Equality in Distribution of Random Sums for Introducing Selfdecomposability

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Abstract. It constitutes a general recognition that discrete Poisson random sums are very strong tools of probability theory with significant applications in a very wide variety of important practical disciplines. The paper makes use of an equality in distribution for the investigation of the structure of a particularly significant class of discrete Poisson random sums.

Keywords. Random sum; Equality in distribution; Probability generating function

MSC. 97K60; 60E05

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1. Introduction

We consider the discrete random variable $R$ with values in the set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and probability generating function $P_R(z)$. The discrete random variable $S$ with values in the set $\mathbb{N}_0$ and probability generating function

$$P_S(z) = \exp \{ \lambda [P_R(z) - 1] \}, \quad \lambda > 0$$

is called Poisson random sum [2,7].

We also consider the discrete random variable $V$ with values in the set $\mathbb{N}_0$ and probability generating function $P_V(z)$. The discrete random variable $J$ with values in the set $\mathbb{N}_0$ and
probability generating function

\[ P_J(z) = \frac{\alpha}{(1-z)^\alpha} \int_0^1 P_V(w)(1-w)^{a-1}dw, \quad \alpha > 0 \]

is called \( \alpha \)-monotone [5].

Moreover, the discrete random variable \( T \) with values in the set \( \mathbb{N}_0 \) and probability generating function

\[ P_T(z) = \exp \left\{ \lambda \left[ \frac{\alpha}{(1-z)^\alpha} \int_z^1 P_V(w)(1-w)^{a-1}dw - 1 \right] \right\} \]

is called Poisson random sum of \( \alpha \)-monotone random variables [1].

The present paper is mainly devoted on the contribution of discrete Poisson random sums and Poisson random sums of \( \alpha \)-monotone random variables to the structure of the very important class of discrete selfdecomposable random variables.

## 2. An Integral Equation for Probability Generating Functions

The present section is devoted to the formulation and investigation of equality in distribution for the establishment of a characterization of the extremely important class of discrete selfdecomposable distributions.

**Theorem.** Let \( X \) be a discrete random variable with values in the set \( \mathbb{N}_0 \) and probability generating function \( P_X(z) \) and let \( Y \) be a Poisson random sum with probability generating function

\[ P_Y(z) = \exp \{ \lambda [P_X(z) - 1] \}, \quad \lambda > 0. \]

We suppose that \( C \) is a discrete random variable with values in the set \( \mathbb{N} = \{1, 2, \ldots \} \) and probability generating function \( P_C(z) \) and \( L \) is the Poisson random sum of \( \alpha \)-monotone random variables with probability generating function

\[ P_L(z) = \exp \left\{ \lambda \left[ \frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w)P_C(w)(1-w)^{a-1}dw - 1 \right] \right\} \]

then, \( X \) is a selfdecomposable random variable with probability generating function

\[ P_X(z) = \exp \left\{ -\alpha \int_z^1 \frac{1-P_C(w)}{1-w}dw \right\} \]

if, and only if,

\[ Y \overset{d}{=} L, \quad (2.1) \]

where \( \overset{d}{=} \) denotes equality in distribution.

**Proof.** Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. If we use the probability generating function \( P_Y(z) \) and the probability generating function \( P_L(z) \) in (2.1) we get the integral equation

\[ \exp\{\lambda[P_X(z)-1]\} = \exp\left\{ \lambda \left[ \frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w)P_C(w)(1-w)^{a-1}dw - 1 \right] \right\}. \quad (2.2) \]

It is readily shown that the integral equation in (2.2) can be written in the form

\[ P_X(z) = \frac{\alpha}{(1-z)^\alpha} \int_z^1 P_X(w)P_C(w)(1-w)^{a-1}dw. \quad (2.3) \]
By multiplying both sides of the integral equation in (2.3) by \((1 - z)^\alpha\) and then differentiating we get the differential equation

\[-\alpha(1 - z)^\alpha P_X(z) - (1 - z)^\alpha \frac{dP_X(z)}{dz} = -\alpha P_X(z)P_C(z)(1 - z)^{\alpha - 1}.\]  
(2.4)

If

\[z \neq 1\]

then the differential equation in (2.4) can be written in the form

\[(1 - z) \frac{1}{P_X(z)} \frac{dP_X(z)}{dz} = \alpha (1 - P_C(z)).\]
(2.5)

If we integrate in (2.5) with due regard to the boundary conditions

\[P_X(1) = 1\]

and

\[P_C(1) = 1\]

we obtain the function

\[P_X(z) = \exp \left\{ -\alpha \int_z^1 \frac{1 - P_C(w)}{1 - w} dw \right\}.\]
(2.6)

Since

\[P_C(0) = 0\]

then the function \(P_X(z)\) in (2.6) is the probability generating function of a selfdecomposable distribution [3,4,6].

### 3. Conclusion

The presentation of a Poisson random sum as a Poisson random sum of \(\alpha\)-monotone random variables is readily recognized as a useful stochastic model for the description, analysis and solution of real problems.

### Competing Interests

The authors declare that they have no competing interests.

### Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

### References


