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When Every Gorenstein Projective (Resp. Flat) Module is Strongly Gorenstein Projective (Resp. Flat)

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Abstract. In [5], the authors discuss the rings over which all modules are strongly Gorenstein projective. In this paper, we are interesting to an extension of this idea. Thus, we discuss the rings over which every Gorenstein projective (resp. flat) module is strongly Gorenstein projective (resp. flat). Our aim is to give examples of rings with different Gorenstein global dimension satisfied this condition.

1. **Introduction**

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Setup and notation:

Let R be a ring, and let M be an R-module. As usual we use $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ to denote, respectively, the classical projective, injective and flat dimensions of M. By $gldim(R)$ and $wdim(R)$ we denote, respectively, the classical global and weak dimensions of R.

It is by now a well-established fact that even if R to be non-Noetherian, there exists Gorenstein projective, injective and flat dimensions of M, which are usually denoted by $Gpd_R(M)$, $Gid_R(M)$, and $Gfd_R(M)$, respectively. Some references are [2, 3, 8, 9, 11, 12, 13, 15].

Recently in [3], the authors started the study of global Gorenstein dimensions of rings, which are called, for a commutative ring R, Gorenstein global projective, injective, and weak dimensions of R, denoted by $GPD(R)$, $GID(R)$, and $G.wdim(R)$, respectively, and, respectively, defined as follows:

(i) $GPD(R) = \sup\{Gpd_R(M) \mid M\ R\text{-module}\}$.
(ii) $GID(R) = \sup\{Gid_R(M) \mid M\ R\text{-module}\}$.

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(iii) $G.wdim(R) = \sup\{Gfd_R(M) \mid M\ R\text{-module}\}$.

They proved that, for any ring R, $G.wdim(R) \leq GID(R) = GPD(R)$ ([3, Theorems 2.1 and 2.11]). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of $GPD(R)$ and $GID(R)$ is called Gorenstein global dimension of R, and denoted by $G.gldim(R)$.

They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That is: $G.gldim(R) \leq gldim(R)$ and $G.wdim(R) \leq wdim(R)$ with equality if $wdim(R)$ is finite ([3, Propositions 2.12]).

In [2], the authors studied a particulars cases of a Gorenstein projective, injective and flat modules which they call a strongly Gorenstein projective, injective and flat modules respectively, and defined as follows:

Definition 1.1. (i) A module M is said to be strongly Gorenstein projective (SG-projective for short), if there exists an exact sequence of projective modules of the form:

$$\mathbf{P} = \cdots \to P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$$

such that $M \cong \mathfrak{I}(f)$ and such that $\mathrm{Hom}(-, P)$ leaves the sequence $\mathbf{P}$ exact whenever P is projective.

The exact sequence $\mathbf{P}$ is called a strongly complete projective resolution.

(ii) The strongly Gorenstein injective modules are defined dually.

(iii) A module M is said to be strongly Gorenstein flat (SG-flat for short), if there exists an exact sequence of flat module of the form:

$$\mathbf{F} = \cdots \to F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \to \cdots$$

such that $M \cong \mathfrak{I}(f)$ and such that $- \otimes I$ leaves $\mathbf{F}$ exact whenever I is injective.

The exact sequence $\mathbf{F}$ is called a strongly complete flat resolution.

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of a Gorenstein projective and injective modules, respectively, as follows:

Theorem 1.2 ([2, Theorem 2.7]). *A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*

Using [2, Theorem 3.5] together with [15, Theorem 3.7], we have the next result:

Proposition 1.3. *Let R be a coherent ring. A module is Gorenstein flat if, and only if, it is a direct summand of a strongly Gorenstein flat module.*

Notation. By $\mathscr{P}(R)$, $\mathscr{I}(R)$ and $\mathscr{F}(R)$ we denote the classes of all projective, injective and fat R-modules respectively and by $\mathscr{G}\mathscr{P}(R)$, $\mathscr{G}\mathscr{I}(R)$ and $\mathscr{G}\mathscr{F}(R)$ denote the classes of all strongly Gorenstein projective, injective and fat R-modules respectively. Furthermore, we let $\mathscr{SGP}(R), \mathscr{SGI}(R)$ and $\mathscr{SGF}(R)$ denote the classes of all Gorenstein projective, injective and flat R-modules, respectively.

In many place of this paper we use the notion of resolving class. This notion was be introduced by Holm in [15] as follows:

Definition 1.4. (Definition 1.1,[15]). For any class $\mathscr{X}$ of R-modules.

a/: We call $\mathscr{X}$ projectively resolving if $\mathscr{P}(R) \subseteq \mathscr{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathscr{X}$ the conditions $X' \in \mathscr{X}$ and $X \in \mathscr{X}$ are equivalent.

b/: We call $\mathscr{X}$ injectively resolving if $\mathscr{I}(R) \subseteq \mathscr{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathscr{X}$ the conditions $X'' \in \mathscr{X}$ and $X \in \mathscr{X}$ are equivalent.

In [15] again, Holm prove that the class $\mathscr{GP}(\mathscr{R})$ is projectively resolving and closed under arbitrary direct sums and under direct summands ([15, Theorem 2.5]), and dually, the class $\mathscr{GI}(\mathscr{R})$ is injectively resolving and closed under arbitrary direct products and under direct summands ([15, Theorem 2.6]). He also prove that, if R is coherent, then the class $\mathscr{GF}(\mathscr{R})$ is projectively resolving and closed under direct summands ([15, Theorem 3.7]).

For any ring R, it is clear that we have the following inclusions of classes:

$$\mathscr{P}(R) \subseteq \mathscr{SGP}(R) \subseteq \mathscr{GP}(R).$$

We will see after that the inverse inclusions are not true in general. Indeed, using [2, Theorem 2.7], the inclusion $\mathscr{SGP}(R) \subseteq \mathscr{P}(R)$ means that $G.gldim(R) = gldim(R)$ and so we are in the classical case.

In this paper, we are interesting to discuss the equality of classes $\mathscr{SGP}(R) = \mathscr{GP}(R)$ i.e., when every Gorenstein projective module is strongly Gorenstein projective. A trivial example is when the global dimension of R is finite. In the second part of the second section, we are also interesting to the equality $\mathscr{SGF}(R) = \mathscr{GF}(R)$ over a coherent ring R. In many place of this papers, we use the following Lemma.

Lemma 1.5. *Consider the following diagram of modules aver a ring R.*

$$(\star)\qquad \begin{array}{ccccccccc} 0 & \to & M & \stackrel{\alpha}{\to} & P & \stackrel{\beta}{\to} & M & \to & 0 \\ & & {\scriptstyle u}\downarrow & & & & {\scriptstyle u}\downarrow & & \\ 0 & \to & Q & \stackrel{\iota}{\to} & Q \oplus Q & \stackrel{j}{\to} & Q & \to & 0 \end{array}$$

where M is a Gorenstein projective module, P and Q are projective and ι and j are the canonical injection and projection respectively. Then, there is a morphism $\gamma : P \to Q \oplus Q$ which complete $(\star)$ and make it commutative.

Proof. If we apply the functor $\mathrm{Hom}(-, Q)$ to the short exact sequence

$$(*)\qquad 0 \to M \stackrel{\alpha}{\to} P \stackrel{\beta}{\to} M \to 0$$

we obtain the short exact sequence:

$$(**)\qquad 0 \to \mathrm{Hom}(M,Q) \xrightarrow{\circ\beta} \mathrm{Hom}(P,Q) \xrightarrow{\circ\alpha} \mathrm{Hom}(M,Q) \to 0\,.$$

Since $\mathrm{Ext}(M,Q) = 0$ ([15, Proposition 2.3]). On the other hand, $u \in \mathrm{Hom}(M,Q)$. Then, from the exactness of $(**)$, there is a morphism $\upsilon : P \to Q$ such that $\upsilon \circ \alpha = u$. Consequently, we can verified that the morphism $\gamma : P \to Q \oplus Q$ defined by $\gamma(p) := (\upsilon(p), u \circ \beta(p))$ whenever $p \in P$ is the desired morphism. □

Dually, one can find easily the injective version of Lemma 1.5.

The aim of this paper is to construct a family of rings $\{R_i\}_i$ over which every Gorenstein projective module is strongly Gorenstein projective and such that $G.gldim(R_i) = i$ and $wdim(R_i) = \infty$.

2. **Main results**

Now, we give our main first result.

Theorem 2.1. *Let R be a ring. The following are equivalents:*

(i) *Every Gorenstein projective module is strongly Gorenstein projective.*

(ii) *For every module M with* $\mathrm{Gpd}(M) \leq 1$*, there exists a short exact sequence*

$$0 \to M \to Q \to M \to 0$$

where $\mathrm{pd}(Q) \leq 1$.

Proof. Assume (i) and we claim (ii). Let M be a module such that $Gpd(M) \leq 1$ and pick a short exact sequence $0 \to G \to Q \to M \to 0$ where Q is projective and G is Gorenstein projective module (then strongly Gorenstein projective by the hypothesis condition). Thus, from [2, Proposition 2.9], there exists a short exact sequence $0 \to G \to P \to G \to 0$ where P is projective. So, using Lemma 1.5, there is a morphism $\gamma : P \to Q \oplus Q$ such the following diagram is commutative:

$$(\star)\qquad \begin{array}{ccccccccc} 0 & \to & G & \to & P & \to & G & \to & 0 \\ & & \downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow & & \\ 0 & \to & Q & \xrightarrow{\iota} & Q \oplus Q & \xrightarrow{j} & Q & \to & 0 \end{array}$$

Hence, applying the Snake Lemma to this diagram and since $M \cong \mathrm{coker}(G \to Q)$, we obtain a short exact sequence of the form $0 \to M \to X \to M \to 0$ where $X \cong \mathrm{coker}(\gamma)$. Clearly, we have $pd(X) \leq 1$, as desired.

Conversely, we assume (ii) and we claim (i). Let M be a Gorenstein projective module. By the hypothesis condition, there is an exact sequence $(\star)$ $0 \to M \to Q \to M \to 0$ where $pd(Q) \leq 1$. But the class $\mathcal{GP}(R)$ is projectively resolving (by [15, Theorem 2.5]). Hence, from $(\star)$, Q is also Gorenstein projective. Consequently, from [15, Proposition 2.27], Q is projective. On the other hand, since M is Gorenstein projective, for any projective module P and any integer $i > 0$, $\mathrm{Ext}^i(M,P) = 0$. Thus, from [2, Proposition 2.9], M is strongly Gorenstein projective, as desired. □

In the next, we give example of rings over which $\mathcal{SGP}(-) = \mathcal{GP}(-)$ or different.

Example 2.2. Let D be a principal ideal domain and $\mathcal{P}$ a nonzero prime ideal of D. Consider the rings $R = D/\mathcal{P}^2$ and $S = D/\mathcal{P}^3$. Then,

(i) $\mathcal{SGP}(R) = \mathcal{GP}(R)$, and

(ii) $\mathcal{SGP}(S) \nsubseteq \mathcal{GP}(S)$.

Proof. (i) Follows immediately from [5, Corollary 3.9]

(ii) From [5, Corollary 3.10], $\mathcal{SGP}(S) \neq \mathcal{M}(S) = \mathcal{GP}(S)$ (where $\mathcal{M}(S)$ is the class of all S-modules). So, we have the desired strict inclusion. □

From [5, Corollary 3.9 and 3.10], the rings R and S of Example 2.2 have zero Gorenstein Global dimension. In order to move up this $G.gldim$, we past by the direct product of rings as shown by the next result.

Theorem 2.3. *Let $\{R_i\}_{i=1,\dots,m}$ be a family of rings with finite Gorenstein global dimensions. Then, $\mathcal{SGP}\left(\prod_{i=1}^{m} R_i\right) = \mathcal{GP}\left(\prod_{i=1}^{m} R_i\right)$, if and only if, $\mathcal{SGP}(R_i) = \mathcal{GP}(R_i)$ for each $i = 1, \dots, m$.*

Proof. By induction on m, we may assume $m = 2$.

Assume that $\mathcal{SGP}(R_1 \times R_2) = \mathcal{GP}(R_1 \times R_2)$ and let M be a G-projective R_1-module. We claim that M is a strongly Gorenstein projective R_1-module. Obviously, $M \times 0$ is an $R_1 \times R_2$-module (see [6, p. 101]). First, we claim that $M \times 0$ is a G-projective $R_1 \times R_2$-module. The R_1-module M is a direct summand of an SG-projective R_1-module N ([2, Theorem 2.7]). For such module, there is a short exact sequence of R_1-modules $0 \to N \to P \to N \to 0$ where P is a projective R_1-module ([2, Proposition 2.9]). Hence, we have a short sequence of $R_1 \times R_2$-modules $(*)\quad 0 \to N \times 0 \to P \times 0 \to N \times 0 \to 0$ and $P \times 0$ is a projective $R_1 \times R_2$-module. But $G.gldim(R_1 \times R_2)$ is finite (by [4, Theorem 3.1]). Then, there is an integer $i > 0$ such that $\mathrm{Ext}^i_{R_1 \times R_2}(N \times 0, Q) = 0$ for each projective $R_1 \times R_2$-module Q (since $Gpd_{R_1 \times R_2}(N \times 0) < \infty$ and by [15, Theorem 2.20]). From $(*)$ we deduce that $\mathrm{Ext}_{R_1 \times R_2}(N \times 0, Q) = 0$. Then, by [2, Proposition 2.9], $N \times 0$ is an SG-projective $R_1 \times R_2$-module. So, $M \times 0$ is a G-projective $R_1 \times R_2$-module (since it is a direct summand of $N \times 0$ as an $R_1 \times R_2$-modules and by [15, Theorem 2.5]. Now, we claim that M is an SG-projective R_1-module. By hypothesis, the $R_1 \times R_2$-module $M \times 0$ is SG-projective (since $M \in \mathcal{GP}(R_1 \times R_2) = \mathcal{SGP}(R_1 \times R_2)$). Thus, there exists a short exact sequence of $R_1 \times R_2$-modules $(\star)\quad 0 \to M \times 0 \to P \to M \times 0 \to 0$ where P is a projective $R_1 \times R_2$-module. Now, tensor $(\star)$ by $- \otimes R_1$ we obtain the short exact sequence of R_1-modules (see that R_1 is a projective $R_1 \times R_2$-module)

$$(\star\star)\qquad 0 \to (M \times 0) \otimes_{R_1 \times R_2} R_1 \to P \otimes_{R_1 \times R_2} R_1 \to M \times 0 \otimes_{R_1 \times R_2} R_1 \to 0\,.$$

But $(M \times 0) \otimes_{R_1 \times R_2} R_1 \cong (M \times 0) \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M \times 0 \cong_R M$ (isomorphism of R-modules). Then, we can write $(\star\star)$ as $0 \to M \to P \otimes_{R_1 \times R_2} R_1 \to M \to 0$. It is clear that $P \otimes_{R_1\times R_2} R_1$ is a projective R_1-module. Furthermore, by [15, Theorem 2.20], $\mathrm{Ext}_{R_1}(M,F)=0$ for every R_1-module projective F since M is a G-projective R_1-module. So, by [2, Proposition 2.9], M is an SG-projective R_1-module, as desired.

Similarly, we can prove that $\mathscr{SGP}(R_2)=\mathscr{GP}(R_2)$.

Conversely, assume that $\mathscr{SGP}(R_i)=\mathscr{GP}(R_i)$ for $i=1,2$ and let M be a G-projective $R_1\times R_2$-module. We claim that M is an SG-projective $R_1\times R_2$-module. We have the isomorphism of $R_1\times R_2$-modules:

$$M \cong M \otimes_{R_1\times R_2} (R_1\times R_2) \cong M \otimes_{R_1\times R_2} (R_1\times 0 \oplus 0\times R_2) \cong M_1\times M_2$$

where $M_i = M\otimes_{R_1\times R_2} R_i$ for $i=1,2$ (for more details see [6, p.102]). By [4, Lemma 3.2], for each $i=1,2$, M_i is a G-projective R_i-module. Then, by hypothesis, M_i is an SG-projective R_i-module for $i=1,2$. On the other hand, the family $\{R_i\}_{i=1,2}$ of rings satisfies the conditions of [4, Lemma 3.3](by [3, Corollary 2.10] since $G.gldim(R_i)$ is finite for each $i=1,2$). Thus, $M=M_1\times M_2$ is an SG-projective $R_1\times R_2$-module, as desired. □

Now, we are able to construct a non-Noetherian family of rings $\{R_i\}$ over which every Gorenstein projective module is strongly Gorenstein projective and such that $G.gldim(R_i)=i$ and $wdim(R_i)=\infty$.

Example 2.4. Consider a non-semisimple quasi-Frobenius ring $R=K[X]/(X^2)$ where K is a field, and a non-Noetherian hereditary ring S. Then, for every positive integer n, we have:

(i) $\mathscr{SGP}(R\times S[X_1,X_2,\dots,X_n])=\mathscr{GP}(R\times S[X_1,X_2,\dots,X_n])$,
(ii) $G.gldim(R\times S[X_1,X_2,\dots,X_n])=n+1$ and $wdim(R\times S[X_1,X_2,\dots,X_n])=\infty$.

Proof. From [4, Example 3.4], only the first assertion need an argument. It is clear that $\mathscr{P}(S[X_1,X_2,\dots,X_n])=\mathscr{SGP}(S[X_1,X_2,\dots,X_n])=\mathscr{GP}(S[X_1,X_2,\dots,X_n])$ since $wdim(S[X_1,X_2,\dots,X_n])$ is finite (by the Hilbert Syzygies's Theorem) and by using [15, Proposition 2.27]. On the other hand, from Example 2.2, $\mathscr{SGP}(R)=\mathscr{GP}(R)$. Thus, Theorem 2.3 finish the proof. □

Remark 2.5. Similarly as in Example 2.4 and by using the ring S of Example 2.2 and Theorem 2.3, we can construct a family of rings $\{S_i\}_{i\geq 0}$ (Noetherian or not) and with any Gorenstein global dimensions, such that $\mathscr{SGP}(S_i)\subsetneq\mathscr{GP}(R_i)$.

In the rest of this paper we will be interesting to discuss and give examples of rings satisfies the equality of classes $\mathscr{SGF}(-)=\mathscr{GF}(-)$. Before that, the next rest result give a particular case where the equality $\mathscr{SGP}(-)=\mathscr{GP}(-)$, $\mathscr{SGI}(-)=\mathscr{GI}(-)$ and $\mathscr{SGF}(-)=\mathscr{GF}(-)$ are equivalents.

Theorem 2.6. *Let R be a commutative ring with Gorenstein global dimension ≤ 1. We consider the following assertions:*

(i) *Every Gorenstein projective module is strongly Gorenstein projective.*

(ii) *Every Gorenstein injective module is strongly Gorenstein injective.*

(iii) *Every Gorenstein flat module is strongly Gorenstein flat.*

Then (i) *and* (ii) *are equivalent. If R is coherent, all assertions are equivalent.*

Proof. (i)$\Rightarrow$(ii). Assume (i) and let M be a G-injective R-module. We claim that M is SG-injective. By hypothesis, $Gpd_R(M) \leq 1$. Then, there exists a short exact sequence of R-modules $(\star)$ $0 \to M \to Q \to M \to 0$ where $pd_R(Q) \leq 1$ (by Theorem 2.1). By [3, Corollary 2.10], $id_R(Q) \leq 1$. On the other hand, $\mathcal{GI}(R)$ is injectively resolving ([15, Theorem 2.6]). Then, Q is a G-injective R-module. Thus, by the injective version of [15, Proposition 2.27], we conclude that Q is injective. Moreover, for every injective R-module E we have $\mathrm{Ext}_R(E,M) = 0$ (since M is G-injective). Consequently, from [2, Remark 2.10], M is SG-injective, as desired.

(ii)$\Rightarrow$(i). By the injective version of Theorem 2.1, the argument of this implication is similar to the proof of the first one.

Now, we suppose that R is coherent and prove the equivalence (i)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(iii). Let M be a G-flat module. By hypothesis $Gpd_R(M) \leq 1$. Thus, from Theorem 2.1, there is an exact sequence $0 \to M \to X \to M \to 0$ where $pd(X) \leq 1$. Then, $id(\mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}/\mathbb{Z})) = fd(X) \leq pd(X) \leq 1$. Furthermore, from [15, Theorem 3.7], X is Gorenstein flat module since $\mathcal{GF}(R)$ is projectively resolving. Hence, by [15, Proposition 3.11], $\mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Consequently, by the dual of [15, Proposition 2.27], $\mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{Q}/\mathbb{Z})$ is injective. Then, from [14, Theorem 1.2.1]), X is flat. Hence, M is immediately SG-flat (by [2, Proposition 3.6] and since for any injective module I, we have $Tor(M,I) = 0$ since M is G-flat).

(iii)$\Rightarrow$(i). Let I be an injective R-module. From [3, Corollary 2.10], $fd_R(I) \leq 1$. Then, from [10, Theorem 3.8], R is an $1-FC$ ring (i.e., coherent ring with $\mathrm{Ext}^2_R(P,R) = 0$ for each finitely presented R-module P). Now, let M be a G-projective R-module. Then, M embeds in projective R-module. So, from [7, Theorem 7], M is G-flat. Then, by hypothesis M becomes SG-flat. Hence, there exists a short exact sequence $0 \to M \to F \to M \to 0$ where F is flat. By the resolving of the class $\mathcal{GP}(R)$ and from the short exact sequence above we deduce that F is G-projective (since M is G-projective). On the other hand, $pd_R(F) < \infty$ (by [3, Corollary 2.10] and since F is flat). Therefore, F is projective by [15, Proposition 2.27]. So M is SG-projective (by [2, Proposition 2.9] and since $\mathrm{Ext}(M,P) = 0$ for every projective module P because M is G-projective). □

We have a similar result with the perfect rings as shown by the next result. Recall that a ring R is called perfect if every flat R-module is projective [1].

Proposition 2.7. *Let R be a coherent ring with finite Gorenstein global dimension. If every Gorenstein flat module is strongly Gorenstein flat, then every Gorenstein projective module is strongly Gorenstein projective with equivalence if R is perfect.*

Proof. Assume that every Gorenstein flat module is strongly Gorenstein flat and let M be a G-projective module. By [15, Theorem 2.20], $\mathrm{Ext}_R(M,Q) =$ for every projective module Q. So, to prove that M is SG-projective it suffices to find a short exact sequence of R-modules $0 \to M \to P \to M \to 0$ where P is projective (by [2, Proposition 2.9]). From [15, Proposition 3.4 and Theorem 3.24], M is also G-flat (since $G.gldim(R)$ is finite). Then, by the hypothesis condition, M becomes SG-flat. Thus, from [2, Proposition 3.6], there exists a short exact sequence $0 \to N \to F \to M \to 0$ where F is flat. So, from [3, Corollary 2.10], $pd_R(F)$ is finite. On the other hand, by [Theorem 3.7], $\mathscr{GF}(R)$ is projectively resolving and then, from the short exact sequence above, F is G-projective since M is G-projective. Therefore, from [15, Proposition 2.27], F is projective. Consequently, we have the desired short exact sequence.

Now, assume that R is a perfect ring with finite Gorenstein global dimension and such that every Gorenstein projective module is strongly Gorenstein projective. Let M a G-flat module. We claim that it is strongly Gorenstein flat. By [15, Theorem 3.14], $\mathrm{Tor}_R(M,I) = 0$ for every injective module I. So, from [2, Proposition 3.6], it stays to prove the existence of a short exact sequence $0 \to M \to F \to M \to 0$ where F is flat. By [2, Theorem 3.5] M is a direct summand of an SG-flat module N. For such module, by [2, Proposition 3.6], there exists a short exact sequence $(*)$ $0 \to N \to F \to N \to 0$ where F is flat (then projective since R is perfect). Now, let P be a projective module. We have $id_R(P) \leq n$ where $n = G.gldim(R)$ (by [3, Corollary 2.10]). Then, $\mathrm{Ext}_R^{n+1}(N,P) = 0$ and so, from the short exact sequence $(*)$ we deduce that $\mathrm{Ext}_R(N,P) = 0$. So, from [2, Proposition 2.9], N is SG-projective. Consequently, M is G-projective (since it is a direct summand of N and by [2, Theorem 2.7]). Then, by hypothesis, M becomes SG-projective. Hence, by [2, Proposition 2.9], there exists a short exact sequence $0 \to M \to P \to M \to 0$ where P is projective (then flat), and this is the desired short exact sequence. $\square$

Remark 2.8. Seen Theorem 2.6, with the rings R and S of Example 2.2, we have $S\mathscr{GF}(R) = \mathscr{GF}(R)$ and $\mathscr{SGF}(S) \subsetneq \mathscr{GF}(S)$.

In what follows, we give the flat version of Theorem 2.3 as:

Theorem 2.9. *Let $\{R_i\}_{i=1,\dots,m}$ be a family of coherent rings with finite Gorenstein weak dimensions. Then, $\mathscr{SGF}\left(\prod_{i=1}^{m} R_i\right) = \mathscr{GF}\left(\prod_{i=1}^{m} R_i\right)$, if and only if, $\mathscr{SGF}(R_i) = \mathscr{GF}(R_i)$ for each $i = 1,\dots,m$.*

Proof. By induction on m, we may assume $m = 2$.

First note that $R_1 \times R_2$ is coherent since R_1 and R_2 are coherents.

Assume that every Gorenstein flat $R_1 \times R_2$-module is strongly Gorenstein flat and let M be a G-flat R_1-module. We claim that M is an SG-flat R_1-module. Clearly $M \times 0$ is an $R_1 \times R_2$-module (see [6, p. 101]). First, we claim that $M \times 0$ is a G-flat $R_1 \times R_2$-module. The R_1-module M is a direct summand of an SG-flat R_1-module N ([3, Theorem 3.5.]). For such module, there is a short exact sequence of R_1-modules $0 \to N \to F \to N \to 0$ where F is a flat R_1-module. Hence, we have a short sequence of $R_1 \times R_2$-modules $(*) \quad 0 \to N \times 0 \to F \times 0 \to N \times 0 \to 0$ and $F \times 0$ will be a flat $R_1 \times R_2$-module ([4, Lemma 3.7]). But $G.wdim(R_1 \times R_2)$ is finite ([4, Theorem 3.5]). Then, there is an integer $i > 0$ such that $Tor^i_{R_1 \times R_2}(N \times 0, I) = 0$ for each injective $R_1 \times R_2$-module I. From $(*)$ we deduce that $Tor_{R_1 \times R_2}(N \times 0, I) = 0$. Then, by [2, Proposition 3.6], $N \times 0$ is an SG-flat $R_1 \times R_2$-module. So, from [15, theorem 3.7], $M \times 0$ is a G-flat $R_1 \times R_2$-module (since it is a direct summand of $N \times 0$ as $R_1 \times R_2$-modules and since $R_1 \times R_2$ is coherent). Now, we claim that M is an SG-flat R_1-module. The $R_1 \times R_2$-module $M \times 0$ is SG-flat (by the hypothesis condition and since we have proved that $R_1 \times R_2$-module $M \times 0$ is G-flat). Then, there exists a short exact sequence of $R_1 \times R_2$-modules $(\star) \quad 0 \to M \times 0 \to F \to M \times 0 \to 0$ where F is a flat $R_1 \times R_2$-module. Now, we tensor $(\star)$ by $- \otimes_{R_1 \times R_2} R_1$ (see that R_1 is a projective $R_1 \times R_2$-module), we obtain the short exact sequence of R_1-modules:

$$(\star\star) \qquad 0 \to M \times 0 \otimes_{R_1 \times R_2} R_1 \to F \otimes_{R_1 \times R_2} R_1 \to M \times 0 \otimes_{R_1 \times R_2} R_1 \to 0$$

But $(M \times 0) \otimes_{R_1 \times R_2} R_1 \cong (M \times 0) \otimes_{R_1 \times R_2} (R_1 \times R_2)/(0 \times R_2) \cong M \times 0 \cong_R M$ (isomorphism of R-modules). Then, we can write $(\star\star)$ as $0 \to M \to F \otimes_{R_1 \times R_2} R_1 \to M \to 0$. It is clear that $F \otimes_{R_1 \times R_2} R_1$ is a flat R_1-module. So, M is an SG-flat R_1-module (since $G.wdim(R_1)$ is finite and by the same argument as above), as desired.

Similarly, we can prove that every G-flat R_2-module is SG-flat.

Conversely, assume that every G-flat R_i-module is SG-flat R_i-module for, $i = 1, 2$. Let M be a G-flat $R_1 \times R_2$-module. We claim that M is strongly Gorenstein flat. We have the isomorphism of $R_1 \times R_2$-modules:

$$M \cong M \otimes_{R_1 \times R_2} (R_1 \times R_2) \cong M \otimes_{R_1 \times R_2} (R_1 \times 0 \oplus 0 \times R_2) \cong M_1 \times M_2$$

where $M_i = M \otimes_{R_1 \times R_2} R_i$ for $i = 1, 2$ (see [6, p. 102]). By [15, Proposition 3.10], for each $i = 1, 2, M_i$ is a G-flat R_i-module. Then, M_i is an SG-flat R_i-module (by the hypothesis condition).

Let I be an injective R_1-module and set $n = G.wdim(R_1)$. Then, using [15, Theorem 3.14], for every R_1-module K we have $Tor^{n+1}_{R_1}(K, I) = 0$ since $Gfd_R(K) \leq n$. Therefore, $fd_{R_1}(I) \leq n$. Similarly, we can prove that every injective R_2-module has a finite flat dimension. Thus, the family $\{R_i\}_{i=1,2}$ of rings satisfies the conditions of [4, Lemma 3.6]. Hence, $M = M_1 \times M_2$ is an SG-flat $R_1 \times R_2$-module, as desired. $\square$

Now we are able to construct a family of non-Noetherian coherent rings $\{R_i\}_{i>0}$ over which every G-flat module is strongly Gorenstein flat and such that $i = G.wdim(R_i) < G.gldim(R_i)$ and $wdim(R_i) = \infty$.

Example 2.10. Consider a non-semisimple quasi-Frobenius ring R and a semihereditary ring S which is not hereditary. Then, for every positive integer n, we have:

(i) $\mathscr{SGF}(R \times S[X_1, X_2, \dots, X_n]) = \mathscr{GF}(R \times S[X_1, X_2, \dots, X_n])$, and

(ii) $n + 1 = G.wdim(R \times S[X_1, X_2, \dots, X_n]) < G.gldim(R \times S[X_1, X_2, \dots, X_n])$ and $wdim(R \times S[X_1, X_2, \dots, X_n]) = \infty$.

Proof. Since S is semihereditary, the ring $T := S[X_1, X_2, \dots, X_n]$ is coherent and from [4, Example 3.8] only the first assertion need an argument. Since $wdim(T) < \infty$ (by the Hilbert Syzygies's Theorem), every Gorenstein flat T-module is flat (by [15, Corollary 3.8]). Hence, $\mathscr{F}(T) = \mathscr{SGF}(T) = \mathscr{GF}(T)$. On the other hand, from Remark 2.8, $\mathscr{SGF}(R) = \mathscr{GF}(R)$. Thus, since $G.wdim(R) = G.gldim(R) = 0$ and R is Noetherian (since R is quasi-Frobenius), Theorem 2.9 finish the proof. $\square$

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