On Acyclic Coloring of Mycielskians

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Abstract. An acyclic coloring of a graph G is a proper vertex coloring such that the induced subgraph of any two color classes is acyclic. The minimum number of colors needed to acyclically color the vertices of a graph G is called as acyclic chromatic number and is denoted by \( \chi_a(G) \). In this paper, we give the exact value of the acyclic chromatic number of Mycielskian graph of cycles, paths, complete graphs and complete bipartite graphs.

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1. Introduction

A proper vertex coloring (or proper coloring) of a graph G is a mapping \( \phi: V \rightarrow N^+ \) such that if \( a \) and \( b \) are adjacent vertices, then \( \phi(a) \neq \phi(b) \). The chromatic number of a graph G is the
minimum number of colors required in any proper coloring of $G$. The notion of acyclic coloring was introduced by Branko Grünbaum [3] in 1973. An acyclic coloring of a graph $G$ is a proper vertex coloring such that the induced subgraph of any two color classes is acyclic, i.e., disjoint collection of trees. The minimum number of colors needed to acyclically color the vertices of a graph $G$ is called as acyclic chromatic number and is denoted by $\chi_a(G)$.

2. Preliminaries

We consider only finite, undirected, loopless graphs without multiple edges. The open neighborhood of a vertex $x$ in a graph $G$, denoted by $N_G(x)$, is the set of all vertices of $G$, which are adjacent to $x$. Also, $N_G[x] = N_G(x \cup \{x\})$ is called the closed neighborhood of $x$ in the graph $G$.

In this paper, by $G$ we mean a connected graph. From a graph $G$, by Mycielski’s construction [2,5,6], we get the Mycielskian $\mu(G)$ of $G$ with $V(\mu(G)) = V \cup U \cup \{z\}$, where $V = V(G) = \{x_1, \ldots, x_n\}$, $U = \{y_1, \ldots, y_n\}$ and $E(\mu(G)) = E(G) \cup \{y_ix : x \in N_G(x_i) \cup \{z\}, i = 1, \ldots, n\}$.

Definition 2.1 ([3]). An acyclic coloring of a graph $G$ is a proper coloring such that the union of any two color classes induces a forest.

Additional graph theory terminology used in this paper can be found in [1,4].

3. Main Results

In the following subsections, we find the the acyclic chromatic number of Mycielskian graph of cycles, paths, complete graphs and complete bipartite graphs.

3.1 Acyclic Coloring on Mycielskian of Cycles

Theorem 3.1. For any cycle $C_n$, $n \geq 3$, the acyclic chromatic number of Mycielskian $\chi_a(\mu(C_n)) = \begin{cases} 4 & \text{if } n = 3m, m = 1, 2, 3 \ldots \\ 5 & \text{otherwise.} \end{cases}$

Proof. Let $V(\mu(C_n)) = \{X, Y, z\} = \{u_1, u_2, \ldots, u_n; u'_1, u'_2, \ldots, u'_n; z\}$ for a total of $2n + 1$.

Case 1: $n = 3m, m = 1, 2, 3 \ldots$

Let $\sigma$ be a mapping defined as $V(\mu(C_n)) \rightarrow c_i$ for $1 \leq i \leq 4$ as follows:

- For $1 \leq i \leq 3$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $4 \leq i \leq n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \mod 3$ for $k = 1, 2$ and $\sigma(u_i) = \sigma(u'_i) = c_3$ if $i \equiv 0 \mod 3$.
- $\sigma(z) = c_4$.

The color class of $c_1$ is $\{u_{3k-2}, u'_{3k-2} : 1 \leq k \leq \frac{n}{3}\}$.
Case 2: \( n \neq 3m \) and \( n \equiv 0 \mod 4 \).

Let \( \sigma \) be a mapping defined as \( V(\mu(C_n)) \rightarrow c_i \) for \( 1 \leq i \leq 5 \) as follows:

- For \( 1 \leq i \leq 4 \), \( \sigma(u_i) = \sigma(u'_i) = c_i \).
- For \( 5 \leq i \leq n \), \( \sigma(u_i) = c_k \) if \( i \equiv k \mod 3 \) for \( k = 1, 2, 3 \) and \( \sigma(u_i) = c_4 \) if \( i \equiv 0 \mod 4 \).
- \( \sigma(u_n) = c_5 \).

The color class of

- \( c_1 \) is \( \{ u_{4k-3}, u'_{4k-3} : 1 \leq k \leq \frac{n}{4} \} \)
- \( c_2 \) is \( \{ u_{4k-2}, u'_{4k-2}, u'_n : 1 \leq k \leq \frac{n}{4} \} \)
- \( c_3 \) is \( \{ u_{4k-1}, u'_{4k-1} : 1 \leq k \leq \frac{n}{4} \} \)
- \( c_4 \) is \( \{ u_{4k}, u'_{4k} : 1 \leq k \leq \frac{n}{4} \} \)
- \( c_5 \) is \( \{ z, u_n \} \).

Case 3: For \( n \neq 3m \) and \( n \equiv 1 \mod 4 \)

Let \( \sigma \) be a mapping defined as \( V(\mu(C_n)) \rightarrow c_i \) for \( 1 \leq i \leq 5 \) as follows:

- For \( 1 \leq i \leq 4 \), \( \sigma(u_i) = \sigma(u'_i) = c_i \).
- For \( 5 \leq i \leq n-1 \), \( \sigma(u_i) = c_k \) if \( i \equiv k \mod 3 \) for \( k = 1, 2, 3 \) and \( \sigma(u_i) = c_4 \) if \( i \equiv 0 \mod 4 \).
- \( \sigma(u_n) = c_5 \).
- For \( 5 \leq i \leq n-1 \), \( \sigma(u'_i) = c_k \) if \( i \equiv k \mod 3 \) for \( k = 1, 2, 3 \) and \( \sigma(u'_i) = c_4 \) if \( i \equiv 0 \mod 4 \).
- \( \sigma(u'_n) = c_2 \).
- \( \sigma(z) = c_5 \).

The color class of

- \( c_1 \) is \( \{ u_{4k-3}, u'_{4k-3} : 1 \leq k \leq \frac{n-1}{4} \} \)
Case 4: $n \neq 3m$ and $n \equiv 2 \mod 4$.
Let $\sigma$ be a mapping defined as $V(\mu(C_n)) \rightarrow c_i$ for $1 \leq i \leq 5$ as follows:

- For $1 \leq i \leq 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$
- For $5 \leq i \leq n - 1$, $\sigma(u_i) = c_k$ if $i \equiv k \mod 3$ for $k = 1, 2, 3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \mod 4$.
- $\sigma(u_n) = c_5$.
- For $5 \leq i \leq n$, $\sigma(u'_i) = c_k$ if $i \equiv k \mod 3$ for $k = 1, 2, 3$ and $\sigma(u_i) = c_4$ if $i \equiv 0 \mod 4$.
- $\sigma(z) = c_5$.

The color class of

- $c_1$ is $\{u_{4k-3}, u'_{4k-3} : 1 \leq k \leq \frac{n+2}{4}\}$
- $c_2$ is $\{u_{4k-2}, u'_{4k-2}, u_n' : 1 \leq k \leq \frac{n-2}{4}\}$
- $c_3$ is $\{u_{4k-1}, u'_{4k-1} : 1 \leq k \leq \frac{n-2}{4}\}$
- $c_4$ is $\{u_{4k}, u'_{4k} : 1 \leq k \leq \frac{n-2}{4}\}$
- $c_5$ is $\{u_n, z\}$.

Case 5: $n \neq 3m$ and $n \equiv 3 \mod 4$.
Let $\sigma$ be a mapping defined as $V(\mu(C_n)) \rightarrow c_i$ for $1 \leq i \leq 5$ as follows:

- For $1 \leq i \leq 4$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- For $5 \leq i \leq n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \mod 3$ for $k = 1, 2, 3$ and $\sigma(u_i) = \sigma(u'_i) = c_4$ if $i \equiv 0 \mod 4$.
- $\sigma(z) = c_5$.

The color class of

- $c_1$ is $\{u_{4k-3}, u'_{4k-3} : 1 \leq k \leq \frac{n+1}{4}\}$
- $c_2$ is $\{u_{4k-2}, u'_{4k-2} : 1 \leq k \leq \frac{n+1}{4}\}$
• $c_3$ is $\{u_{4k-1}, u'_{4k-1} : 1 \leq k \leq \frac{n+1}{4}\}$
• $c_4$ is $\{u_{4k}, u'_{4k} : 1 \leq k \leq \frac{n-3}{4}\}$
• of $c_5$ is $\{z\}$.

From the Cases 2, 3, 4 and 5 and by the Definition 2.1, $\sigma$ is a proper acyclic coloring and $\chi_a(\mu(P_n)) = 4$. For if $\chi_a(\mu(C_n)) < 5$, then there exists any one bicolored cycle $C_4$. A contradiction to proper star coloring. Hence, $\chi_a(\mu(C_n)) = 5$.

3.2 Acyclic Coloring on Mycielskian of Paths

Theorem 3.2. For $n \geq 4$, $\chi_a(\mu(P_n)) = 4$.

Proof. Let $V(\mu(P_n)) = \{u_1, u_2, \ldots, u_n\} \cup \{u'_1, u'_2, \ldots, u'_n\} \cup \{z\}$.

Let $\sigma$ be a mapping defined as $V(\mu(P_n)) \rightarrow c_i$ for $1 \leq i \leq 4$ such that

• For $1 \leq i \leq 3$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
• For $4 \leq i \leq n$, $\sigma(u_i) = \sigma(u'_i) = c_k$ if $i \equiv k \mod 3$ for $k = 1,2$ and $\sigma(u_i) = \sigma(u'_i) = c_3$ if $i \equiv 0 \mod 3$.
• $\sigma(z) = c_4$.

To prove that $\sigma$ is a proper acyclic coloring consider the discussion of the following cases:

Case 1: Consider the colors $c_1$ and $c_2$. The color class of $c_1$ is $\{u_{3k-2}, u'_{3k-2}\}$ and that of $c_2$ is $\{u_{3k-1}, u'_{3k-1}\}$. The induced subgraph of these color classes is a forest, containing stars $K_{1,2}$.

Case 2: Consider the colors $c_2$ and $c_3$. The color class of $c_2$ is $\{u_{3k-1}, u'_{3k-1}\}$ and that of $c_3$ is $\{u_{3k}, u'_{3k}\}$. The induced subgraph of these color classes is a forest, whose components are $K_{1,2}$.

Case 3: Consider the colors $c_3$ and $c_1$. The color class of $c_3$ is $\{u_{3k}, u'_{3k}\}$ and that of $c_1$ is $\{u_{3k-2}, u'_{3k-2}\}$. The induced subgraph of these color classes is a forest, whose components are $K_{1,2}$. 

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Case 4: Consider the colors $c_4$ and $c_i : 1 \leq i \leq 3$. The color class of $c_4$ is
\[\{z\}\]
and that of $c_i$ is
\[\{u_i, u'_i : 1 \leq k \leq n\}\].
The induced subgraph of these color classes is a forest whose components are $K_{1,3}$.

From the above cases and by the definition 2.1, $\sigma$ is a proper acyclic coloring and hence,
\[\chi_a(\mu(P_n)) = 4.\]
For if $\chi_a(\mu(P_n)) < 4$, then there exists bicolored cycles. Hence,
\[\chi_a(\mu(P_n)) = 4.\]

3.3 Acyclic Coloring on Mycielskian of Complete Graphs

Theorem 3.3. For $n \geq 4$, $\chi_a(\mu(K_n)) = n + 1$.

Proof. Let
\[V(\mu(K_n)) = \{u_1, u_2, \ldots, u_n\} \cup \{u'_1, u'_2, \ldots, u'_n\} \cup \{z\}.
\]
Let $\sigma$ be a mapping defined as $V(\mu(K_n)) \rightarrow c_i$ for $1 \leq i \leq 4$ such that
- For $1 \leq i \leq n$, $\sigma(u_i) = \sigma(u'_i) = c_i$.
- $\sigma(z) = c_{n+1}$.

It is clear that for a complete graph $K_n$, $\chi(K_n) = n$ for proper coloring. Thus, $\chi_a(\mu(K_n)) \geq n$. Suppose that one of the existing $n$ colors is assigned to the left out vertex $z$. A contradiction to proper coloring, since each $y_i : 1 \leq i \leq n$ is adjacent to $z$. Hence, $\chi_a(\mu(K_n)) = n + 1$. \qed

3.4 Acyclic Coloring on Mycielskian of Complete Bipartite Graphs

Theorem 3.4. Let $n$ and $m$ be positive integers, then
\[\chi_a(\mu(K_{m,n})) = 2(\min(m,n) + 1).\]

Proof. For a complete bipartite graph $K_{m,n}$ with vertex set
\[V(K_{m,n}) = \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\},\]
its Mycielskian graph is defined as follows: The vertex set of $\mu(K_{m,n})$ is
\[V(\mu(K_{m,n})) = \{U_m, V_n, U'_m, V'_n, z\},\]
where
\[U_m = \{u_1, u_2, \ldots, u_m\}, V_n = \{v_1, v_2, \ldots, v_n\}, U'_m = \{u'_1, u'_2, \ldots, u'_m\}, V'_n = \{v'_1, v'_2, \ldots, v'_n\}.
\]
for a total of $2m + 2n + 1$ vertices.

Case 1: $m < n$

Assign $\sigma$ as acyclic coloring for $\mu(K_{m,n})$ as follows:
- Assign the color $c_1$ to the vertex $z$. 

• For \(1 \leq i \leq m\), assign the color \(c_{i+1}\) for the vertices \(u'_i\).
• For \(1 \leq i \leq n\), assign the color \(c_{m+2}\) for the vertices \(v'_i\).
• For \(1 \leq i \leq m\), assign the color \(c_{m+2+i}\) for the vertices \(u_i\).
• For \(1 \leq i \leq n\), assign the color \(c_{m+2}\) for the vertices \(v_i\).

Thus \(\chi_a(\mu(K_{m,n})) = 2m + 2 = 2(m + 1) = 2(\min(m, n) + 1)\). Suppose that \(\chi_a(\mu(K_{m,n})) < 2m + 2\), say \(2m + 1\). Then the vertex \(u_m\) should be colored either with \(c(u'_i)\), \(1 \leq i \leq m\) or with \(c(u_i)\), \(1 \leq i \leq m - 1\) which results in bicolored cycles. This is a contradiction to proper acyclic coloring, acyclic coloring with \(2m + 1\) colors is impossible. Thus, \(\chi_a(\mu(K_{m,n})) = 2m + 2\).

**Case 2: \(n < m\)**

Assign \(\sigma\) as acyclic coloring as follows:

• Assign the color \(c_1\) to the vertex \(z\).
• For \(1 \leq i \leq m\), assign the color \(c_2\) for the vertices \(u'_i\).
• For \(1 \leq i \leq n\), assign the color \(c_{2+i}\) for the vertices \(v'_i\).
• For \(1 \leq i \leq m\), assign the color \(c_2\) for the vertices \(u_i\).
• For \(1 \leq i \leq n\), assign the color \(c_{n+2+i}\) for the vertices \(v_i\).

Thus \(\chi_a(\mu(K_{m,n})) = 2n + 2 = 2(n + 1) = 2(\min(m, n) + 1)\).

Suppose that \(\chi_a(\mu(K_{m,n})) < 2n + 2\), say \(2n + 1\). Then the vertex \(v_n\) should be colored either with \(c(v'_i)\), \(1 \leq i \leq n\) or with \(c(v_i)\), \(1 \leq i \leq n - 1\) which results in bicolored cycles. This is a contradiction to proper acyclic coloring, acyclic coloring with \(2n + 1\) colors is impossible. Thus, \(\chi_a(\mu(K_{m,n})) = 2n + 2\).

**Competing Interests**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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