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Research Article

# Acceleration of the Modified *S*-Algorithm to Search for a Fixed Point of a Nonexpansive Mapping

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**Abstract.** The purpose of this paper is to present accelerations of the $S$-algorithm. We first apply the Picard algorithm to the smooth convex minimization problem and point out that the Picard algorithm is the steepest descent method for solving the minimization problem. Next, we provide the accelerated Picard algorithm by using the ideas of conjugate gradient methods that accelerated the steepest descent method. Then, based on the accelerated Picard algorithm, we present accelerations of the $S$-algorithm. Under certain assumptions, our algorithm strongly converges to a fixed point with the $S$-algorithm and show that it dramatically reduces the running time and iteration needed to find a fixed point compared with that algorithm.

**Keywords.** Demicontractive mappings; Common fixed point

**MSC.** 47H09; 47H10

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# 1. Introduction

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Suppose that $C \subset \mathscr{H}$ is nonempty, closed and convex. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for al $x, y \in C$. The set of fixed point of $T$ is defined by $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$.

In this paper, we consider the following fixed point problem:

**Problem 1.1.** *Suppose that $T : C \to C$ is nonexpansive with* $\mathrm{Fix}(T) \neq \emptyset$. *Then*

$$\text{find } x^* \in C \text{ such that } Tx^* = x^*.$$

The fixed point problems for nonexpansive mapping [5, 14, 18] have been investigated in many practical application, and they include convex feasibility problem, convex optimization problems, problems of finding the zeros of monotone operators, and monotone variational inequalities.

We first apply the Picard algorithm to the smooth convex mininmization problem and illustrate that the Picard algorithm is the steepest descent method [12] have been widely seen as an efficient accelerated version of most gradient methods, we introduce an accelerated Picard algorithm by combining the conjugate gradient methods and the Picard algorithm. Finally, based on the accelerated Picard algorithm, we present accelerations of the $S$-algorithm.

In this paper, we propose two accelerated algorithms for finding a fixed point of a nonexpansive mapping and prove the convergence of the algorithms. Finally, the numerical examples are presented to demonstrate the effectiveness and fast convergence of the accelerated $S$-algorithm.

# 2. Preliminaries

## 2.1 Picard Algorithm and Our Algorithm

The Picard algorithm generates the sequence $\{x_n\}_{n=0}^{\infty}$ as follows: given $x_0 \in \mathscr{H}$,

$$x_{n+1} = Tx_n, \quad n \ge 0. \tag{2.1}$$

The Picard algorithm (2.1) converges to a fixed point of the mapping $T$ if $T : C \to C$ is contraction.

When Fix($T$) is the set of all minimizers of a convex, continuously Frechet differentiable functional $f$ over $\mathscr{H}$, that algorithm (2.1) is the steepest descent method [9, 12] to minimize $f$ over $\mathscr{H}$. Suppose that the gradient of $f$, denoted by $\nabla f$, is Lipschitz continuous with a constant $L > 0$ and define $T^f : \mathscr{H} \to \mathscr{H}$ by

$$T^f := I - \lambda \nabla f, \tag{2.2}$$

where $\lambda \in (0, 2/L)$ and $I : \mathscr{H} \to \mathscr{H}$ stands for the identity mapping. Accordingly, $T^f$ satisfies the contraction condition [7] and

$$\mathrm{Fix}(T^f) = \operatorname*{argmin}_{x\in\mathscr{H}} f(x) := \left\{x^* \in \mathscr{H} : f(x^*) = \min_{x\in\mathscr{H}} f(x)\right\}.$$

Therefore, algorithm (2.1) with $T := T^f$ can be expressed as follows:

$$\begin{cases} d_{n+1}^f := -\nabla f(x_n), \\ x_{n+1} := T^f(x_n) = x_n - \lambda \nabla f(x_n) = x_n + \lambda d_{n+1}^f. \end{cases} \tag{2.3}$$

The conjugate gradient methods [12] are popular acceleration methods of the steepest descent method. The conjugate gradient direction of $f$ at $x_n (n \geq 0)$ is

$$d_{n+1}^{f,CGD} := -\nabla f(x_n) + \beta_n d_n^{f,CGD},$$

where $d_0^{f,CGD} := -\nabla f(x_0)$ and $\{\beta_n\}_{n=0}^{\infty} \subset (0,\infty)$, which, together with (2.2), implies that

$$d_{n+1}^{f,CGD} = \frac{1}{\lambda}(T^f(x_n) - x_n) + \beta_n d_n^{f,CGD}. \tag{2.4}$$

By replacing $d_{n+1}^{f} := -\nabla f(x_n)$ in algorithm (2.3) with $d_{n+1}^{f,CGD}$ defined by (2.4), we get the accelerated Picard algorithm as follows:

$$\begin{cases} d_{n+1}^{f,CGD} := \frac{1}{\lambda}(T^f(x_n) - x_n) + \beta_n d_n^{f,CGD}, \\ x_{n+1} := x_n + \lambda d_{n+1}^{f,CGD}. \end{cases} \tag{2.5}$$

The convergence condition of Picard algorithm is very restrictive and if does not converges for general nonexpansive mapping [21]. So, In 2007, Agarwal, O'Regan and Sahu [1] introduced the $S$-iteration process

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n \end{cases} \tag{2.6}$$

and showed that the sequence generated by it converges to a fixed point of a nonexpansive mapping. In this paper, we combine (2.5)-(2.6) to present novel algorithm.

## 2.2 Some Lemmas

We will use the following notations:

**Lemma 2.1** ([6]). *Suppose that $C \subset \mathcal{H}$ is nonempty, closed and convex, $T : C \to C$ is nonexpansive mapping, and $x \in \mathcal{H}$. Then* Fix$(T)$ *is closed and convex.*

**Lemma 2.2** ([3]). *Suppose that $C \subset \mathcal{H}$ is nonempty, closed and convex, $T : C \to C$ is nonexpansive mapping, and $x \in \mathcal{H}$. Then $\hat{x} = P_C x$ if and only if $\langle x - \hat{x}, y - \hat{x}\rangle \leq 0 (y \in C)$.*

**Lemma 2.3.** *Let $\mathcal{H}$ be a real Hilbert space. There hold the following identities:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \ \forall \ x,y \in \mathcal{H}$,

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$.

**Lemma 2.4** ([19]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq a_n + u_n, \quad n \geq 0,$$

*where $\{u_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} u_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 2.5** ([13]). *Suppose that $\{x_n\}$ weakly converges to $x \in \mathcal{H}$ and $y \neq x$. Then*

$$\liminf_{n\to\infty} \|x_n - x\| \leq \liminf_{n\to\infty} \|x_n - y\|.$$

**Lemma 2.6** ([2]). *Let $\{\Psi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be the sequence in $[0,+\infty)$ such that $\Psi_{n+1} \leq \Psi_n + \alpha_n(\Psi_n - \Psi_{n-1}) + \delta_n$ for each $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number $\alpha$ with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$. Then the following conditions hold:*

(1) $\sum_{n\geq 1}[\Psi_n - \Psi_{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$*;*

(2) *there exists* $\Psi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \Psi_n = \Psi^*$.

**Lemma 2.7** ([3])**.** *Let* $D$ *be a nonempty closed convex subset of* $\mathscr{H}$ *and* $T : D \to \mathscr{H}$ *be a nonexpansive mapping. Let* $\{x_n\}$ *be a sequence in* $D$ *and* $x \in \mathscr{H}$ *such that* $x_n \rightharpoonup x$ *and* $Tx_n - x_n \to 0$ *as* $n \to +\infty$*. Then* $x \in \mathrm{Fix}(T)$.

**Lemma 2.8** ([3])**.** *Let* $C$ *be a nonempty subset of* $\mathscr{H}$ *and* $\{x_n\}$ *be a sequence in* $\mathscr{H}$ *such that the following two condition hold:*

(1) *for all* $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ *exists;*

(2) *every sequential weak cluster point of* $\{x_n\}$ *is in* $C$.

*Then the sequence* $\{x_n\}$ *converges weakly to a point in* $C$.

**Lemma 2.9** ([17])**.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* $X$*and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*. Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$*. Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Lemma 2.10** ([20])**.** *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n$$

*where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then,* $\lim_{n\to\infty} a_n = 0$.

## 3. Main Results

In this section, we present the accelerated $S$-algorithm and give its convergence.

**Algorithm 3.1.**

Step 0: Choose $\lambda > 0$ and $x_0 \in \mathscr{H}$ arbitrarily and set $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\gamma_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset [0,\infty)$. Compute $d_0 := (Tx_0 - x_0)/\alpha$.

Step 1: Given $x_n, d_n \in \mathscr{H}$, compute $d_{n+1} \in \mathscr{H}$ by

$$d_{n+1} := \frac{1}{\lambda}(Tx_n - x_n) + \beta_n d_n.$$

Step 2: Compute $x_{n+1} \in \mathscr{H}$ as follows

$$\begin{cases} y_n = x_n + \lambda d_{n+1}, \\ z_n = \gamma_n x_n + (1-\gamma_n)y_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)Tz_n. \end{cases}$$

Step 3: Put $n := n+1$, and go to Step 1.

We can check that Algorithm 3.1 coincides with the $S$-algorithm (2.6) when $\beta_n := 0$ ($n \in \mathbb{N}$).

In this section, we make the following assumption:

**Assumption 3.2.** *The sequence* $\{\beta_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *satisfy*

(A1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$;

(A2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(A3) $\sum_{n=0}^{\infty} \beta_n < \infty$ *and* $\beta_n \le \alpha_n^2$;

(A4) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Before doing the convergence analysis of Algorithm 3.1, we first show the four lemmas:

**Lemma 3.3.** *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \varnothing$ *and that Assumption* 3.2 *holds. Then* $\{d_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *are bounded.*

*Proof.* We have from (A3) that $\lim_{n\to\infty} \beta_n = 0$. Accordingly, there exists $n_0 \in \mathbb{N}$ such that $\beta_n \le \frac{1}{2}$ for all $n \ge n_0$. Define

$$M_1 = \max\left\{ \max_{1\le k\le n_0} \|d_k\|, (2/\lambda) \sup_{n\in\mathbb{N}} \|Tx_n - x_n\| \right\}.$$

Then (A3) implies that $M_1 < \infty$. Assume that $\|d_n\| \le M_1$ for some $n \ge n_0$. The triangle inequality ensure that

$$\begin{aligned}
\|d_{n+1}\| &= \left\| \frac{1}{\lambda}(Tx_n - x_n) + \beta_n d_n \right\| \\
&\le \frac{1}{\lambda}\|Tx_n - x_n\| + \beta_n \|d_n\| \\
&\le M_1
\end{aligned}$$

which means that $\|d_n\| \le M_1$ for all $n \ge 0$, i.e., $\{d_n\}_{n=0}^{\infty}$ is bounded. The definition of $\{y_n\}_{n=0}^{\infty}$ implies that

$$\begin{aligned}
y_n &= x_n + \lambda\left( \frac{1}{\lambda}(Tx_n - x_n) + \beta_n d_n \right) \\
&= Tx_n + \lambda \beta_n d_n.
\end{aligned} \tag{3.1}$$

The nonexpansive of $T$ and (3.1) implies that, for any $p \in \mathrm{Fix}(T)$ and for all $n \ge n_0$,

$$\begin{aligned}
\|y_n - p\| &= \|Tx_n + \lambda\beta_n d_n - p\| \\
&\le \|Tx_n - p\| + \lambda\beta_n \|d_n\| \\
&\le \|x_n - p\| + \lambda M_1 \beta_n
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\|z_n - p\| &= \|\gamma_n x_n - (1-\gamma_n) y_n - p\| \\
&= \|\gamma_n (x_n - p) + (1-\gamma_n)(y_n - p)\| \\
&\le \gamma_n \|x_n - p\| + (1-\gamma_n)\|y_n - p\|.
\end{aligned} \tag{3.3}$$

Therefore, we find

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n y_n+(1-\alpha_n)Tz_n-p\| \\
&= \|\alpha_n(y_n-p)+(1-\alpha_n)(Tz_n-p)\| \\
&\le \alpha_n\|y_n-p\|+(1-\alpha_n)\|z_n-p\| \\
&\le \alpha_n\|y_n-p\|+(1-\alpha_n)\gamma_n\|x_n-p\|+(1-\alpha_n)(1-\gamma_n)\|y_n-p\| \\
&\le (1-(1-\alpha_n)\gamma_n))\|y_n-p\|+(1-\alpha_n)\gamma_n\|x_n-p\| \\
&\le (1-\alpha_n)\gamma_n\|x_n-p\|+(1-(1-\alpha_n)\gamma_n))(\|x_n-p\|+\lambda M_1\beta_n) \\
&\le \|x_n-p\|+\lambda M_1\beta_n
\end{aligned} \tag{3.4}$$

which implies

$$\|x_n-p\|\le\|x_0-p\|+\lambda M_1\sum_{n=0}^{n-1}\beta_k<\infty.$$

So, we get that $\{x_n\}_{n=0}^{\infty}$ is bounded. From (3.2) and (3.3) it follows that $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ are bounded. □

In addition, using Lemma 2.4, (A3) and (3.4), we obtain $\lim_{n\to\infty}\|x_n-p\|$ exists.

**Lemma 3.4.** *Suppose that $T:\mathscr{H}\to\mathscr{H}$ is nonexpansive with* $\mathrm{Fix}(T)\neq\varnothing$ *and Assumption* 3.2 *holds. Then*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

*Proof.* Equation (3.1), the triangle inequality, and the nonexpansive of $T$ imply that, for all $n\in\mathbb{N}$,

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|Tx_{n+1}-Tx_n+\lambda(\beta_{n+1}d_{n_1}-\beta_n d_n)\| \\
&\le \|Tx_{n+1}-Tx_n\|+\lambda\|\beta_{n+1}d_{n_1}-\beta_n d_n\| \\
&\le \|x_{n+1}-x_n\|+\lambda(\beta_{n+1}\|d_{n_1}\|-\beta_n\|d_n\|)
\end{aligned}$$

which, together with $\|d_n\|\le M_1(n\ge n_0)$ and (A3), implies that, for all $n\ge n_0$,

$$\|y_{n+1}-y_n\|\le\|x_{n+1}-x_n\|+\lambda M_1(\alpha_{n+1}^2+\alpha_n^2). \tag{3.5}$$

On the other hand, from $\alpha_n\le|\alpha_{n+1}-\alpha_n|\le\alpha_{n+1}$ and $\alpha_n<1$ ($n\in\mathbb{N}$), we have that, for all $n\in\mathbb{N}$,

$$\begin{aligned}
\alpha_{n+1}^2+\alpha_n^2 &\le \alpha_{n+1}^2+\alpha_n(|\alpha_{n+1}-\alpha_n|+\alpha_{n+1}) \\
&\le (\alpha_{n+1}+\alpha_n)\alpha_{n+1}+|\alpha_{n+1}-\alpha_n|.
\end{aligned} \tag{3.6}$$

Next, by Algorithm 3.1, the triangle inequality, and the nonexpansive of $T$ imply that, for all $n\in\mathbb{N}$,

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|\gamma_{n+1}x_{n+1}+(1-\gamma_{n+1})Ty_{n+1}-\gamma_n x_n-(1-\gamma_n)Ty_n\| \\
&= \|\gamma_{n+1}x_{n+1}-\gamma_{n+1}x_n+\gamma_{n+1}x_n+(1-\gamma_{n+1})Ty_{n+1} \\
&\quad -(1-\gamma_{n+1})Ty_n+(1-\gamma_{n+1})Ty_n-\gamma_n x_n-(1-\gamma_n)Ty_n\| \\
&\le \gamma_{n+1}\|x_{n+1}-x_n\|+(1-\gamma_{n+1})\|y_{n+1}-y_n\|+|\gamma_{n+1}-\gamma_n|\|x_n\|+|\gamma_{n+1}-\gamma_n|\|y_n\| \\
&= \gamma_{n+1}\|x_{n+1}-x_n\|+(1-\gamma_{n+1})\|y_{n+1}-y_n\|+|\gamma_{n+1}-\gamma_n|(\|x_n\|+\|y_n\|) \\
&\le \gamma_{n+1}\|x_{n+1}-x_n\|+(1-\gamma_{n+1})\|y_{n+1}-y_n\|+M_2|\gamma_{n+1}-\gamma_n|
\end{aligned} \tag{3.7}$$

which $M_2:=\sup_{n\in\mathbb{N}}(\|x_n\|+\|y_n\|)<\infty$.

Setting $x_{n+1} = (1-\beta_n)w_n + \beta_n x_n$ for all $n \geq 0$, we see that $w_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, then

$$\begin{aligned}
\|w_{n+1}-w_n\| &= \left\| \frac{x_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{1}{1-\beta_{n+1}} \Big(\alpha_{n+1}y_{n+1} - (1-\alpha_{n+1})Tz_{n+1} - \beta_{n+1}\Big(\alpha_n y_n - (1-\alpha_n)Tz_n\Big)\Big) \right. \\
&\quad \left. - \frac{1}{1-\beta_n}\Big(\alpha_n y_n + (1-\alpha_n)Tz_n + \beta_n x_n\Big) \right\| \\
&= \left\| \frac{1}{1-\beta_{n+1}} \Big(\alpha_{n+1}y_{n+1} - \alpha_{n+1}y_n + \alpha_{n+1}y_n - (1-\alpha_{n+1})Tz_{n+1} \right. \\
&\quad + (1-\alpha_{n+1})Tz_n - (1-\alpha_{n+1})Tz_n - \beta_{n+1}\Big(\alpha_n y_n - (1-\alpha_n)Tz_n\Big)\Big) \\
&\quad \left. - \frac{1}{1-\beta_n}\Big(\alpha_n y_n + (1-\alpha_n)Tz_n + \beta_n x_n\Big) \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\| + \frac{(1-\alpha_{n+1})}{1-\beta_{n+1}}\|z_{n+1}-z_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\beta_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_{n+1}} \right| \|y_n\| \\
&\quad + \left| \frac{(1-\alpha_{n+1})}{1-\beta_{n+1}} - \frac{(1-\alpha_n)\beta_{n+1}}{1-\beta_{n+1}} - \frac{(1-\alpha_n)}{1-\beta_{n+1}} \right| \|Tz_n\| + \frac{\beta_n}{1-\beta_n}\|x_n\| \qquad (3.8)
\end{aligned}$$

From (3.5) and (3.7), we have

$$\begin{aligned}
\|w_{n+1}-w_n\| &\leq \frac{\beta_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \|x_{n+1}-x_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\beta_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_{n+1}} \right| \|y_n\| \\
&\quad + \left| \frac{(1-\alpha_{n+1})}{1-\beta_{n+1}} - \frac{(1-\alpha_n)\beta_{n+1}}{1-\beta_{n+1}} - \frac{(1-\alpha_n)}{1-\beta_{n+1}} \right| \|Tz_n\| + \frac{\beta_n}{1-\beta_n}\|x_n\| \\
&\quad + \lambda M_1(1-(1-\alpha_{n+1})\gamma_{n+1})\Big((\alpha_{n+1}-\alpha_n)\alpha_{n+1} + |\alpha_{n+1}-\alpha_n|\Big). \qquad (3.9)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\| &\leq \frac{\beta_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\
&\quad + \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\beta_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_{n+1}} \right| \|y_n\| \\
&\quad + \left| \frac{(1-\alpha_{n+1})}{1-\beta_{n+1}} - \frac{(1-\alpha_n)\beta_{n+1}}{1-\beta_{n+1}} - \frac{(1-\alpha_n)}{1-\beta_{n+1}} \right| \|Tz_n\| + \frac{\beta_n}{1-\beta_n}\|x_n\| \\
&\quad + \lambda M_1(1-(1-\alpha_{n+1})\gamma_{n+1})\Big((\alpha_{n+1}-\alpha_n)\alpha_{n+1} + |\alpha_{n+1}-\alpha_n|\Big). \qquad (3.10)
\end{aligned}$$

It follows from the condition (A1), (A2) and (A4), that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\|) \leq 0. \qquad (3.11)$$

Applying Lemma 2.9, we obtain $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and we also have

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|w_n - x_n\| \to 0, n \to \infty. \qquad (3.12)$$

Hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

□

**Lemma 3.5.** *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \varnothing$ *and Assumption* 3.2 *holds. Then*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

*Proof.* By Algorithm 3.1, and the nonexpansive of $T$, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|\alpha_n y_n + (1-\alpha_n)Tz_n - y_n\| \\ &= (1-\alpha_n)\|Tz_n - y_n\| \\ &\leq (1-\alpha_n)M_3\end{aligned}$$

by (A1) means that $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. Since the triangle inequality ensures that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|$$

for all $n \in \mathbb{N}$, we find from (3.13) that

$$\lim_{n\to\infty} \|d_{n+1}\| = \frac{1}{\lambda} \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.14}$$

From the definition of $d_{n+1} (n \in \mathbb{N})$, we have, for all $n \geq n_0$,

$$0 \leq \frac{1}{\lambda}\|Tx_n - x_n\| \leq \|d_{n+1}\| + \beta_n \|d_n\| \leq \|d_{n+1}\| + M_1\beta_n.$$

Since equation (3.14), and $\lim_{n\to\infty} \beta_n = 0$ guarantee that the right side of above inequality converges to 0, we find that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.15}$$

□

**Lemma 3.6.** *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \varnothing$ *and Assumption* 3.2 *holds. Then*

$$\limsup_{n\to\infty} \langle x_n - p, y_n - p\rangle \leq 0 \quad \textit{where} \quad p = P_{\mathrm{Fix}(T)} x_n.$$

*Proof.* From the limit superior of $\{\langle x_n - p, y_n - p\rangle\}_{n=0}^{\infty}$, there exists $\{y_{n_k}\}_{n=0}^{\infty}$ such that

$$\limsup_{n\to\infty} \langle x_n - p, y_n - p\rangle = \lim_{k\to\infty} \langle x_n - p, y_{n_k} - p\rangle. \tag{3.16}$$

Moreover, since $\{y_{n_k}\}_{n=0}^{\infty}$ is bounded, there exists $\{y_{n_{k_i}}\}_{n=0}^{\infty}$ which weakly converges to some point $q \in \mathscr{H}$. Equation (3.14), guarantees that $\{x_{n_{k_i}}\}_{n=0}^{\infty}$ weakly converges to $q$.

We shall show that $q \in \mathrm{Fix}(T)$. Assume that $q \notin \mathrm{Fix}(T)$ that is $q \neq Tq$. Lemma 2.5, (3.15), and the nonexpansive of $T$ ensure that

$$\begin{aligned}\liminf_{i\to\infty} \left\|x_{n_{k_i}} - q\right\| &< \liminf_{i\to\infty} \left\|x_{n_{k_i}} - Tq\right\| \\ &= \liminf_{i\to\infty} \left\|x_{n_{k_i}} - Tx_{n_{k_i}} + Tx_{n_{k_i}} - Tq\right\| \\ &= \liminf_{i\to\infty} \left\|Tx_{n_{k_i}} - Tq\right\| \\ &\leq \liminf_{i\to\infty} \left\|Tx_{n_{k_i}} - q\right\|.\end{aligned}$$

This is a contradiction. Hence, $q \in \mathrm{Fix}(T)$. Hence, (3.16), and Lemma 2.2 guarantee that

$$\limsup_{n\to\infty}\langle x_n - p, y_n - p\rangle = \lim_{k\to\infty}\langle x_n - p, y_{n_k} - p\rangle = \langle x_n - p, q - p\rangle \le 0.$$

This completes the proof. □

**Theorem 3.7.** *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is nonexpansive with* $\mathrm{Fix}(T) \neq \varnothing$ *and that Assumption* 3.2 *holds. Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *strongly converges to a fixed point of* $T$.

*Proof.* The inequality, $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x-y\rangle (x,y \in \mathscr{H})$, and the nonexpansive of $T$ imply that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - p\|^2 &= \|Tx_n - p + \lambda\beta_n d_n\| \\ &\le \|Tx_n - p\|^2 + 2\lambda\beta_n\langle y_n - p, d_n\rangle \\ &\le \|x_n - p\| + M_4\alpha_n^2,\end{aligned}$$

where $\beta_n \le \alpha_n^2 (n \in \mathbb{N})$ and $M_4 := \sup_{n\in\mathbb{N}} 2\lambda|\langle y_n - p, d_n\rangle| < \infty$ and

$$\begin{aligned}\|Tz_n - p\|^2 &\le \|z_n - p\|^2 \\ &\le \|\gamma_n x_n + (1-\gamma_n)y_n - p\|^2 \\ &\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|y_n - p\|^2 + 2\gamma_n(1-\gamma_n)\langle x_n - p, y_n - p\rangle.\end{aligned}$$

We have that, for all $n \in \mathbb{N}$

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n y_n + (1-\alpha_n)Tz_n - p\|^2 \\ &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n(1-\alpha_n)\langle y_n - p, Tz_n - p\rangle \\ &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle y_n - p, (x_{n+1} - p) - \alpha_n(y_n - p)\rangle \\ &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle - 2\alpha_n^2\|y_n - p\|^2 \\ &\le (\alpha_n - 2\alpha_n^2)\|y_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle \\ &\le (\alpha_n - 2\alpha_n^2)\|y_n - p\|^2 + (1-\alpha_n)\gamma_n\|x_n - p\|^2 + (1-\alpha_n)(1-\gamma_n)\|y_n - p\|^2 \\ &\quad + 2(1-\alpha_n)\gamma_n(1-\gamma_n)\langle x_n - p, y_n - p\rangle + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle \\ &\le (1 - 2\alpha_n^2 - \gamma_n + \alpha_n\gamma_n)\|y_n - p\|^2 + (1-\alpha_n)\gamma_n\|x_n - p\|^2 \\ &\quad + 2(1-\alpha_n)\gamma_n(1-\gamma_n)\langle x_n - p, y_n - p\rangle + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle \\ &\le (1 - 2\alpha_n^2 - \gamma_n + \alpha_n\gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\gamma_n\|x_n - p\|^2 \\ &\quad + M_4(1 - 2\alpha_n^2 - \gamma_n + \alpha_n\gamma_n)\alpha_n^2 + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle \\ &\quad + 2(1-\alpha_n)\gamma_n(1-\gamma_n)\langle x_n - p, y_n - p\rangle \\ &\le (1 - 2\alpha_n^2)\|x_n - p\|^2 + M_4(1 - 2\alpha_n^2 - \gamma_n + \alpha_n\gamma_n)\alpha_n^2 \\ &\quad + 2\alpha_n\langle y_n - p, x_{n+1} - p\rangle + 2(1-\alpha_n)\gamma_n(1-\gamma_n)\langle x_n - p, y_n - p\rangle. \qquad (3.17)\end{aligned}$$

From Lemma 2.10, (A2) and Lemma 3.6 lead one to deduce that

$$\lim_{n\to\infty}\|x_{n+1} - p\|^2 = 0. \qquad (3.18)$$

This guarantees that $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 3.1 strongly converges to $p = P_{\mathrm{Fix}(T)}x_n$. □

## 4. Numerical Examples and Conclusion

In this section, we compare the original algorithm and the accelerated algorithm. The codes were written in MATLAB 8.0 and run personal computer.

Firstly, we apply the $S$-algorithm (2.6) and Algorithm 3.1 to the following *convex feasibility problem* (CFP).

**Problem 4.1** (From [15]). Given a nonempty

$$\text{find } x^* \in C := \bigcap_{i=0}^{m} C_i,$$

where one assume that $C \neq \varnothing$. Define a mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ by

$$T := P_0\Big(\frac{1}{m}\sum_{i=1}^{m} P_i\Big), \tag{4.1}$$

where $P_i = P_{C_i}$ $(i = 0, 1, \dots, m)$ stands for the metric projection onto $C_i$. Since $P_i$ $(i = 0, 1, 2, \dots, m)$ is nonexpansive, $T$ defined by (4.1) is also nonexpansive. Moreover, we find that

$$\mathrm{Fix}(T) = \mathrm{Fix}(P_0)\bigcap_{i=1}^{m}\mathrm{Fix}(P_i) = C_0\bigcap_{i=1}^{m} C_i = C.$$

Set $\lambda = 0.5, \alpha_n = 1/(n+1), \gamma_n = 1/(2n+1) (n \geq 0)$ and $\beta_n = (3n+1)$ in Algorithm 3.1 in the $S$-algorithm (2.6). In the experiment, we set $C_i$ $(i = 0, 1, \dots, m)$ as a closed ball with center $c_i \in \mathbb{R}^N$ and radius $r_i > 0$. Thus, $P_i$ $(i = 0, 1, \dots, m)$ can be computed with

$$P_i(x) := \begin{cases} c_i + \frac{r_i}{\|c_i - x\|}(x - c_i), & \text{if } \|c_i - x\| > r_i, \\ x, & \text{if } \|c_i - x\| \leq r_i. \end{cases}$$

We set $r_i := 1$ $(i = 0, 1, \dots, m)$, $c_0 := 0$ and $c_i \in (-1/\sqrt{N}, 1/\sqrt{N})^N$ $(i = 1, \dots, m)$ were randomly chosen. Set $e := (1, 1, \dots, 1)$. In Table 1, "Iter." and "Sec." denote the number of iterations and the cpu time in second, respectively. We took $\|Tx_n - x_n\| < \epsilon = 10^{-6}$ as the stopping criterion.

Table 1 illustrates that, with a few exceptions, Algorithm 3.1 significantly reduces the running time and iteration needed to find a fixed point compared with the $S$-algorithm. The advantage is more obvious, as the parameters $N$ and $m$ become larger. It is worth further research on the reason of emergence of a few exceptions.

**Table 1.** Computational results for Problem 4.1 with different dimensions

| | Initial point | | rand$(N,1)$ | 200×rand$(N,1)$ | 5e | 5,000e |
|---|---|---|---|---|---|---|
| $N = 5$ | Algorithm 3.1 | Iter | 2 | 50 | 48 | 48 |
| $m = 5$ | | Sec. | 0.0156 | 0 | 0.0156 | 0 |
| | $S$-Iteration | Iter | 5158 | 131082 | 90978 | 64604 |
| | | Sec. | 0.5313 | 1.9219 | 1.7031 | 1.1563 |
| $N = 100$ | Algorithm 3.1 | Iter | 565 | 681 | 677 | 566 |
| $m = 50$ | | Sec. | 0.3750 | 0.4688 | 0.2656 | 0.1719 |
| | $S$-Iteration | Iter | 321565 | 239882 | 247063 | 248627 |
| | | Sec. | 71.9844 | 60.6406 | 79.2031 | 81.9844 |
| $N = 50$ | Algorithm 3.1 | Iter | 379 | 347 | 339 | 354 |
| $m = 100$ | | Sec. | 0.0781 | 0.0781 | 0.0625 | 0.0781 |
| | $S$-Iteration | Iter | 181627 | 195724 | 182802 | 181870 |
| | | Sec. | 31.8125 | 36.8438 | 33.9688 | 31.2656 |

In the experiment, we compare the error (Err) values under the different number of iterations. Figure 1, Figure 2 and Figure 3 show that, comparing with Algorithm 3.1, the $S$-algorithm (2.6) has obvious advantages in computing.

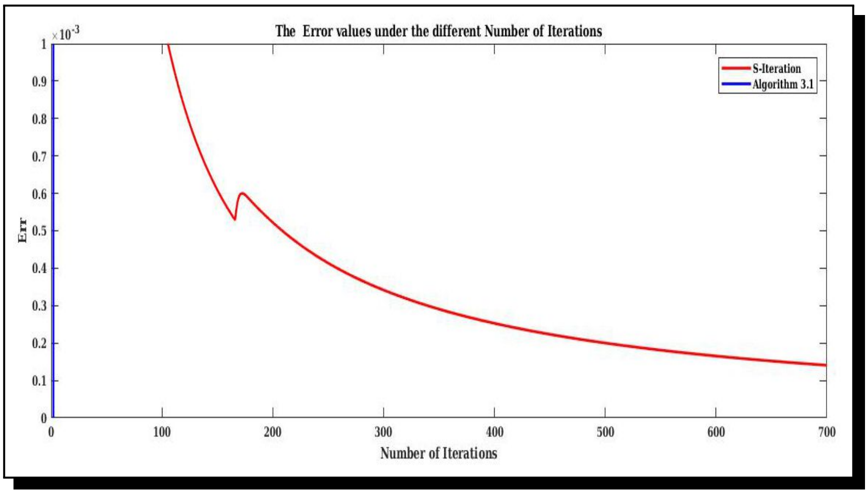


**Figure 1.** Comparison of the number of iterations of $N = 5$, $m = 5$

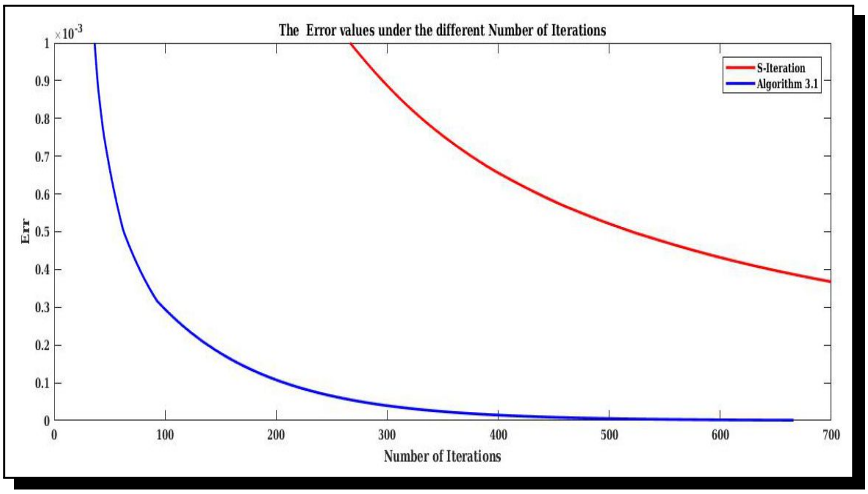


**Figure 2.** Comparison of the number of iterations of $N = 50$, $m = 100$

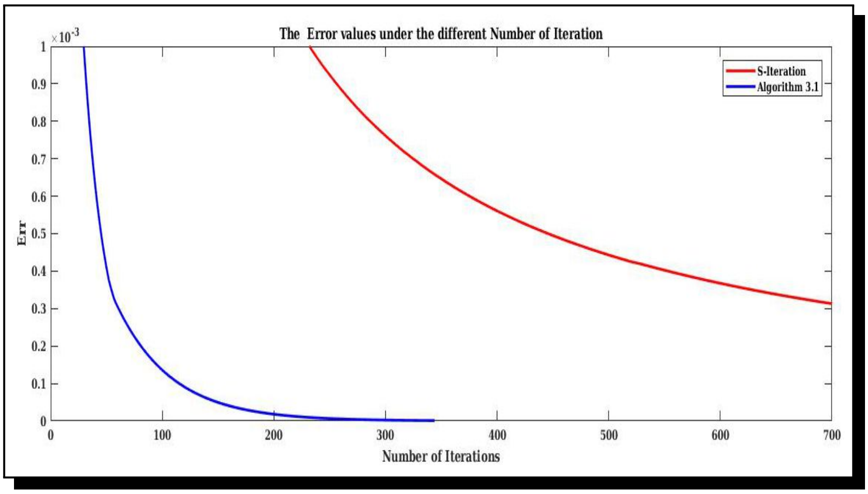


**Figure 3.** Comparison of the number of iterations of $N = 100$, $m = 50$

## 5. Conclusion

In this paper, we accelerated $S$-algorithm. Then we present the strong convergence of the accelerated $S$-algorithm and the strong convergence under some conditions. The numerical example illustrate that the acceleration of the $S$-algorithm is effective.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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