Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 835–842, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

On Acyclic Coloring of Mycielskians

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Abstract. An acyclic coloring of a graph G is a proper vertex coloring such that the induced subgraph of any two color classes is acyclic. The minimum number of colors needed to acyclically color the vertices of a graph G is called as acyclic chromatic number and is denoted by $\chi_a(G)$. In this paper, we give the exact value of the acyclic chromatic number of Mycielskian graph of cycles, paths, complete graphs and complete bipartite graphs.

Keywords. Acyclic coloring; Mycielskian

MSC. 05C15; 05C75

Received: January 9, 2017 **Accepted:** March 17, 2017

1. Introduction

A proper vertex coloring (or proper coloring) of a graph G is a mapping $\phi: V \to N^+$ such that if a and b are adjacent vertices, then $\phi(a) \neq \phi(b)$. The chromatic number of a graph G is the minimum number of colors required in any proper coloring of G. The notion of acyclic coloring was introduced by Branko Grünbaum [3] in 1973. An acyclic coloring of a graph G is a proper vertex coloring such that the induced subgraph of any two color classes is acyclic, i.e., disjoint collection of trees. The minimum number of colors needed to acyclically color the vertices of a graph G is called as acyclic chromatic number and is denoted by $\chi_a(G)$.

2. Preliminaries

We consider only finite, undirected, loopless graphs without multiple edges. The open neighborhood of a vertex x in a graph G, denoted by $N_G(x)$, is the set of all vertices of G, which are adjacent to x. Also, $N_G[x] = N_G(x \cup \{x\}$ is called the closed neighborhood of x in the graph G.

In this paper, by G we mean a connected graph. From a graph G, by Mycielski's construction [2,5,6], we get the *Mycielskian* $\mu(G)$ of G with $V(\mu(G)) = V \cup U \cup \{z\}$, where $V = V(G) = \{x_1,\dots,x_n\}$, $U = \{y_1,\dots,y_n\}$ and $E(\mu(G)) = E(G) \cup \{y_i x : x \in N_G(x_i) \cup \{z\}, i = 1,\dots,n\}$.

Definition 2.1 ([3]). An aylic coloring of a graph G is a proper coloring such that the union of any two color classes induces a forest.

Additional graph theory terminology used in this paper can be found in [1,4].

3. Main Results

In the following subsections, we find the the acyclic chromatic number of Mycielskian graph of cycles, paths, complete graphs and complete bipartite graphs.

3.1 Acyclic Coloring on Mycielskian of Cycles

Theorem 3.1. *For any cycle C_n, $n \geq 3$, the acyclic chromatic number of Mycielskian*

$$\chi_a(\mu(C_n)) = \begin{cases} 4 & \textit{if } n = 3m, m = 1,2,3\dots \\ 5 & \textit{otherwise.} \end{cases}$$

Proof. Let $V(\mu(C_n)) = \{X,Y,z\} = \{u_1,u_2,\dots,u_n;u'_1,u'_2,\dots,u'_n;z\}$ for a total of $2n+1$.

Case 1: $n = 3m$, $m = 1,2,3\dots$

Let σ be a mapping defined as $V(\mu(C_n)) \to c_i$ for $1 \leq i \leq 4$ as follows:

- For $1 \leq i \leq 3$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $4 \leq i \leq n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2$ and $\sigma(u_i) = \sigma(u'_i) = c_3$ if $i \equiv 0 \bmod 3$.
- $\sigma(z) = c_4$.

The color class of

- c_1 is $\left\{u_{3k-2}, u'_{3k-2} : 1 \leq k \leq \dfrac{n}{3}\right\}$

- c_2 is $\left\{u_{3k-1}, u'_{3k-1} : 1 \le k \le \frac{n}{3}\right\}$
- c_3 is $\left\{u_{3k}, u'_{3k} : 1 \le k \le \frac{n}{3}\right\}$
- c_4 is $\{z\}$.

The induced subgraph of any two of these color classes is a forest whose components are star graphs say $K_{1,2}$, $K_{1,3}, \dots$. Thus, by Definition 2.1, σ is a proper acyclic coloring and $\chi_a(\mu(C_n)) = 4$. For if $\chi_a(\mu(C_n)) < 4$, then there exists any one bicolored cycle C_4. A contradiction to proper star coloring. Hence, $\chi_a(\mu(C_n)) = 4$.

Case 2: $n \ne 3m$ and $n \equiv 0 \bmod 4$.
Let σ be a mapping defined as $V(\mu(C_n)) \to c_i$ for $1 \le i \le 5$ as follows:

- For $1 \le i \le 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $5 \le i \le n$, $\sigma(u_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \bmod 4$.
- For $5 \le i \le n$, $\sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(z) = c_5$.

The color class of

- c_1 is $\left\{u_{4k-3}, u'_{4k-3} : 1 \le k \le \frac{n}{4}\right\}$
- c_2 is $\left\{u_{4k-2}, u'_{4k-2}, u'_n : 1 \le k \le \frac{n}{4}\right\}$
- c_3 is $\left\{u_{4k-1}, u'_{4k-1} : 1 \le k \le \frac{n}{4}\right\}$
- c_4 is $\left\{u_{4k}, u'_{4k} : 1 \le k \le \frac{n}{4}\right\}$
- c_5 is $\{z, u_n\}$.

Case 3: For $n \ne 3m$ and $n \equiv 1 \bmod 4$
Let σ be a mapping defined as $V(\mu(C_n)) \to c_i$ for $1 \le i \le 5$ as follows:

- For $1 \le i \le 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $5 \le i \le n-1$, $\sigma(u_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(u_n) = c_5$.
- For $5 \le i \le n-1$, $\sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u'_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(u'_n) = c_2$.
- $\sigma(z) = c_5$.

The color class of

- c_1 is $\left\{u_{4k-3}, u'_{4k-3} : 1 \le k \le \frac{n-1}{4}\right\}$

- c_2 is $\left\{u_{4k-2}, u'_{4k-2}, u'_n : 1 \le k \le \dfrac{n-1}{4}\right\}$
- c_3 is $\left\{u_{4k-1}, u'_{4k-1} : 1 \le k \le \dfrac{n-1}{4}\right\}$
- c_4 is $\left\{u_{4k}, u'_{4k} : 1 \le k \le \dfrac{n-1}{4}\right\}$
- c_5 is $\{u_n, z\}$.

Case 4: $n \ne 3m$ and $n \equiv 2 \bmod 4$.
Let σ be a mapping defined as $V(\mu(C_n)) \to c_i$ for $1 \le i \le 5$ as follows:

- For $1 \le i \le 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$
- For $5 \le i \le n-1$, $\sigma(u_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(u_n) = c_5$.
- For $5 \le i \le n$, $\sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(z) = c_5$.

The color class of

- c_1 is $\left\{u_{4k-3}, u'_{4k-3} : 1 \le k \le \dfrac{n+2}{4}\right\}$
- c_2 is $\left\{u_{4k-2}, u'_{4k-2}, u'_n : 1 \le k \le \dfrac{n-2}{4}\right\}$
- c_3 is $\left\{u_{4k-1}, u'_{4k-1} : 1 \le k \le \dfrac{n-2}{4}\right\}$
- c_4 is $\left\{u_{4k}, u'_{4k} : 1 \le k \le \dfrac{n-2}{4}\right\}$
- c_5 is $\{z, u_n\}$.

Case 5: $n \ne 3m$ and $n \equiv 3 \bmod 4$.
Let σ be a mapping defined as $V(\mu(C_n)) \to c_i$ for $1 \le i \le 5$ as follows:

- For $1 \le i \le 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $5 \le i \le n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2,3$ and $\sigma(u_i) = \sigma(u'_i) = c_4$ if $i \equiv 0 \bmod 4$.
- $\sigma(z) = c_5$.

The color class of

- c_1 is $\left\{u_{4k-3}, u'_{4k-3} : 1 \le k \le \dfrac{n+1}{4}\right\}$
- c_2 is $\left\{u_{4k-2}, u'_{4k-2} : 1 \le k \le \dfrac{n+1}{4}\right\}$

- c_3 is $\left\{u_{4k-1}, u'_{4k-1} : 1 \le k \le \dfrac{n+1}{4}\right\}$
- c_4 is $\left\{u_{4k}, u'_{4k} : 1 \le k \le \dfrac{n-3}{4}\right\}$
- of c_5 is $\{z\}$.

From the Cases 2, 3, 4 and 5 and by the Definition 2.1, σ is a proper acyclic coloring and $\chi_a(\mu(P_n)) = 4$. For if $\chi_a(\mu(C_n)) < 5$, then there exists any one bicolored cycle C_4. A contradiction to proper star coloring. Hence, $\chi_a(\mu(C_n)) = 5$. □

3.2 Acyclic Coloring on Mycielskian of Paths

Theorem 3.2. *For $n \ge 4$, $\chi_a(\mu(P_n)) = 4$.*

Proof. Let $V(\mu(P_n)) = \{u_1, u_2, \dots, u_n\} \cup \{u'_1, u'_2, \dots, u'_n\} \cup \{z\}$.

Let σ be a mapping defined as $V(\mu(P_n)) \to c_i$ for $1 \le i \le 4$ such that

- For $1 \le i \le 3$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $4 \le i \le n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \bmod 3$ for $k = 1,2$ and $\sigma(u_i) = \sigma(u'_i) = c_3$ if $i \equiv 0 \bmod 3$.
- $\sigma(z) = c_4$.

To prove that σ is a proper acyclic coloring consider the discussion of the following cases:

Case 1: Consider the colors c_1 and c_2. The color class of c_1 is

$$\{u_{3k-2}, u'_{3k-2}\}$$

and that of c_2 is

$$\{u_{3k-1}, u'_{3k-1}\}.$$

The induced subgraph of these color classes is a forest, containing stars $K_{1,2}$

Case 2: Consider the colors c_2 and c_3. The color class of c_2 is

$$\{u_{3k-1}, u'_{3k-1}\}$$

and that of c_3 is

$$\{u_{3k}, u'_{3k}\}.$$

The induced subgraph of these color classes is a forest, whose components are $K_{1,2}$.

Case 3: Consider the colors c_3 and c_1. The color class of c_3 is

$$\{u_{3k}, u'_{3k}\}$$

and that of c_1 is

$$\{u_{3k-2}, u'_{3k-2}\}.$$

The induced subgraph of these color classes is a forest, whose components are $K_{1,2}$.

Case 4: Consider the colors c_4 and $c_i : 1 \le i \le 3$. The color class of c_4 is

$$\{z\}$$

and that of c_i is

$$\{u_i, u'_i : 1 \le k \le n\}.$$

The induced subgraph of these color classes is a forest whose components are $K_{1,3}$.

From the above cases and by the definition 2.1, σ is a proper acyclic coloring and hence, $\chi_a(\mu(P_n)) = 4$. For if $\chi_a(\mu(P_n)) < 4$, then there exists bicolored cycles. Hence,

$$\chi_a(\mu(P_n)) = 4.$$ □

3.3 Acyclic Coloring on Mycielskian of Complete Graphs

Theorem 3.3. *For $n \ge 4$, $\chi_a(\mu(K_n)) = n + 1$.*

Proof. Let

$$V(\mu(K_n)) = \{u_1, u_2, \dots, u_n\} \cup \{u'_1, u'_2, \dots, u'_n\} \cup \{z\}.$$

Let σ be a mapping defined as $V(\mu(K_n)) \to c_i$ for $1 \le i \le 4$ such that

- For $1 \le i \le n$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- $\sigma(z) = c_{n+1}$.

It is clear that for a complete graph K_n, $\chi(K_n) = n$ for proper coloring. Thus, $\chi_a(\mu(K_n)) \ge n$. Suppose that one of the existing n colors is assigned to the left out vertex z. A contradiction to proper coloring, since each $y_i : 1 \le i \le n$ is adjacent to z. Hence, $\chi_a(\mu(K_n)) = n + 1$. □

3.4 Acyclic Coloring on Mycielskian of Complete Bipartite Graphs

Theorem 3.4. *Let n and m be positive integers, then*

$$\chi_a(\mu(K_{m,n})) = 2\{\min(m,n) + 1\}.$$

Proof. For a complete bipartite graph $K_{m,n}$ with vertex set

$$V(K_{m,n}) = \{u_1, u_2, \dots, u_m\} \cup \{v_1, v_2, \dots, v_n\},$$

its Mycielskian graph is defined as follows: The vertex set of $\mu(K_{m,n})$ is

$$V(\mu(K_{m,n})) = \{U_m, V_n, U'_m, V'_n, z\},$$

where

$$U_m = \{u_1, u_2, \dots u_m\},\ V_n = \{v_1, v_2, \dots v_n\},\ U'_m = \{u'_1, u'_2, \dots, u'_m\},\ V'_n = \{v'_1, v'_2, \dots, v'_n\}.$$

for a total of $2m + 2n + 1$ vertices.

Case 1: $m < n$

Assign σ as acyclic coloring for $\mu(K_{m,n})$ as follows:

- Assign the color c_1 to the vertex z.

- For $1 \le i \le m$, assign the color c_{i+1} for the vertices u'_i.
- For $1 \le i \le n$, assign the color c_{m+2} for the vertices v'_i.
- For $1 \le i \le m$, assign the color c_{m+2+i} for the vertices u_i.
- For $1 \le i \le n$, assign the color c_{m+2} for the vertices v_i.

Thus $\chi_a(\mu(K_{m,n})) = 2m+2 = 2(m+1) = 2\{\min(m,n)+1\}$. Suppose that $\chi_a(\mu(K_{m,n})) < 2m+2$, say $2m+1$. Then the vertex u_m should be colored either with $c(u'_i)$, $1 \le i \le m$ or with $c(u_i)$, $1 \le i \le m-1$ which results in bicolored cycles. This is a contradiction to proper acyclic coloring, acyclic coloring with $2m+1$ colors is impossible. Thus, $\chi_a(\mu(K_{m,n})) = 2m+2$.

Case 2: $n < m$

Assign σ as acyclic coloring as follows:

- Assign the color c_1 to the vertex z.
- For $1 \le i \le m$, assign the color c_2 for the vertices u'_i.
- For $1 \le i \le n$, assign the color c_{2+i} for the vertices v'_i.
- For $1 \le i \le m$, assign the color c_2 for the vertices u_i.
- For $1 \le i \le n$, assign the color c_{n+2+i} for the vertices v_i.

Thus $\chi_a(\mu(K_{m,n})) = 2n+2 = 2(n+1) = 2\{\min(m,n)+1\}$.

Suppose that $\chi_a(\mu(K_{m,n})) < 2n+2$, say $2n+1$. Then the vertex v_n should be colored either with $c(v'_i)$, $1 \le i \le n$ or with $c(v_i)$, $1 \le i \le n-1$ which results in bicolored cycles. This is a contradiction to proper acyclic coloring, acyclic coloring with $2n+1$ colors is impossible. Thus, $\chi_a(\mu(K_{m,n})) = 2n+2$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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