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Research Article

Equality in Distribution of Random Sums for Introducing Selfdecomposability

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Abstract. It constitutes a general recognition that discrete Poisson random sums are very strong tools of probability theory with significant applications in a very wide variety of important practical disciplines. The paper makes use of an equality in distribution for the investigation of the structure of a particularly significant class of discrete Poisson random sums.

Keywords. Random sum; Equality in distribution; Probability generating function

MSC. 97K60; 60E05

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1. Introduction

We consider the discrete random variable R with values in the set $\mathbf{N}_0 = \{0,1,2,\dots\}$ and probability generating function $P_R(z)$. The discrete random variable S with values in the set $\mathbf{N}_0$ and probability generating function

$$P_S(z) = \exp\{\lambda[P_R(z)-1]\}, \quad \lambda > 0$$

is called Poisson random sum [2, 7].

We also consider the discrete random variable V with values in the set $\mathbf{N}_0$ and probability generating function $P_V(z)$. The discrete random variable J with values in the set $\mathbf{N}_0$ and probability generating function

$$P_J(z) = \frac{\alpha}{(1-z)^\alpha} \int_0^1 P_V(w)(1-w)^{\alpha-1} dw, \quad \alpha > 0$$

is called α-monotone [5].

Moreover, the discrete random variable T with values in the set $\mathbf{N}_0$ and probability generating function

$$P_T(z) = \exp\left\{\lambda \left[\frac{\alpha}{(1-z)^\alpha} \int_z^1 P_V(w)(1-w)^{\alpha-1} dw - 1\right]\right\}$$

is called Poisson random sum of α-monotone random variables [1].

The present paper is mainly devoted on the contribution of discrete Poisson random sums and Poisson random sums of α-monotone random variables to the structure of the very important class of discrete selfdecomposable random variables.

2. An Integral Equation for Probability Generating Functions

The present section is devoted to the formulation and investigation of equality in distribution for the establishment of a characterization of the extremely important class of discrete selfdecomposable distributions.

Theorem. *Let X be a discrete random variable with values in the set $\mathbf{N}_0$ and probability generating function $P_X(z)$ and let Y be a Poisson random sum with probability generating function*

$$P_Y(z) = \exp\{\lambda[P_X(z) - 1]\}, \quad \lambda > 0.$$

We suppose that C is a discrete random variable with values in the set $\mathbf{N} = \{1, 2, \ldots\}$ and probability generating function $P_C(z)$ and L is the Poisson random sum of α-monotone random variables with probability generating function

$$P_L(z) = \exp\left\{\lambda \left[\frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w) P_C(w)(1-w)^{\alpha-1} dw - 1\right]\right\}$$

then, X is a selfdecomposable random variable with probability generating function

$$P_X(z) = \exp\left\{-\alpha \int_z^1 \frac{1 - P_C(w)}{1-w} dw\right\}$$

if, and only if,

$$Y \stackrel{d}{=} L, \tag{2.1}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. If we use the probability generating function $P_Y(z)$ and the probability generating function $P_L(z)$ in (2.1) we get the integral equation

$$\exp\{\lambda[P_X(z) - 1]\} = \exp\left\{\lambda \left[\frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w) P_C(w)(1-w)^{\alpha-1} dw - 1\right]\right\}. \tag{2.2}$$

It is readily shown that the integral equation in (2.2) can be written in the form

$$P_X(z) = \frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w) P_C(w)(1-w)^{\alpha-1} dw. \tag{2.3}$$

By multiplying both sides of the integral equation in (2.3) by $(1-z)^\alpha$ and then differentiating we get the differential equation

$$-\alpha(1-z)^\alpha P_X(z)-(1-z)^\alpha \frac{dP_X(z)}{dz}=-\alpha P_X(z)P_C(z)(1-z)^{\alpha-1}. \tag{2.4}$$

If

$$z \neq 1$$

then the differential equation in (2.4) can be written in the form

$$(1-z)\frac{1}{P_X(z)}\frac{dP_X(z)}{dz}=\alpha(1-P_C(z)). \tag{2.5}$$

If we integrate in (2.5) with due regard to the boundary conditions

$$P_X(1)=1$$

and

$$P_C(1)=1$$

we obtain the function

$$P_X(z)=\exp\left\{-\alpha\int_z^1 \frac{1-P_C(w)}{1-w}dw\right\}. \tag{2.6}$$

Since

$$P_C(0)=0$$

then the function $P_X(z)$ in (2.6) is the probability generating function of a selfdecomposable distribution [3, 4, 6].

3. Conclusion

The presentation of a Poisson random sum as a Poisson random sum of α-monotone random variables is readily recognized as a useful stochastic model for the description, analysis and solution of real problems.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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