Iterative Methods for Solving the Proximal Split Feasibility Problems

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Abstract. In this work, we study the proximal split feasibility problem. We introduce a new algorithm with inertial technique for solving this problem in Hilbert spaces. We also prove the strong convergence theorem under some suitable conditions. Finally, we give some numerical experiments to support our results.

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1. Introduction

Let $H_1$ and $H_2$ be two real Hilbert spaces. Suppose that $f: H_1 \to \mathbb{R} \cup \{+\infty\}$, $g: H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex and lower semi-continuous functions and $A: H_1 \to H_2$ is a bounded linear operator. In this research, we shall consider the following Proximal Split Feasibility Problem (PSFP):
Find a solution $x^* \in H_1$ such that

$$\min_{x \in H_1} f(x) + g(x),$$

where $g_A : H_2 \to \mathbb{R} \cup \{+\infty\}$, is the Moreau-Yosida approximate \cite{11} defined by

$$g_A(y) := \min_{u \in H_2} g(u) + \frac{1}{2\lambda} \|u - y\|^2.$$  

Problem (1.1) includes many nonlinear problems in applied sciences, engineering and economics. For example, if we take $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], the indicator function of nonempty, closed and convex subset $C$ of $H_1$ and $g = \delta_Q$, the indicator function of nonempty, closed and convex subset $Q$ of $H_2$, then Problem (1.1) reduces to the following Split Feasibility Problem (SFP):

Find

$$x \in C$$

such that $Ax \in Q$. The SPFP attracts the attention of many authors due to its application in signal processing, medical image reconstruction and modeling inverse problems which arise from phase retrievals. Various algorithms have been invented to solve it (see, e.g., \cite{1,4,5,10,13,17,19,21} and references therein). If $A = I$, $H_1 = H_2 = H$ and $g$ be differentiable, where $I$ is an identity mapping on $H$, then Problem (1.1) reduces to the following minimization problem:

$$\min_{x \in H} f(x) + g(x).$$

Suppose that the problem (1.1) has at least a solution and denote by $\Gamma$ the solution set of (1.1).

Set

$$\theta(x) := \sqrt{\|\nabla h(x)\|^2 + \|\nabla l(x)\|^2}$$

with $h(x) = \frac{1}{2}\|I - \prox_{\lambda g}Ax\|^2$, $l(x) = \frac{1}{2}\|(I - \prox_{\lambda f})x\|^2$, $x \in H_1$, where $\prox_{\lambda g}(y) = \arg\min_{u \in H_1} \left\{ g(u) + \frac{1}{2\lambda} \|u - y\|^2 \right\}$. Moudafi and Thakur \cite{8} introduced the following split proximal algorithm for solving (1.1):

**Split Proximal Algorithm.** Given an initial point $x_1 \in H_1$. Assume that $x_n$ has been constructed and $\theta(x_n) \neq 0$, then compute $x_{n+1}$ via the rule

$$x_{n+1} = \prox_{\lambda u_n f}(x_n - \mu_n A^*(I - \prox_{\lambda g}Ax_n), \quad n \geq 1,$$

where step size $\mu_n := \rho_n \frac{h(x_n)}{\theta^2(x_n)}$ and $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (1.1) and the iterative process stop. Otherwise, we set $n := n + 1$ and go to (1.4). Using the split proximal algorithm (1.4), Moudafi and Thakur \cite{8} proved a weak convergence theorem for approximating a solution of (1.1) under the condition that $\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n) + l(x_n)} - \epsilon$ for some $\epsilon > 0$.

**Remark 1.1.** It is observed in Shehu and Iyiola \cite{12} that the above condition means that the convergence is ensured only in the very restrictive case when a condition imposed on the sequence $\{x_n\}$ itself is fulfilled. Thus, it is of practical computational importance to introduce a new iterative scheme in which this condition is avoided and replaced with a much simple condition on the step size and convergence result is still achieved.

Ochs et al. \cite{9} proposed the following forward-backward splitting algorithm by using the inertial technique for solving convex minimization problem for the sum of a smooth convex function $g$ and a non-smooth convex function $f : x_0, x_1 \in H_1$

$$x_{n+1} = \prox_{\alpha f}(x_n - \alpha \nabla g(x_n) + \beta(x_n - x_{n-1})), \quad n \geq 1,$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 2$.

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where \( \alpha > 0 \) and \( \beta \in [0,1) \). Recently, Shehu and Iyiola \cite{12} obtained weak convergence results for solving PSFP by replacing the condition imposed in \( \rho \) with a much simpler condition on the step size. They studied convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method. They introduced the following inertial extrapolation split proximal algorithm:

Given initial points \( x_0, x_1 \in H_1 \). Assume that \( x_n \) has been constructed and \( \theta(y_n) \neq 0 \), then compute \( x_{n+1} \) via the rule

\[
\begin{align*}
    y_n &= x_n + \beta_n(x_n - x_{n-1}), \\
    z_n &= y_n - \rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)}(\nabla h(y_n) + \nabla l(y_n)), \\
    x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n z_n, \quad n \geq 1
\end{align*}
\]  

(1.6)

where \( 0 < \rho_n < 4 \). If \( \nabla h(y_n) = 0 = \nabla l(y_n) \) and \( y_n = x_n \), then the sequence \( \{x_n\} \) is a solution of PSFP and the iterative process stops. Otherwise, we set \( n := n + 1 \) and go to (1.6).

In this research, we introduce a new algorithm for solving the proximal split feasibility problem. We prove strong convergence theorems under some suitable conditions. Some numerical experiments are shown in Section 4.

### 2. Preliminaries and Lemmas

In this section, we give some preliminaries which will be used in our proof. Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \) and let \( C \) be a nonempty, closed and convex subset of \( H \). A mapping \( T : C \to C \) is said to be nonexpansive if

\( \|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \)

For any point \( u \in H \), there exists a unique point \( P_C u \in C \) such that

\( \|u - P_C u\| \leq \|u - y\|, \quad \forall y \in C. \)

\( P_C \) is called the metric projection of \( H \) onto \( C \). We know that \( P_C \) is a nonexpansive mapping of \( H \) onto \( C \). It is also known that \( P_C \) satisfies

\[
\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.1}
\]

for all \( x, y \in C \). Furthermore, \( P_C x \) is characterized by the properties \( P_C x \in C \) and

\[
\langle x - P_C x, P_C x - y \rangle \geq 0 \tag{2.2}
\]

for all \( y \in C \). A mapping \( T : H \to H \) is said to be firmly nonexpansive if and only if \( 2T - I \) is nonexpansive, or equivalently

\( \langle x - Tx, Tx - y \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H. \)

For example, projections and proximal mappings are firmly nonexpansive.

Let the proximal operator of the scaled function \( \lambda f \), where \( \lambda > 0 \), which can be expressed as \( f : H \to \mathbb{R} \cup \{+\infty\} \) be a proper, convex and lower semicontinuous.

\[
prox_{\lambda f}(v) = \arg\min_{x \in H} \{f(x) + (1/2\lambda)\|x - v\|^2\}. \tag{2.3}
\]

This is also called the proximal operator of \( f \) with parameter \( \lambda \). In a real Hilbert space \( H \), we have the following equality:

\[
\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2 \tag{2.4}
\]
and the subdifferential inequality
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \]
for all \( x, y \in H \).

**Theorem 3.1**. Let \( \Gamma_n \) be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence \( \{ \Gamma_{n_i} \} \) of \( \{ \Gamma_n \} \) which satisfies \( \Gamma_{n_i} < \Gamma_{n_i+1} \) for all \( i \in \mathbb{N} \). Define the sequence \( \{ \psi(n) \}_{n \geq n_0} \) of integers as follows:
\[ \psi(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1} \}. \]
where \( n_0 \in \mathbb{N} \) such that \( \{ k \leq n_0 : \Gamma_k < \Gamma_{k+1} \} \neq \emptyset \). Then, the following hold:
1. \( \Gamma(n_0) \leq \Gamma(n_0 + 1) \leq \ldots \) and \( \Gamma(n) \to \infty \);
2. \( \Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1} \) and \( \Gamma_n \leq \Gamma_{\psi(n)+1}, \forall \ n \geq n_0 \).

**Lemma 2.1** ([7]). Let \( \{ \Gamma_n \} \) be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence \( \{ \Gamma_{n_i} \} \) of \( \{ \Gamma_n \} \) which satisfies \( \Gamma_{n_i} < \Gamma_{n_i+1} \) for all \( i \in \mathbb{N} \). Define the sequence \( \{ \psi(n) \}_{n \geq n_0} \) of integers as follows:
\[ \psi(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1} \}. \]
where \( n_0 \in \mathbb{N} \) such that \( \{ k \leq n_0 : \Gamma_k < \Gamma_{k+1} \} \neq \emptyset \). Then, the following hold:
1. If \( \psi(n_0) \leq \Gamma(n_0 + 1) \leq \ldots \) and \( \Gamma(n) \to \infty \);
2. \( \Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1} \) and \( \Gamma_n \leq \Gamma_{\psi(n)+1}, \forall \ n \geq n_0 \).

**Lemma 2.2** ([6],[18]). Let \( \{ a_n \} \) and \( \{ c_n \} \) be sequences of nonnegative real numbers such that
\[ a_{n+1} \leq (1 - \delta_n) a_n + b_n + c_n, \quad n \geq 1, \]
where \( \{ \delta_n \} \) is a sequence in \( (0,1) \) and \( \{ b_n \} \) is a real sequence. Assume \( \sum_{n=1}^{\infty} c_n < \infty \). Then the following results hold:
1. If \( b_n \leq \delta_n M \) for some \( M \geq 0 \), then \( \{ a_n \} \) is a bounded sequence.
2. If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \limsup_{n \to \infty} b_n/\delta_n \leq 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

### 3. Main Results

In this section, we give strong convergence result using inertial extrapolation for solving (1.1), which is the main result of this paper. Now, set \( \theta(x) := \sqrt{\| \nabla h(x) + \nabla l(x) \|^2} \) with \( h(x) = \frac{1}{2}\| (I - \text{prox}_{\lambda g}) Ax \|^2 \), \( l(x) = \frac{1}{2}\| (I - \text{prox}_{\lambda f}) x \|^2 \), \( x \in H_1 \) and we introduce the following inertial extrapolation split proximal algorithm:

**Algorithm 3.1.** Given initial points \( x_0, x_1, u \in H \). Assume that \( x_n \) has been constructed and and \( \theta(y_n) \neq 0 \), then compute \( x_{n+1} \) by the following manner:
\[
\begin{align*}
    y_n &= x_n + \beta_n (x_n - x_{n-1}) \\
    z_n &= y_n - \rho_n \frac{h(y_n) + l(y_n)}{\theta(y_n)} (\nabla h(y_n) + \nabla l(y_n)) \\
    x_{n+1} &= a_n u + (1 - a_n) z_n, \quad n \geq 1
\end{align*}
\]
where \( 0 < a_n < 1, 0 \leq \beta_n < 1 \) and \( 0 < \rho_n < 4 \). If \( \nabla h(y_n) = 0 = \nabla l(y_n) \) and \( y_n = x_n \) then \( x_n \) is a solution of (1.1) and then iterative process stops. Otherwise, we set \( n := n + 1 \) and go to (3.1).

**Theorem 3.2.** Assume that \( \{ a_n \} \subset (0,1), \{ \rho_n \} \subset (0,4), \{ \beta_n \} \subset [0, \beta] \), where \( \beta \in [0,1] \) satisfy the following conditions:
1. \( \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=1}^{\infty} a_n = \infty; \)
2. \( \inf_{n \in \mathbb{N}} \rho_n (4 - \rho_n) > 0; \)
3. \( \lim_{n \to \infty} \frac{\beta_n}{a_n} \| x_n - x_{n-1} \| = 0 \).

Then the sequence \( \{ x_n \} \) generated by (3.1) strongly converges to \( x^* = P_{\Gamma} u \) which is a solution in \( \Gamma \).

**Proof.** Let \( x^* = P_{\Gamma} u \in \Gamma \). Observe that \( \nabla h(x) = A^*(I - \text{prox}_{\lambda g}) Ax \) and \( \nabla l(x) = (I - \text{prox}_{\lambda f}) x \). Since \( \text{prox}_{\lambda g} \) is firmly nonexpansive, the mapping \( (I - \text{prox}_{\lambda g}) \) is also firmly nonexpansive. It follows
that
\[ \langle \nabla h(y_n), y_n - x^* \rangle = (I - \text{prox}_{\lambda g})A_{n}y_n, Ay_n - Ax^* \rangle \geq \| (I - \text{prox}_{\lambda g})A_{n}y_n \|^2 = 2h(y_n). \]
From (3.1), we obtain
\[
\| z_n - x^* \|^2 = \left\| y_n - x^* - \rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} (\nabla h(y_n) + \nabla l(y_n)) \right\|^2
\[ = \| y_n - x^* \|^2 + \rho_n^2 \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} (\nabla h(y_n) + \nabla l(y_n)) \| y_n - x^* \|^2 \]
\[ - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \langle \nabla h(y_n) + \nabla l(y_n), y_n - x^* \rangle \]
\[ = \| y_n - x^* \|^2 + \rho_n^2 \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \]
\[ - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} (A^*(I - \text{prox}_{\lambda g})Ay_n + (I - \text{prox}_{\lambda f})y_n, y_n - x^*) \]
\[ \leq \| y_n - x^* \|^2 - 2\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} (2h(y_n) + 2l(y_n)) + \rho_n^2 \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \]
\[ = \| y_n - x^* \|^2 - 4\rho_n \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} + \rho_n^2 \frac{h(y_n) + l(y_n)}{\theta^2(y_n)} \]
\[ = \| y_n - x^* \|^2 - \rho_n (4 - \rho_n) \left( \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \right) \]
\[ \leq \| y_n - x^* \|^2. \quad (3.2) \]
So, we obtain \( \| z_n - x^* \| \leq \| y_n - x^* \| \) for all \( n \geq 0 \). Moreover, we have
\[ \| y_n - x^* \| = \| x_n - x^* + \beta_n (x_n - x_{n-1}) \| \leq \| x_n - x^* \| + \beta_n \| x_n - x_{n-1} \|. \quad (3.3) \]
It follows that, by (3.3)
\[
\| x_{n+1} - x^* \| = \| (\alpha_n u + (1 - \alpha_n)z_n) - x^* \|
\[ \leq \alpha_n \| u - x^* \| + (1 - \alpha_n) \| y_n - x^* \| \]
\[ \leq \alpha_n \| u - x^* \| + (1 - \alpha_n) \| x_n - x^* \| + \beta_n \| x_n - x_{n-1} \| \]
\[ = \alpha_n \| u - x^* \| + (1 - \alpha_n) \| x_n - x^* \| + (1 - \alpha_n) \beta_n \| x_n - x_{n-1} \| \]
\[ = (1 - \alpha_n) \| x_n - x^* \| + \alpha_n \left( \| u - x^* \| + (1 - \alpha_n) \frac{\beta_n}{\alpha_n} \| x_n - x_{n-1} \| \right). \quad (3.4) \]
By Lemma 2.2, we conclude that \( (x_n) \) is bounded. By (2.4), we have
\[
\langle x_n - x^*, x_n - x_{n-1} \rangle = \frac{1}{2} \| x_n - x^* \|^2 + \frac{1}{2} \| x_n - x_{n-1} \|^2 - \frac{1}{2} \| x_n - x^* - x_n + x_{n-1} \|^2
\[ = \frac{1}{2} \| x_n - x^* \|^2 + \frac{1}{2} \| x_n - x_{n-1} \|^2 - \frac{1}{2} \| x_{n-1} - x^* \|^2. \]
Therefore, we get
\[
\| y_n - x^* \|^2 = \| x_n - x^* + \beta_n (x_n - x_{n-1}) \|^2
\[ = \| x_n - x^* \|^2 + \beta_n^2 \| x_n - x_{n-1} \|^2 + 2\beta_n \langle x_n - x^*, x_n - x_{n-1} \rangle
\[ = \| x_n - x^* \|^2 + \beta_n^2 \| x_n - x_{n-1} \|^2 + 2\beta_n \left( \frac{1}{2} \| x_n - x^* \|^2 + \frac{1}{2} \| x_n - x_{n-1} \|^2 - \frac{1}{2} \| x_{n-1} - x^* \|^2 \right)
\[ = \| x_n - x^* \|^2 + \beta_n^2 \| x_n - x_{n-1} \|^2 + \beta_n \| x_n - x^* \|^2 + \beta_n \| x_n - x_{n-1} \|^2 - \beta_n \| x_{n-1} - x^* \|^2
We next consider two cases. Set

\[ (1 - \alpha_n) \beta_n (\|x_n - x^*\|^2 - |x_{n-1} - x^*|^2) + 2 \beta_n \|x_n - x_{n-1}\|^2. \]  

(3.5)

Now, using (2.5) and (3.1), we obtain

\[
\|x_{n+1} - x^*\|^2 = \|\alpha_n (u - x^*) + (1 - \alpha_n) (z_n - x^*)\|^2 \\
\leq (1 - \alpha_n)^2 \|z_n - x^*\|^2 + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
\leq (1 - \alpha_n) \|z_n - x^*\|^2 + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
\leq (1 - \alpha_n) \|y_n - x^*\|^2 - (1 - \alpha_n) \rho_n (4 - \rho_n) \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
\leq (1 - \alpha_n) \|x_n - x^*\|^2 + (1 - \alpha_n) \beta_n \|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2 \\
+ 2(1 - \alpha_n) \beta_n \|x_n - x_{n-1}\|^2 - (1 - \alpha_n) \rho_n (4 - \rho_n) \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle.
\]

(3.6)

We next consider two cases. Set \( \Gamma_n = \|x_n - x^*\|^2 \).

**Case 1:** Suppose that there exists a natural number \( N \) such that \( \Gamma_{n+1} \leq \Gamma_n \) for all \( n \geq N \). In this case, \( (\Gamma_n) \) is convergent. From (C1) and (C2), we can find a constant \( \sigma \) such that \( (1 - \alpha_n) \rho_n (4 - \rho_n) \geq \sigma > 0 \) for all \( n \in \mathbb{N} \), it follows that, by (3.6)

\[
\Gamma_{n+1} \leq (1 - \alpha_n) \Gamma_n + (1 - \alpha_n) \beta_n (\Gamma_n - \Gamma_{n-1}) + 2(1 - \alpha_n) \beta_n \|x_n - x_{n-1}\|^2 - (1 - \alpha_n) \rho_n (4 - \rho_n) \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle,
\]

(3.7)

which gives

\[
\sigma \frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \leq (1 - \alpha_n) \Gamma_n + (1 - \alpha_n) \beta_n (\Gamma_n - \Gamma_{n-1}) - \Gamma_{n+1} \\
+ 2(1 - \alpha_n) \beta_n \|x_n - x_{n-1}\|^2 + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle \\
\leq (\Gamma_n - \Gamma_{n+1}) + (1 - \alpha_n) \beta_n (\Gamma_n - \Gamma_{n-1}) + 2(1 - \alpha_n) \beta_n \|x_n - x_{n-1}\|^2 + 2 \alpha_n \langle u - x^*, x_{n+1} - x^* \rangle.
\]

(3.8)

We see that (C3) implies \( \beta_n \|x_n - x_{n-1}\|^2 \to 0 \) since \( \{\alpha_n\} \) is bounded. Since \( (\Gamma_n) \) converges and \( \alpha_n \to 0 \),

\[
\frac{(h(y_n) + l(y_n))^2}{\theta^2(y_n)} \to 0.
\]

(3.9)

Consequently, we have

\[
\lim_{n \to \infty} (h(y_n) + l(y_n)) = 0 \quad \text{and} \quad \lim_{n \to \infty} h(y_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} l(y_n) = 0,
\]

since \( \theta^2(x_n) = \|\nabla h(x_n) + \nabla l(x_n)\|^2 \) is bounded. This follows from the fact that \( \nabla h \) is Lipschitz continuous with constant \( \|A\|^2 \), \( \nabla l \) is nonexpansive and \( \{y_n\} \) is bounded. Indeed

\[
\|\nabla h(y_n)\| = \|\nabla h(y_n) - \nabla h(x^*)\| \leq \|A\|^2 \|y_n - x^*\| \quad \text{and} \quad \|\nabla l(y_n)\| = \|\nabla l(y_n) - \nabla l(x^*)\| \leq \|y_n - x^*\|.
\]

Now, let \( z \) be a weak cluster point of \( \{x_n\} \). So there exists a subsequence \( \{x_{n_j}\} \) which weakly converges to \( z \). Since \( x_{n_j} \to z \), \( j \to \infty \) and \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \), we have \( y_{n_j} \to z \), \( j \to \infty \). The lower-semicontinuity of \( h \) implies that

\[
0 \leq h(z) \leq \liminf_{j \to \infty} h(y_{n_j}) = \lim_{n \to \infty} h(y_n) = 0.
\]

This shows that \( h(z) = \frac{1}{2} \| (I - \text{prox}_{\lambda g})Az \| = 0 \). Thus \( Az \) is a minimizer of \( g \).
Also, the lower-semicontinuity of $l$ implies that
\[0 \leq l(z) \leq \liminf_{j \to \infty} l(y_{n_j}) = \lim_{n \to \infty} l(y_n) = 0.\]

Hence $l(z) = \frac{1}{2} \| (I - \text{prox}_{\mu_y} \lambda_f) z \| = 0$ and $z \in \Gamma$. From (2.6) it follows that
\[
\limsup_{n \to \infty} \langle x - x^*, x_{n+1} - x^* \rangle = \limsup_{n \to \infty} \langle u - x^*, x_n - x^* \rangle
\]
\[\leq 0. \quad (3.10)\]

From (3.7), it follows that
\[\Gamma_{n+1} \leq (1 - \alpha_n) \Gamma_n + 2(1 - \alpha_n) \beta_n \| x_n - x_{n-1} \|^2 + 2\alpha_n \langle u - x^*, x_{n+1} - x^* \rangle. \quad (3.11)\]

Applying Lemma 2.2(ii) and using (3.10) and (3.11) and the conditions (C1) and (C3), we conclude that $\Gamma_n = \| x_n - x^* \|^2 \to 0$ and thus $x_n \to x^*$ as $n \to \infty$.

**Case 2:** Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_{i+1}}$ for all $i \in \mathbb{N}$. In this case, we define $\psi : \mathbb{N} \to \mathbb{N}$ as in (2.6). Then, by Lemma 2.2 we have $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$. From (3.9), it follows that
\[
\Gamma_{\psi(n)+1} \leq (1 - \alpha_{\psi(n)}) \Gamma_{\psi(n)} + (1 - \alpha_{\psi(n)}) \beta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) + 2(1 - \alpha_{\psi(n)}) \beta_{\psi(n)} \| x_{\psi(n)} - x_{\psi(n)-1} \|^2
\]
\[- \sigma \frac{(h(y_{\psi(n)}) + l(y_{\psi(n)}))^2}{\theta^2(y_{\psi(n)})} + 2\alpha_{\psi(n)} \langle u - x^*, x_{\psi(n)+1} - x^* \rangle,
\]
which gives
\[
\sigma \frac{(h(y_{\psi(n)}) + l(y_{\psi(n)}))^2}{\theta^2(y_{\psi(n)})} \leq (1 - \alpha_{\psi(n)}) \beta_{\psi(n)}(\Gamma_{\psi(n)} - \Gamma_{\psi(n)-1}) + 2(1 - \alpha_{\psi(n)}) \beta_{\psi(n)} \| x_{\psi(n)} - x_{\psi(n)-1} \|^2
\]
\[+ 2\alpha_{\psi(n)} \langle u - x^*, x_{\psi(n)+1} - x^* \rangle + 2(1 - \alpha_{\psi(n)}) \beta_{\psi(n)} \| x_{\psi(n)} - x_{\psi(n)-1} \|^2
\]
\[+ 2\alpha_{\psi(n)} \langle u - x^*, x_{\psi(n)+1} - x^* \rangle \to 0.
\]

This shows that $h(y_{\psi(n)}) \to 0$ and $l(y_{\psi(n)}) \to 0$ as $n \to \infty$. Moreover, we have
\[
\| y_{\psi(n)} - x_{\psi(n)} \| = \| x_{\psi(n)} - \beta_{\psi(n)}(x_{\psi(n)} - x_{\psi(n)-1}) - x_{\psi(n)} \|
\]
\[= \beta_{\psi(n)} \| x_{\psi(n)} - x_{\psi(n)-1} \|
\]
\[\to 0 \text{ as } n \to \infty. \quad (3.12)\]

From (3.1), we have
\[
\| z_{\psi(n)} - x_{\psi(n)} \| \leq \| y_{\psi(n)} - x_{\psi(n)} \| + \rho_{\psi(n)} \frac{h(y_{\psi(n)}) + l(y_{\psi(n)})}{\theta^2(y_{\psi(n)})} \| \nabla h(y_{\psi(n)}) + \nabla l(y_{\psi(n)}) \|
\]
\[\to 0 \text{ as } n \to \infty. \quad (3.13)\]

It follows that
\[
\| x_{\psi(n)+1} - x_{\psi(n)} \| \leq \alpha_{\psi(n)} \langle u - x_{\psi(n)} \| + (1 - \alpha_{\psi(n)}) \| z_{\psi(n)} - x_{\psi(n)} \|
\]
\[\to 0 \text{ as } n \to \infty.
\]
We aim to find $x^*$ which implies $x = x^*$ as $n \to \infty$.

Now repeating the argument of the proof in Case 1, we obtain

$$
\limsup_{n \to \infty} \langle u - x^*, x_{\psi(n)} - x^* \rangle \leq 0.
$$

Since $\|x_{\psi(n+1)} - x_{\psi(n)}\| \to 0$, we obtain $\limsup_{n \to \infty} \langle u - x^*, x_{\psi(n+1)} - x^* \rangle \leq 0$. From (3.14) it follows that

$$
\limsup_{n \to \infty} \Gamma_{\psi(n)} \leq 0.
$$

This implies that

$$
\lim_{n \to \infty} \Gamma_{\psi(n)} = \lim_{n \to \infty} \|x_{\psi(n)} - x^*\|^2 = 0.
$$

Hence $x_{\psi(n)} \to x^*$ as $n \to \infty$. On the other hand, we see that

$$
\|x_{\psi(n+1)} - x^*\| \leq \|x_{\psi(n+1)} - x_{\psi(n)}\| + \|x_{\psi(n)} - x^*\| \to 0
$$
as $n \to \infty$. By Lemma [2.1] we have $\Gamma_n \leq \Gamma_{\psi(n)+1}$ and thus

$$
\Gamma_n = \|x_n - x^*\|^2 \leq \|x_{\psi(n)+1} - z\|^2 \to 0.
$$

So we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. \qed

4. Examples and Numerical Results

In this section, we give numerical experiments to support our main result this paper.

**Example 4.1.** Let $H_1 = H_2 = \mathbb{R}^5$. Let $f(x) := \|x\|_2$ and $g(x) := -\sum_{i=1}^{5} \log x_i$, $x = (x_1, x_2, \ldots, x_5) \in \mathbb{R}^5$. Let

$$
A = \begin{pmatrix}
5 & 7 & 10 & 5 & 8 \\
3 & 10 & 7 & 2 & 4 \\
6 & 7 & 8 & 9 & 11 \\
13 & 7 & 5 & 9 & 11 \\
11 & 13 & 15 & 3 & 7
\end{pmatrix}.
$$

We aim to find $x^* \in \arg\min f$ such that $Ax^* \in \arg\min g$.

Choose $\alpha_n = \frac{1}{\sqrt{200n+1}}$, $\rho_n = 3.95$ and $\beta_n = \min \{ 0.5, \frac{1}{n^{1.5} \|x_n - x_{n-1}\|^1} \}$ if $x_n \neq x_{n-1}$ and $\beta_n = 0.5$ if $x_n = x_{n-1}$. The stopping criteria is defined by $E_n = \|x_n - x_{n-1}\|_2 < 10^{-5}$.

Then, we obtain the following results.
Table 1. Numerical results of Example 4.1

<table>
<thead>
<tr>
<th>Case</th>
<th>Initial points</th>
<th>Target point</th>
<th>Error plots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$x_0 = (1,3,5,7,9)$, $x_1 = (11,5,3,13,7)$, $u = (3,5,11,7,11)$</td>
<td>$E_n$ for Case 1 in Example 4.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2434 CPU time</td>
<td>0.1977</td>
<td></td>
</tr>
<tr>
<td>Case 2</td>
<td>$x_0 = (11,15,5,7,19)$, $x_1 = (11,15,5,13,3)$, $u = (13,25,11,6,11)$</td>
<td>$E_n$ for Case 2 in Example 4.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2271 CPU time</td>
<td>0.1801</td>
<td></td>
</tr>
<tr>
<td>Case 3</td>
<td>$x_0 = (25,13,27,3,23)$, $x_1 = (13,25,7,15,5)$, $u = (9,11,15,8,17)$</td>
<td>$E_n$ for Case 3 in Example 4.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2262 CPU time</td>
<td>0.1714</td>
<td></td>
</tr>
<tr>
<td>Case 4</td>
<td>$x_0 = (13,13,11,7,15)$, $x_1 = (25,5,33,31,5)$, $u = (11,13,9,8,21)$</td>
<td>$E_n$ for Case 4 in Example 4.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2234 CPU time</td>
<td>0.1723</td>
<td></td>
</tr>
</tbody>
</table>

The convergence behavior of error $E_n$ for each cases are shown in Figures 1-4 respectively.
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**Figure 3.** Error plotting $E_n$ for Case 3 in Example 4.1

**Figure 4.** Error plotting $E_n$ for Case 4 in Example 4.1

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**Competing Interests**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.
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