Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 873–887, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

A SIR Epidemic Model with Primary Immunodeficiency and Time Delay

Elizabeth Sebastian and Priyanka Victor*

Department of Mathematics, Auxilium College, Vellore, Tamil Nadu, India
***Corresponding author:** priyankavictor2@gmail.com

Abstract. In this paper, we have proposed a SIR (Susceptible-Infected-Recovered) epidemic model incorporating Primary Immunodeficiency and distributed delays. We discretize the model using a variation of Backward Euler method. We divide the susceptible population into two groups based on their immunity levels and apply the transmission rate for these two populations. We derive a threshold value known as the basic reproduction number denoted by R_0. We have two equilibria namely the disease free and endemic equilibrium. We analyze the global stability of the disease free and endemic equilibrium using Lyapunov functional techniques. Finally, We prove our theoretical results using numerical simulations through MATLAB.

Keywords. Difference equations; Basic reproduction number; Time delay; Global stability

MSC. 39A10; 92D30

Received: January 8, 2017 **Accepted:** March 14, 2017

1. Introduction

Our immune system is a complicated network of cells, tissues and organs to keep us healthy and fight off diseases and infection. The immune system is composed of two major parts: the innate immune system and adaptive immune system. The macrophages and neutrophils of the innate immune system provide a first line of defense against many common micro-organisms and are essential for the control of common bacterial infections. However, they cannot always eliminate infectious organisms and there are some pathogens that they cannot recognize. Moreover, because there is a delay of 4-7 days before the initial adaptive immune response takes effect, the innate immune response has a critical role in controlling infections during this period.

Immunodeficiency (or immune deficiency) is a state in which the immune system's ability to fight infectious disease is compromised or entirely absent. Primary immunodeficiencies are disorders in which part of the body's immune system is missing or does not function normally. Most of the primary immunodeficiencies are genetic disorders. Majority are diagnosed in children under the age of one, although milder forms may not be recognized until adulthood. About 1 in 500 people in the United States are born with primary immunodeficiency. People with primary immunodeficiencies are more prone to infections. So in case of an epidemic, people with primary immunodeficiencies are more likely to be infected than other people [5], [6], [7], [8], [9].

The delays or lags can represent gestation times, incubation periods, transport delays or can simply lump complicated biological processes together. Such models have the advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes for a single system. Delay models are becoming more common, appearing in many branches of biological modelling. They have been used for describing several aspects of infectious disease dynamics namely primary infection, drug therapy and immune response. Delays have also appeared in the study of chemostat models, epidemiology, the respiratory system, tumor growth and neural networks. In the context of epidemiology, delays can be caused by a variety of factors. The most common reasons for a delay are (i) the latency of the infection in a vector and (ii) the latency of the infection in an infected host. In these cases, some time should elapse before the level of infection in the infected host or the vector reaching a sufficiently high level to transmit the infection further [10].

In Section 2, we have formulated the mathematical model. Basic properties of the model are discussed in Section 3. In Section 4, the global stability of the disease-free equilibrium is analysed. Global stability of endemic equilibrium is discussed in Section 5. Numerical simulations of the mathematical model are given using MATLAB in Section 6. Numerical simulations are discussed in Section 7.

2. Mathematical Model

Let us consider the following continuous time SIR (*Susceptible-Infected-Recovered*) model with distributed delays. Let $S(t)$, $A(t)$, $I(t)$ and $R(t)$ denote the proportion of population susceptible to disease without primary immunodeficiency disorders, population susceptible to disease with primary immunodeficiency disorders, population of infective members and members who have been removed from the possibility of infection at time t, respectively.

$$\begin{aligned}
\frac{dS(t)}{dt} &= bpQ - \mu S(t) - \beta\alpha q S(t)\int_0^h f(\tau)I(t-\tau)d\tau,\\
\frac{dA(t)}{dt} &= b(1-p)Q - \mu A(t) - \beta_1\alpha_1 A(t)\int_0^h f(\tau)I(t-\tau)d\tau,\\
\frac{dI(t)}{dt} &= \beta\alpha q S(t)\int_0^h f(\tau)I(t-\tau)d\tau + \beta_1\alpha_1 A(t)\int_0^h f(\tau)I(t-\tau)d\tau - (\mu+\gamma+\delta)I(t),\\
\frac{dR(t)}{dt} &= \mu R(t) + \gamma I(t).
\end{aligned} \tag{1}$$

Let us consider this as system (1), where

- b is the birth rate of the population.
- α is the immunity rate of population S.
- α_1 is the immunity rate of population A.
- p is rate of population without primary immunodeficiency.
- Q denotes the constant population.
- μ is the natural death rate.
- δ is the death rate due to infection.
- q is the rate at which population A comes in contact with population S.
- γ is the recovery rate.
- β is the transmission rate of population S.
- β_1 is the transmission rate of population A.

We assume the following condition:

- $\beta_1 > \beta$, that is the infection rate of the population with primary immunodeficiency is greater than that of population without primary immunodeficiency.

Infectiousness is assumed to vary over time from the initial time of infection until a duration h has passed and the function $f(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ. Here β, β_1 and $f(\tau)$ are chosen such that it is non negative and continuous on $[0,h]$ and assume for

$$\int_0^h f(\tau)d\tau = 1. \tag{2}$$

System (1) has the disease free equilibrium

$$E_0 = \left(\frac{bpQ}{\mu}, \frac{b(1-p)Q}{\mu}, 0, 0\right)$$

Furthermore, if $R_0 > 1$, the system (1) has an unique endemic equilibrium

$$E^* = (S^*, A^*, I^*, R^*),$$

where

$$S^* = \frac{bpQ}{\mu+\beta\alpha q I^*}, \quad A^* = \frac{b(1-p)Q}{\mu+\beta_1\alpha_1 I^*},$$

where I^* is the positive root of the following equation.

$$A_1 I^{*2} + A_2 I^* + A_3 = 0, \tag{3}$$

where

$$A_1 = \beta\beta_1\alpha\alpha_1 q > 0,$$
$$A_2 = \frac{\mu(\beta\alpha q + \beta_1\alpha_1)}{\mu+\gamma+\delta} - \frac{\beta\beta_1\alpha\alpha_1 bqQ}{\mu+\gamma+\delta},$$
$$A_3 = \mu^2 - R_0 < 0.$$

The basic reproduction number (usually denoted by R_0), is a significant epidemiological quantity, which plays an important role in the dynamics of disease transmission. It is a useful metric that helps us to predict whether an infectious disease will spread through a population or not. If it is less than one, the infection will die out in the long run, otherwise, the infection will keep persistent in the population.

The basic reproduction number of the model is given by

$$R_0 = \frac{bQ(\beta\alpha pq + \beta_1\alpha_1(1-p))}{\mu+\gamma+\delta}. \tag{4}$$

We propose the following discrete epidemic model which is derived from system (1), by applying variation of Backward Euler Method:

$$S(t+1) - S(t) = bpQ - \mu S(t+1) - \beta\alpha q S(t+1)\sum_{j=0}^{m} f(j)I(p-j),$$
$$A(t+1) - A(t) = b(1-p)Q - \mu A(t+1) - \beta_1\alpha_1 A(t+1)\sum_{j=0}^{m} f(j)I(p-j),$$
$$I(t+1) - I(t) = [\beta\alpha q S(t+1) + \beta_1\alpha_1 A(t+1)]\sum_{j=0}^{m} f(j)I(p-j) - (\mu+\gamma+\delta)I(t+1),$$
$$R(t+1) - R(t) = \gamma I(t+1) - \mu R(t+1). \tag{5}$$

This can be taken as system (5), where $f(j) \geq 0$, $j = 0,1,2,\ldots,m$. For simplicity, we may assume that $\sum_{j=0}^{m} f(j) = 1$. Similar to the continuous case system, System (5) has a disease free equilibrium,

$$E_0 = \left(\frac{bpQ}{\mu}, \frac{b(1-p)Q}{\mu}, 0, 0\right).$$

Furthermore, if $R_0 > 1$, system (5) has an unique endemic equilibrium

$$E^* = (S^*, A^*, I^*, R^*)$$

with initial conditions

$$S(t) = \phi(t) \geq 0,\ I(t) = \psi(t) \geq 0,\ A(t) = \theta(t) \geq 0,\ R(t) = \sigma(t) \geq 0$$
$$t = -m, -(m-1), \ldots, -1$$
$$S(0) > 0,\ A(0) > 0,\ I(0) > 0,\ R(0) > 0. \tag{6}$$

We have the same threshold value, that is, the basic reproduction number,

$$R_0 = \frac{bQ(\beta\alpha pq + \beta_1\alpha_1(1-p))}{\mu+\gamma+\delta}.$$

3. Basic Properties

For system (5), since the variable R does not appear in the first three equations, it is sufficient to consider the 3 dimensional system:

$$S(t+1)-S(t)=bpQ-\mu S(t+1)-\beta\alpha qS(t+1)\sum_{j=0}^{m}f(j)I(p-j),$$

$$A(t+1)-A(t)=b(1-p)Q-\mu A(t+1)-\beta_1\alpha_1A(t+1)\sum_{j=0}^{m}f(j)I(p-j),$$

$$I(t+1)-I(t)=[\beta\alpha qS(t+1)+\beta_1\alpha_1A(t+1)]\sum_{j=0}^{m}f(j)I(p-j)-(\mu+\gamma+\delta)I(t+1). \tag{7}$$

This can be taken as system (7) with initial conditions

$$S(t)=\phi(t)\geq 0,\ I(t)=\psi(t)\geq 0,\ A(t)=\theta(t)\geq 0$$

$$t=-m,-(m-1),\dots,-1$$

$$S(0)>0,\ A(0)>0,\ I(0)>0. \tag{8}$$

Lemma 3.1. *Let* $(S(t),A(t),I(t))$ *be the solution of system* (7) *with initial condition* (8). *Then* $S(t)>0$, $A(t)>0$ *and* $I(t)>0$.

Proof. Assume that $S(p-j),A(p-j),I(p-j)>0$, $j=0,1,2,\dots,m$. Then system (7) becomes

$$S(t+1)\left\{1+\mu+\beta\alpha q\sum_{j=0}^{m}f(j)I(p-j)\right\}=bpQ+S(t)>0,$$

$$A(t+1)\left\{1+\mu+\beta_1\alpha_1\sum_{j=0}^{m}f(j)I(p-j)\right\}=b(1-p)Q+A(t)>0,$$

$$I(t+1)(\mu+\gamma+\delta)=I(t)+\left\{[\beta\alpha qS(t+1)+\beta_1\alpha_1A(t+1)]\sum_{j=0}^{m}f(j)I(p-j)\right\}>0.$$

From the first equation, we have $S(t+1)>0$ and by the second and third equation, we have $A(t+1)>0$, $I(t+1)>0$. Hence by induction, we prove this lemma.

Lemma 3.2. *Any solution* $(S(t),A(t),I(t))$ *of system* (7) *with initial condition* (8) *satisfies*

$$\limsup_{t\to\infty}(S(t),A(t),I(t))\leq\frac{bQ}{\mu}.$$

Proof. Let $V(t)=S(t)+A(t)+I(t)$. From system (7) we have that

$$V(t+1)-V(t)=bQ-\mu V(t+1)-(\gamma+\delta)I(t+1),$$

$$V(t+1)-V(t)\leq bQ-\mu V(t+1)$$

from which we have that

$$\limsup_{t\to\infty}(S(t),A(t),\ \ I(t))\leq\frac{bQ}{\mu}.$$

Hence the proof is complete.

Now put

$$\check{S}=\liminf_{t\to\infty}S(t),\qquad \hat{S}=\limsup_{t\to\infty}S(t),$$

$$\check{A} = \liminf_{t\to\infty} A(t), \quad \widehat{A} = \limsup_{t\to\infty} A(t),$$
$$\check{I} = \liminf_{t\to\infty} I(t), \quad \widehat{I} = \limsup_{t\to\infty} I(t). \qquad \square$$

Lemma 3.3. *For any solution* $(S(t), A(t), I(t))$ *of system* (7) *with initial condition* (8), *we have that*

$$0 < \frac{bpQ}{\mu + \beta\alpha q \widehat{I}} < \check{S} \le \widehat{S} \le \frac{bpQ}{\mu + \beta\alpha q \check{I}} \le \frac{bQ}{\mu},$$
$$0 < \frac{b(1-p)Q}{\mu + \beta_1\alpha_1 \widehat{I}} < \check{A} \le \widehat{A} \le \frac{b(1-p)Q}{\mu + \beta_1\alpha_1 \check{I}} \le \frac{bQ}{\mu},$$
$$\frac{\beta\alpha q \check{S} + \beta_1\alpha_1 \check{A}}{\mu+\gamma+\delta} \le 1 \quad \text{if } \check{I} > 0,$$
$$\frac{\beta\alpha q \widehat{S} + \beta_1\alpha_1 \widehat{A}}{\mu+\gamma+\delta} \ge 1 \quad \text{if } \widehat{I} > 0.$$

4. Global Stability of the Disease-free Equilibrium

Lemma 4.1. *If* $R_0 \le 1$, *then*

$$\lim_{t\to\infty} S(t) = \frac{bpQ}{\mu}, \quad \lim_{t\to\infty} A(t) = \frac{b(1-p)Q}{\mu}, \quad \lim_{t\to\infty} I(t) = 0$$

and E_0 *is globally asymptotically stable.*

Proof. For any $\epsilon > 0$, there exists a positive integer $t_0 \ge 0$ such that

$$S(t+1) \le \frac{bpQ}{\mu} + \epsilon \qquad \text{for all } t \ge t_0,$$
$$A(t+1) \le \frac{b(1-p)Q}{\mu} + \epsilon \quad \text{for all } t \ge t_0.$$

Consider the following sequence $\{w(t)\}_{t=t_0}^{+\infty}$ defined by

$$w(t) = u(t) + v(t) + I(t) \quad \text{for all } t \ge t_0,$$

where

$$u(t) = \beta\alpha q \sum_{j=0}^{m} f(j) \sum_{k=t-j}^{t} S(j+k+1)I(k) \ \text{ for all } t \ge t_0$$
$$v(t) = \beta_1\alpha_1 \sum_{j=0}^{m} f(j) \sum_{k=t-j}^{t} A(j+k+1)I(k) \ \text{ for all } t \ge t_0$$
$$u(t+1) - u(t) = \beta\alpha q \sum_{j=0}^{m} f(j)S(t+j+2)I(t+1) - S(t+1)I(t-j)$$
$$v(t+1) - v(t) = \beta_1\alpha_1 \sum_{j=0}^{m} f(j)A(t+j+2)I(t+1) - A(t+1)I(t-j)$$
$$w(t+1) - w(t) = \beta\alpha q \sum_{j=0}^{m} f(j)S(t+j+2)I(t+1)$$
$$+ \beta_1\alpha_1 \sum_{j=0}^{m} f(j)A(t+j+2)I(t+1) - (\mu+\gamma+\delta)I(t+1)$$

$$\le \beta\alpha q \sum_{j=0}^{m} f(j)\left(\frac{bpQ}{\mu}+\epsilon\right)I(t+1)$$
$$+\beta_1\alpha_1\sum_{j=0}^{m} f(j)\left(\frac{b(1-p)Q}{\mu}+\epsilon\right)I(t+1)-(\mu+\gamma+\delta)I(t+1)$$
$$\le \beta\alpha q\left(\frac{bpQ}{\mu}+\epsilon\right)I(t+1)+\beta_1\alpha_1\left(\frac{b(1-p)Q}{\mu}+\epsilon\right)I(t+1)-(\mu+\gamma+\delta)I(t+1)$$
$$= \beta\alpha q\left(\frac{bpQ}{\mu}+\epsilon\right)I(t+1)+\beta_1\alpha_1\left(\frac{b(1-p)Q}{\mu}+\epsilon\right)I(t+1)-(\mu+\gamma+\delta)I(t+1)$$
$$=\left\{\frac{bpQ(\beta\alpha q+\beta_1\alpha_1(1-p)Q)}{\mu}-(\mu+\gamma+\delta)\right\}I(t+1)+\left(\beta\alpha q+\beta_1\alpha_1\right)\epsilon I(t+1)$$

Since $\epsilon>0$ is arbitrary, we obtain that if $R_0\le 1$, then

$$w(t+1)-w(t)\le\left\{\frac{bpQ(\beta\alpha q+\beta_1\alpha_1(1-p))}{\mu}\right\}I(t+1)\le 0$$

and the non negative sequence $\{w(t)\}_{t=t_0}^{+\infty}$ is monotone decreasing. Therefore, there exists a non negative constant $\overline{w}$ such that

$$\lim_{t\to\infty} w(t)=\overline{w}.$$

We will prove that $\overline{w}=0$.

If $R_0<1$, then $\left\{\frac{bpQ(\beta\alpha q+\beta_1\alpha_1(1-p))}{\mu}\right\}<0$, we conclude that $\lim_{t\to\infty} I(t)=0$.

Then $\overline{w}=0$ and we obtain, $\lim_{t\to\infty} S(t)=\frac{bpQ}{\mu}$, $\lim_{t\to\infty} A(t)=\frac{b(1-p)Q}{\mu}$, $\lim_{t\to\infty} I(t)=0$.

Suppose $R_0=1$, we can write

$$S(t+1)=\frac{bpQ}{1+\mu}+\frac{S(t)}{1+\mu}-\frac{\beta\alpha q S(t+1)}{1+\mu}\sum_{j=0}^{m} f(j)I(t-j), \tag{9}$$

$$A(t+1)=\frac{b(1-p)Q}{1+\mu}+\frac{A(t)}{1+\mu}-\frac{\beta_1\alpha_1 A(t+1)}{1+\mu}\sum_{j=0}^{m} f(j)I(t-j), \tag{10}$$

$$I(t+1)=\frac{I(t)}{1+\mu+\gamma+\delta}+\frac{\beta\alpha q S(t+1)}{1+\mu+\gamma+\delta}\sum_{j=0}^{m} f(j)I(t-j)+\frac{\beta_1\alpha_1 A(t+1)}{1+\mu+\gamma+\delta}\sum_{j=0}^{m} f(j)I(t-j). \tag{11}$$

It can be written as

$$S(t+1)=\widetilde{b}+\widetilde{c}S(t)-\widetilde{\beta}S(t+1)\sum_{j=0}^{m} f(j)I(t-j), \tag{12}$$

$$A(t+1)=\widetilde{b}_1+\widetilde{c}A(t)-\widetilde{\beta}_1 A(t+1)\sum_{j=0}^{m} f(j)I(t-j), \tag{13}$$

$$I(t+1)=\widetilde{d}I(t)+\widetilde{\beta}'S(t+1)\sum_{j=0}^{m} f(j)I(t-j)+\widetilde{\beta}_1' A(t+1)\sum_{j=0}^{m} f(j)I(t-j), \tag{14}$$

where

$$\widetilde{b}=\frac{bpQ}{1+\mu},\quad \widetilde{b}_1=\frac{b(1-p)Q}{1+\mu},\quad \widetilde{c}=\frac{1}{1+\mu},\quad \widetilde{\beta}=\frac{\beta\alpha q}{1+\mu},$$

$$\widetilde{\beta}'=\frac{\beta\alpha q}{1+\mu+\gamma+\delta},\quad \widetilde{\beta}_1'=\frac{\beta_1\alpha_1}{1+\mu+\gamma+\delta},\quad \widetilde{\beta}_1=\frac{\beta_1\alpha_1}{1+\mu}.$$

We now claim that there is a sequence $\{t_k\}_{k=0}^{+\infty}$ such that each of $I(t_k - j)$, $j = 0,1,2,\dots,m$ converges to 0 as $k \to \infty$.
If $\overline{I} = 0$, then the claim is evident.
Now suppose $\overline{I} > 0$, there exists a sequence $\{t_k\}_{k=0}^{+\infty}$ such that

$$\lim_{t\to\infty} S(t_k+1) = \overline{S}, \quad \lim_{t\to\infty} A(t_k+1) = \overline{A},$$

$$S(t_k+1) = \widetilde{b} + \widetilde{c}S(t_k) - \widetilde{\beta}S(t_k+1)\sum_{j=0}^{m} f(j)I(t_k-j),$$

$$S(t_k+1) = \frac{\widetilde{b} - \widetilde{c}[S(t_k+1)-S(t_k)] - \widetilde{\beta}S(t_k+1)\sum_{j=0}^{m} f(j)I(t_k-j)}{1-\widetilde{c}},$$

$$S(t_k+1) \to \overline{S} \text{ as } k \to +\infty.$$

We easily obtain that,

$$\lim_{k\to+\infty} [\widetilde{c}[S(t_k+1)-S(t_k)] + \widetilde{\beta}S(t_k+1)\sum_{j=0}^{m} f(j)I(t_k-j)] = 0.$$

Therefore

$$\limsup_{k\to+\infty}[S(t_k+1)-S(t_k)] = 0 \quad \text{and} \quad \limsup_{k\to+\infty}\sum_{j=0}^{m} f(j)I(t_k-j) = 0.$$

Thus it holds that

$$\lim_{k\to+\infty} [S(t_k+1)-S(t_k)] = 0 \quad \text{and} \quad \lim_{k\to+\infty}\sum_{j=0}^{m} f(j)I(t_k-j) = 0.$$

Hence, it follows that ,

$$\lim_{k\to+\infty} S(t_k) = \lim_{k\to+\infty} S(t_k+1) = \overline{S}$$

and

$$\lim_{k\to+\infty}\sum_{j=0}^{m} f(j)I(t_k-j) = 0,$$

$$A(t_k+1) = \widetilde{b}_1 - \frac{\widetilde{c}[A(t_k+1)-A(t_k)] + \widetilde{\beta}_1 A(t_k+1)\sum_{j=0}^{m} f(j)I(t_k-j)}{1-\widetilde{c}}$$

$A(t_k+1) \to \overline{A}$ as $k \to +\infty$.
We easily obtain that,

$$\lim_{k\to+\infty} [\widetilde{c}[A(t_k+1)-A(t_k)] + \widetilde{\beta}_1 A(t_k+1)\sum_{j=0}^{m} f(j)I(t_k-j)] = 0.$$

Therefore,

$$\limsup_{k\to+\infty}[A(t_k+1)-A(t_k)] = 0, \quad \limsup_{k\to+\infty}\sum_{j=0}^{m} f(j)I(t_k-j) = 0.$$

Thus it holds that

$$\lim_{k\to+\infty} [A(t_k+1)-A(t_k)] = 0, \lim_{k\to+\infty}\sum_{j=0}^{m} f(j)I(t_k-j) = 0.$$

Hence it follows that,

$$\lim_{k\to+\infty} A(t_k) = \lim_{k\to+\infty} A(t_k+1) = \overline{A}$$

and we can obtain that

$$I(t_k - l - j) = 0, \quad l = 0,1,2,\dots,m.$$

Hence the claim is proved. □

Lemma 4.2. *If* $I(t+1) < \min_{0\le j\le m} I(t-j)$*, then*

$$S(t+1) < S^* \text{ inversely if } S(t+1) \ge S^*, \text{ then } I(t+1) \ge \min_{0\le j\le m} I(t-j).$$

Proof. By the third equation of system (7),we have that

$$I(t+1) = \frac{I(t)-I(t+1)}{\mu+\gamma+\delta} + \frac{\beta\alpha q S(t+1) + \beta_1\alpha_1 A(t+1)}{\mu+\gamma+\delta}\sum_{j=0}^{m} f(j)I(t-j).$$

Therefore, if $I(t+1) < \min_{0\le j\le m} I(t-j)$, then by $I(t)-I(t+1) > 0$ and $\sum_{j=0}^{m} f(j)I(t-j) > I(t+1)$

$$I(t+1) > \frac{\beta\alpha q S(t+1) + \beta_1\alpha_1 A(t+1)}{\mu+\gamma+\delta} I(t+1),$$

$$I(t+1) > \frac{\beta\alpha q S(t+1)}{\mu+\gamma+\delta} I(t+1),$$

$$I(t+1) > \frac{S(t+1)}{S^*} I(t+1),$$

$$S(t+1) < S^*.$$

Inversely, if $S(t+1) \ge S^*$, then $I(t+1) \ge \min_{0\le j\le m} I(t-j)$.

Similarly if $I(t+1) < \min_{0\le j\le m} I(t-j)$, then $A(t+1) < A^*$. Inversely if $A(t+1) \ge A^*$, then $I(t+1) \ge \min_{0\le j\le m} I(t-j)$. □

5. Global Stability of the Endemic Equilibrium

Consider the following Lyapunov function,

$$U(t) = U_S(t) + U_A(t) + U_I(t) + U_+(t),$$

where

$$U_S(t) = g\left(\frac{S(t)}{S^*}\right), \quad U_A(t) = g\left(\frac{A(t)}{A^*}\right),$$

$$U_I(t) = g\left(\frac{I(t)}{I^*}\right), \quad U_+(t) = \sum_{j=0}^{m} f(j) \sum_{k=t-j}^{t} g\left(\frac{I(k)}{I^*}\right)$$

and $g(x) = x - 1 - In(x)$, $x > 0$.

We now show that, $U(t+1) - U(t) \le 0$.

First, we calculate $U_S(t+1) - U_S(t)$,

$$U_S(t+1) - U_S(t) = \frac{S(t+1)-S(t)}{S^*} - In\frac{S(t+1)}{S(t)}$$

$$
\begin{aligned}
&\le \frac{S(t+1)-S(t)}{S^*} - \frac{S(t+1)-S(t)}{S(t+1)} \\
&\le \frac{S(t+1)-S^*}{S^*S(t+1)}[S(t+1)-S(t)] \\
&= \frac{S(t+1)-S^*}{S^*S(t+1)}\left\{-\mu(S(t+1)-S^*)+\beta\alpha q S^* I^* - \beta\alpha q S(t+1)\sum_{j=0}^{m} f(j)I(t-j)\right\} \\
&= -\frac{\mu}{S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)} + \beta\alpha q I^* \sum_{j=0}^{m} f(j)\left[1-\frac{S^*}{S(t+1)}\right]\left[1-\frac{S(t+1)}{S^*}\frac{I(t-j)}{I^*}\right],
\end{aligned}
$$

$$
\begin{aligned}
U_A(t+1)-U_A(t) &= \frac{A(t+1)-A(t)}{A^*} - In\frac{A(t+1)}{A(t)} \\
&\le \frac{A(t+1)-A(t)}{A^*} - \frac{A(t+1)-A(t)}{A(t+1)} \\
&\le \frac{A(t+1)-A^*}{A^*A(t+1)}[A(t+1)-A(t)] \\
&= \frac{A(t+1)-A^*}{A^*A(t+1)}\left\{-\mu(A(t+1)-A^*)+\beta_1\alpha_1 A^* I^* - \beta_1\alpha_1 A(t+1)\sum_{j=0}^{m} f(j)I(t-j)\right\} \\
&= -\frac{\mu}{A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)} + \beta_1\alpha_1 I^* \left[1-\frac{A^*}{A(t+1)}\right]\left[1-\frac{A(t+1)}{A^*}\frac{I(t-j)}{I^*}\right],
\end{aligned}
$$

$$
\begin{aligned}
U_I(t+1)-U_I(t) &= \frac{I(t+1)-I(t)}{I^*} - In\frac{I(t+1)}{I(t)} \\
&\le \frac{I(t+1)-I(t)}{I^*} - \frac{I(t+1)-I(t)}{I(t+1)} \\
&\le \frac{I(t+1)-I^*}{I^*I(t+1)}[I(t+1)-I(t)] \\
&= \frac{I(t+1)-I^*}{I^*I(t+1)}\left[\beta\alpha q S(t+1)+\beta_1\alpha_1 A(t+1)\right]\sum_{j=0}^{m} f(j)I(t-j) - (\mu+\gamma+\delta)I(t+1) \\
&= \sum_{j=0}^{m} f(j)\beta\alpha q S^*\left[1-\frac{I^*}{I(t+1)}\right]\left\{\frac{S(t+1)}{S^*}\frac{I(t-j)}{I^*} - \frac{I(t+1)}{I^*}\right\}.
\end{aligned}
$$

Finally calculating,

$$
\begin{aligned}
U_+(t+1)-U_+(t) &= \sum_{j=0}^{m} f(j)\left\{\sum_{k=t+1-j}^{t+1} g\left(\frac{I(k)}{I^*}\right) - \sum_{k=t-j}^{t} g\left(\frac{I(k)}{I^*}\right)\right\} \\
&= \sum_{j=0}^{m} f(j)\left\{g\left(\frac{I(t+1)}{I^*}\right) - g\left(\frac{I(t-j)}{I^*}\right)\right\} \\
&= \sum_{j=0}^{m} f(j)g\left(\frac{I(t+1)}{I^*}\right) - \sum_{j=0}^{m} f(j)g\left(\frac{I(t-j)}{I^*}\right).
\end{aligned}
$$

Defining,

$$
x_{t+1} = \frac{S(t+1)}{S^*},\quad y_{t+1} = \frac{A(t+1)}{A^*},\quad z_{t+1} = \frac{I(t+1)}{I^*},\quad z_{t,j} = \frac{I(t-j)}{I^*}.
$$

We obtain that

$$\begin{aligned}
U(t+1)-U(t)\leq &\frac{\mu}{S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)}+\beta\alpha q I^*\sum_{j=0}^{m} f(j)\left[1-\frac{S^*}{S(t+1)}\right]\left[1-\frac{S(t+1)}{S^*}\frac{I(t-j)}{I^*}\right]\\
&-\frac{\mu}{A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)}+\beta_1\alpha_1 I^*\sum_{j=0}^{m} f(j)\left[1-\frac{A^*}{A(t+1)}\right]\left[1-\frac{A(t+1)}{A^*}\frac{I(t-j)}{I^*}\right]\\
&+\sum_{j=0}^{m} f(j)\beta\alpha q S^*\left[1-\frac{I^*}{I(t+1)}\right]\left\{\frac{S(t+1)}{S^*}\frac{I(t-j)}{I^*}-\frac{I(t+1)}{I^*}\right\}\\
&+\sum_{j=0}^{m} f(j)\beta_1\alpha_1 A^*\left[1-\frac{I^*}{I(t+1)}\right]\left\{\frac{A(t+1)}{A^*}\frac{I(t-j)}{I^*}-\frac{I(t+1)}{I^*}\right\}\\
&+\sum_{j=0}^{m} f(j)g\left(\frac{I(t+1)}{I^*}\right)-\sum_{j=0}^{m} f(j)g\left(\frac{I(t-j)}{I^*}\right), \qquad (15)
\end{aligned}$$

$$\begin{aligned}
U(t+1)-U(t)\leq &-\frac{\mu}{\beta\alpha q S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)}-\frac{\mu}{\beta_1\alpha_1 A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)}\\
&+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{x_{t+1}}\right][1-x_{t+1}z_{t,j}]+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{y_{t+1}}\right][1-y_{t+1}z_{t,j}]\\
&+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{z_{t+1}}\right][y_{t+1}z_{t,j}-z_{t+1}]+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{z_{t+1}}\right][x_{t+1}z_{t,j}-z_{t+1}]\\
&+\sum_{j=0}^{m} f(j)g(z_{t+1})-\sum_{j=0}^{m} f(j)g(z_{t,j}), \qquad (16)
\end{aligned}$$

$$\begin{aligned}
U(t+1)-U(t)\leq &-\frac{\mu}{\beta\alpha q S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)}-\frac{\mu}{\beta_1\alpha_1 A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)}\\
&+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{x_{t+1}}\right][1-x_{t+1}z_{t,j}]+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{y_{t+1}}\right][1-y_{t+1}z_{t,j}]\\
&+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{z_{t+1}}\right][y_{t+1}z_{t,j}-z_{t+1}]+\sum_{j=0}^{m} f(j)\left[1-\frac{1}{z_{t+1}}\right][x_{t+1}z_{t,j}-z_{t+1}]\\
&+\sum_{j=0}^{m} f(j)g(z_{t+1})-\sum_{j=0}^{m} f(j)g(z_{t,j}) \qquad (17)\\
=&\frac{\mu}{\beta\alpha q S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)}-\frac{\mu}{\beta_1\alpha_1 A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)}\\
&+\sum_{j=0}^{m} f(j)\left[2-\frac{1}{x_{t+1}}-\frac{1}{y_{t+1}}+z_{t,j}-z_{t+1}-\frac{x_{t+1}z_{t,j}}{z_{t+1}}-\frac{y_{t+1}z_{t,j}}{z_{t+1}}-In\, z_{t+1}+In\, z_{t,j}\right]. \qquad (18)
\end{aligned}$$

Finally, we get

$$\begin{aligned}
=&\frac{\mu}{\beta\alpha q S^*}\frac{(S(t+1)-S^*)^2}{S(t+1)}-\frac{\mu}{\beta_1\alpha_1 A^*}\frac{(A(t+1)-A^*)^2}{A(t+1)}\\
&-\sum_{j=0}^{m} f(j)\left[g\left(\frac{1}{x_{t+1}}\right)+g\left(\frac{1}{y_{t+1}}\right)+g\left(\frac{x_{t+1}z_{t,j}}{z_{t+1}}\right)+g\left(\frac{y_{t+1}z_{t,j}}{z_{t+1}}\right)-g(z_{t+1})-g(z_{t,j})\right]. \qquad (19)
\end{aligned}$$

Hence $U(t+1)-U(t) \leq 0$ for all $t \geq 0$. Since $U(t) \leq 0$ is a monotonic decreasing sequence. Then

$$\lim_{t \to \infty} U(t+1)-U(t) = 0. \tag{20}$$

Hence the proof.

6. Numerical Simulation

We take the following set of parametric values. Here we consider the population with primary immunodeficiency as *susceptible 1* (green).

Case 1. $b = 0.75$, $p = 100$, $\mu = 0.5$, $\beta = 0.4$, $\alpha = 0.7$, $q = 0.3$, $p_1 = 10$, $\beta_1 = 0.8$, $\alpha_1 = 0.3$, $\gamma = 0.4$, $\delta = 0.2$, we get $R_0 = 3.02$.

Case 2. In *Case* 2, we have increased the transmission rates. $b = 0.75$, $p = 100$, $\mu = 0.5$, $\beta = 0.7$, $\alpha = 0.7$, $q = 0.3$, $p_1 = 10$, $\beta_1 = 0.9$, $\alpha_1 = 0.3$, $\gamma = 0.1$, $\delta = 0.2$. The R_0 for the above set of values is $R_0 = 7.14$.

Case 3. In *Case* 3, we consider same values as Case 2, but increase the rate of contact between the susceptible population and the people with primary immunodeficiency. $b = 0.75$, $p = 100$, $\mu = 0.5$, $\beta = 0.7$, $\alpha = 0.7$, $q = 0.6$, $p_1 = 10$, $\beta_1 = 0.9$, $\alpha_1 = 0.3$, $\gamma = 0.1$, $\delta = 0.2$, we get $R_0 = 14.03$.

Case 4. In *Case* 4, we consider low transmission rates and high recovery rate. $b = 0.75$, $p = 100$, $\mu = 0.5$, $\beta = 0.3$, $\alpha = 0.7$, $q = 0.1$, $p_1 = 10$, $\beta_1 = 0.5$, $\alpha_1 = 0.3$, $\gamma = 0.7$, $\delta = 0.2$, we get $R_0 = 0.45$.

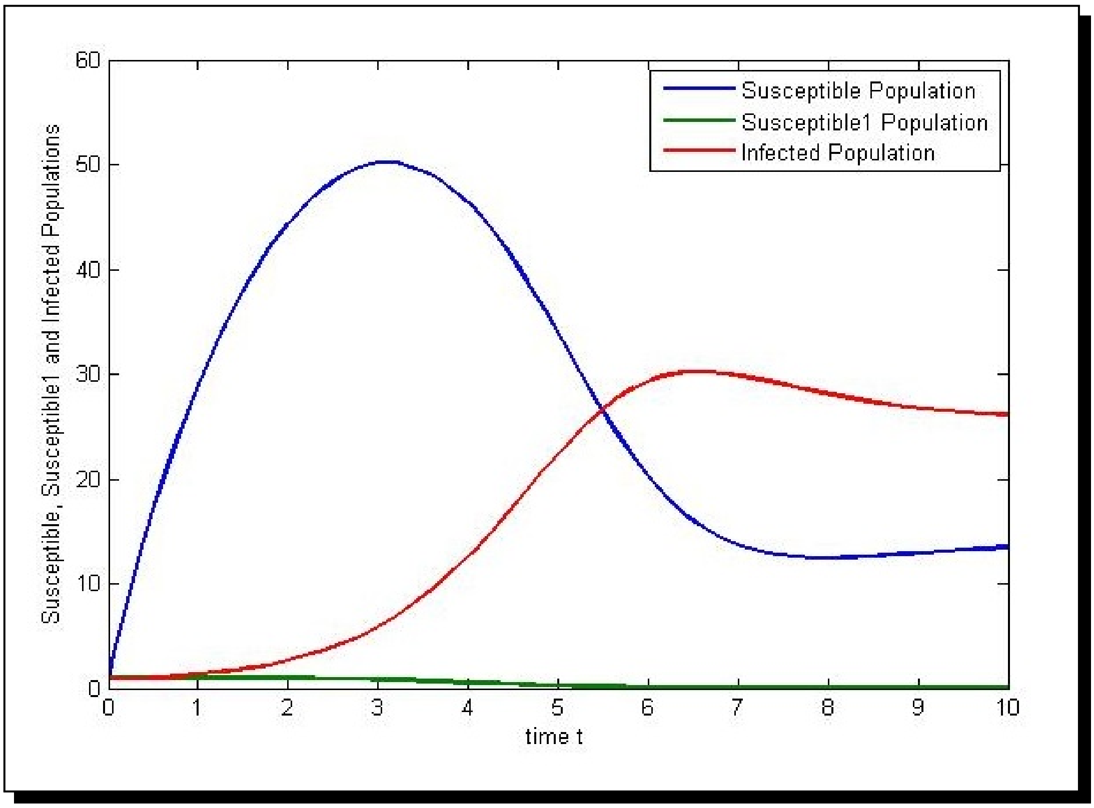

Figure 1. The dynamical behaviours of the system (III) for Case 1

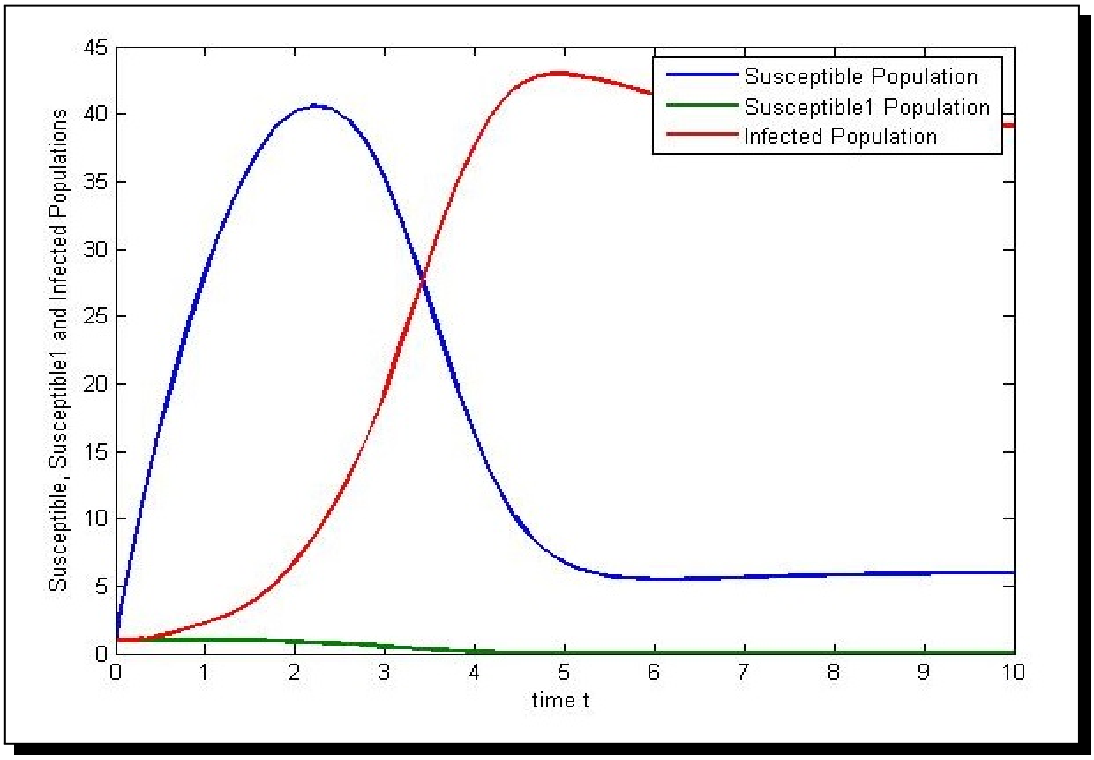

Figure 2. The dynamical behaviours of the system (III) for Case 2

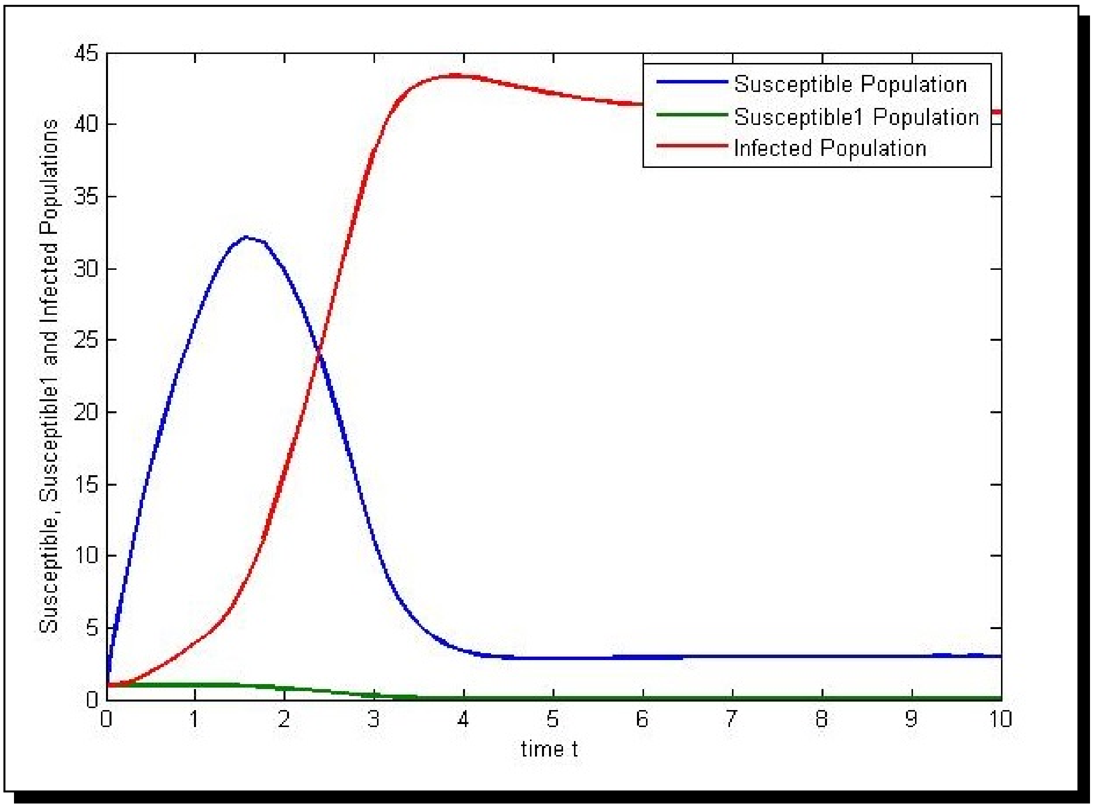

Figure 3. The dynamical behaviours of the system (III) for Case 3

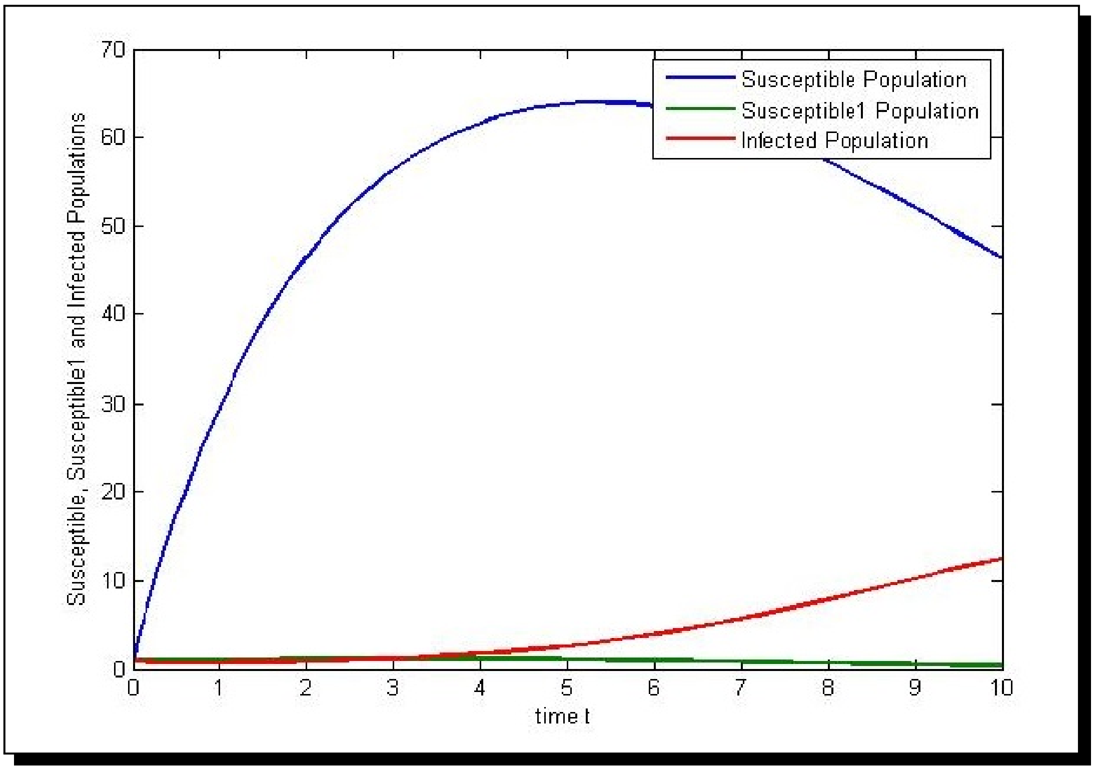

Figure 4. The dynamical behaviours of the system (III) for Case 4

7. Discussion

We have analysed a Discrete-time SIR (*Susceptible-Infected-Recovered*) model with time delay and primary immunodeficiency. We have divided the susceptible in to two populations, a small part of the population is affected with primary immunodeficiency and the other population without primary immunodeficiency. We have considered five cases and observe the changes in the numerical simulations. In *Case* 1, we see the susceptible, susceptible 1 and infected populations. In *Case* 2, we have increased the transmission rates and see that there is a decrease in susceptible population and increase in infected population. In *Case* 3, there is considerable increase in infected population due to the increase in contact rate. In *Case* 4, there is a decrease in infected population and the susceptible population is high. In *Case* 5, the infected population decreases to zero due to low transmission rate.

8. Conclusion

In this paper, we have proposed a SIR (*Susceptible-Infected-Recovered*) epidemic model with primary immunodeficiency and time delay and have analyzed its dynamical behaviour. We have derived the basic reproduction number R_0 of the model. For the discrete-time model, we have analyzed the global asymptotic stability of the disease-free equilibrium and endemic equilibrium respectively. Finally, we have provided a numerical simulations through MATLAB for the model and have discussed about the effect of transmission rates on the spread of the epidemic.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] N. Daldosso and L. Pavesi, *Nanosilicon*, Chapter 1, Vijay Kumar (ed.), Elsevier, New York (2005).

[2] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).

[3] S. Elaydi, *An Introduction to Difference Equations*, 3rd edition, Springer (2004).

[4] Y. Enatsu, Y. Nakata and Y. Muroya, Global stability for a discrete SIS epidemic model with immigration of infectives, *Journal of Difference Equations and Applications* **18** (2012), 1913.

[5] S. Jang and S. Elaydi, Difference equations from discretization of a continuous epidemic model with immigration of infectives, *Canadian Applied Mathematics Quarterly* **11** (Spring 2003), 93 – 105.

[6] P. Zhong, L.M. Agosto, J.B. Munro and W. Mothes, Cell-to-cell transmission of viruses, *Current Opinion in Virology* **3**(1) (2013), 44 – 50

[7] W. Mothes, N.M. Sherer, J. Jin and P. Zhong, Virus cell-to-cell transmission, *Journal of Virology* **84**(17) (September 2010), 8360 – 8368.

[8] X. Lai, Study of virus dynamics on mathematics models, *Electronic Thesis and Dissertation Repository*, Paper number 1978 (2014).

[9] A. S. Perelson, Modelling viral and immune system dynamics, *Immunology* **2**(1) (2002), 28 – 36.

[10] *Mathematical Modeling of Virus Dynamics in Immunology*, `http://ir.lib.uwo.ca/cgi/viewcontent.cgi?article=3391&context=etd`

[11] M. Li and X. Liu, An SIR epidemic model with time delay and general nonlinear incidence rate, *Abstract and Applied Analysis* **2014**, Article ID 131257 (2014).