measure of the Ball $B(w,a)$. The proof is left to readers.

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