A Note on Right Full $k$-Ideals of Seminearrings

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Abstract. This work extends the idea of $k$-ideals of semirings to seminearrings, the concept of $k$-ideals of seminearrings is introduced and investigated, which is an interesting for seminearrings and some interesting characterizations of $k$-ideals of seminearrings are obtained. Also, we prove that the set of all right full $k$-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular in the same way as of the results of Sen and Adhikari.

1. Introduction and Preliminaries


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generalization of semirings introduced and discussed by Rootsebaar [9] in 1963. Therefore, we will study \( k \)-ideals of seminearrings in the same way as of \( k \)-ideals of semirings which was studied by Sen and Adhikari [10].

The purpose of this paper is threefold.

(i) To introduce the concept of \((\text{left}, \text{right})\) \( k \)-ideals of seminearrings.
(ii) To introduce the concept of \((\text{left}, \text{right})\) full \( k \)-ideals of additively inverse seminearrings.
(iii) To characterize the properties of \((\text{left}, \text{right})\) full \( k \)-ideals of seminearrings, and (left, right) full \( k \)-ideals of additively inverse seminearrings.

For the sake of completeness, we state some definitions and notations that are introduced analogously to some definitions and notations in [10].

A seminearring [8] is a system consisting of a nonempty set \( S \) together with two binary operations on \( S \) called addition and multiplication such that

(i) \( S \) together with addition is a semigroup,
(ii) \( S \) together with multiplication is a semigroup, and
(iii) \((a+b)c = ac + bc\) for all \( a, b, c \in S \).

We define a subseminearring \( A \) of a seminearring \( S \) to be a nonempty subset \( A \) of \( S \) such that when the seminearring operations of \( S \) is restricted to \( A \), \( A \) is a seminearring in its own right. A seminearring \( S \) is said to be additively commutative if \( a + b = b + a \) for all \( a, b \in S \). A nonempty subset \( I \) of a seminearring \( S \) is called a right(left) ideal of \( S \) if

(i) \( a + b \in I \) for all \( a, b \in I \), and
(ii) \( ar \in I \) \((ar \in I)\) for all \( r \in S \) and \( a \in I \).

A nonempty subset \( I \) of a seminearring \( S \) is called an ideal of \( S \) if it is both a left and a right ideal of \( S \). A right(left) ideal \( I \) of a seminearring \( S \) is called a right(left) \( k \)-ideal of \( S \) if for any \( a \in I \) and \( x \in S \), \( a + x \in I \) or \( x + a \in I \) implies \( x \in I \). An \( a \) \in \( S \) is said to be additively inverse if for any \( a \in S \), there exists a unique element \( b \in S \) such that \( a = a + b + a \). A seminearring \( S \) is said to be additively regular if for any \( a \in S \), there exists an element \( b \in S \) such that \( a = a + b + a \) and \( b = b + a + b \). In an additively inverse seminearring, the unique inverse \( b \) of an element \( a \) is usually denoted by \( a' \). An element \( a \) of a seminearring \( S \) is called an additive idempotent of \( S \) if \( a + a = a \) and the set of all additive idempotents of \( S \) denoted by \( E^+ \). A right(left) \( k \)-ideal \( I \) of \( S \) is called a right(left) \( k \)-ideal of \( S \) if \( E^+ \subseteq I \). A nonempty subset \( I \) of an additively inverse seminearring \( S \) is called a full \( k \)-ideal of \( S \) if it is both a left and a right full \( k \)-ideal of \( S \). Let \( S \) be a seminearring and \( A \) a right ideal of \( S \). Define the set

\[ \bar{A} = \{ a \in S \mid a + x \in A \text{ for some } x \in A \}. \]
Let $S$ be an additively inverse seminearring. Define the set of all right full $k$-ideals of $S$ by $I(S)$. An equivalence relation $\rho$ on a seminearring $S$ is called a congruence if for any $a, b, c \in S, (a, b) \in \rho$ implies 
\[ (c + a, c + b) \in \rho \quad \text{and} \quad (a + c, b + c) \in \rho \]
and
\[ (ca, cb) \in \rho \quad \text{and} \quad (ac, bc) \in \rho. \]

We can easily prove that the set of all congruence classes $S/\rho$ is a seminearring under addition and multiplication defined by 
\[ (a)_{\rho} + (b)_{\rho} = (a + b)_{\rho} \quad \text{and} \quad (a)_{\rho}(b)_{\rho} = (ab)_{\rho} \]
for all $a, b \in S$.

A lattice $A$ is said to be modular [3] if for any $x, y, z \in A$, $y \leq x$, $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$ implies $x = y$.

2. Lemmas

Before the characterizations of $k$-ideals of seminearrings for the main results, we give some auxiliary results which are necessary in what follows. The following lemma is easy to verify.

**Lemma 2.1.** Let $S$ be a seminearring and $I$ a right(left) ideal of $S$. Then $I$ is a subseminearring of $S$.

**Corollary 2.2.** Let $S$ be a seminearring and $I$ an ideal of $S$. Then $I$ is a subseminearring of $S$.

**Lemma 2.3.** Let $S$ be an additively commutative seminearring, and $A$ and $B$ two right ideals of $S$. Then $A + B$ is a right ideal of $S$.

**Proof.** Let $x, y \in A + B$ and $r \in S$. Then $x = a_1 + b_1$ and $y = a_2 + b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Thus
\[ x + y = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) \in A + B. \]

Since $A$ and $B$ are right ideals of $S$, we have
\[ xr = (a_1 + b_1)r = a_1r + b_1r \in A + B. \]

Hence $A + B$ is a right ideal of $S$. □

**Lemma 2.4.** Let $S$ be a seminearring and $\mathcal{X} = \{ J \mid J$ is a right(left) ideal of $S \}$. Then $\bigcap_{J \in \mathcal{X}} J$ is a right(left) ideal of $S$ where $\bigcap_{J \in \mathcal{X}} J \neq \emptyset$.

**Proof.** Let $x, y \in \bigcap_{J \in \mathcal{X}} J$ and $r \in S$. Then $x, y \in J$ for all $J \in \mathcal{X}$, so $x + y, xr \in J$ for all $J \in \mathcal{X}$. Thus $x + y, xr \in \bigcap_{J \in \mathcal{X}} J$. Hence $\bigcap_{J \in \mathcal{X}} J$ is a right ideal of $S$. □

**Corollary 2.5.** Let $S$ be a seminearring and $\mathcal{X} = \{ J \mid J$ is an ideal of $S \}$. Then $\bigcap_{J \in \mathcal{X}} J$ is an ideal of $S$ where $\bigcap_{J \in \mathcal{X}} J \neq \emptyset$. 
Lemma 2.6. Let $S$ be a seminearring and $\mathcal{X} = \{J \mid J$ is a right(left) $k$-ideal of $S\}$. Then $\bigcap_{J \in \mathcal{X}} J$ is a right(left) $k$-ideal of $S$ where $\bigcap_{J \in \mathcal{X}} J \neq \emptyset$.

Proof. By Lemma 2.4, we have $\bigcap_{J \in \mathcal{X}} J$ is a right ideal of $S$. Let $x \in \bigcap_{J \in \mathcal{X}} J$ and $r \in S$ be such that $x + r \in \bigcap_{J \in \mathcal{X}} J$. Then $x, x + r \in J$ for all $J \in \mathcal{X}$, so $r \in J$ for all $J \in \mathcal{X}$. Thus $r \in \bigcap_{J \in \mathcal{X}} J$. Hence $\bigcap_{J \in \mathcal{X}} J$ is a right $k$-ideal of $S$. \hfill \Box

Corollary 2.7. Let $S$ be a seminearring and $\mathcal{X} = \{J \mid J$ is a $k$-ideal of $S\}$. Then $\bigcap_{J \in \mathcal{X}} J$ is a $k$-ideal of $S$ where $\bigcap_{J \in \mathcal{X}} J \neq \emptyset$.

Lemma 2.8. Let $S$ be a seminearring and $\mathcal{X} = \{J \mid J$ is a right(left) full $k$-ideal of $S\}$. Then $\bigcap_{J \in \mathcal{X}} J$ is a right(left) full $k$-ideal of $S$.

Proof. By Lemma 2.6, we have $\bigcap_{J \in \mathcal{X}} J$ is a right $k$-ideal of $S$. Since $E^+ \subseteq J$ for all $J \in \mathcal{X}$, we have $E^+ \subseteq \bigcap_{J \in \mathcal{X}} J$. Hence $\bigcap_{J \in \mathcal{X}} J$ is a right full $k$-ideal of $S$. \hfill \Box

Corollary 2.9. Let $S$ be a seminearring and $\mathcal{X} = \{J \mid J$ is a full $k$-ideal of $S\}$. Then $\bigcap_{J \in \mathcal{X}} J$ is a full $k$-ideal of $S$.

Lemma 2.10. Let $S$ be a seminearring, and $A$ and $B$ two right $k$-ideals of $S$. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Proof. Let $a \in \overline{A}$. Then $a + x \in A$ for some $x \in A$. Thus $a + x \in A \subseteq B$ for some $x \in A \subseteq B$, so $a \in \overline{B}$. Hence $\overline{A} \subseteq \overline{B}$. \hfill \Box

Lemma 2.11. Let $S$ be an additively regular seminearring in which addition is commutative. Then $E^+$ is a right ideal of $S$.

Proof. Let $x, y \in E^+$ and $r \in S$. Then $x = x + x$ and $y = y + y$. Thus $(x + y) + (x + y) = (x + x) + (y + y) = x + y$ and $xr + xr = (x + x)r = xr$, so $x + y, xr \in E^+$. Hence $E^+$ is a right ideal of $S$. \hfill \Box

Lemma 2.12. For an additively inverse seminearring $S$, $I(S)$ is a partially ordered set under inclusion. Moreover, if $\mathcal{X} = \{J \mid J \in I(S)\}$, then $\bigcap_{J \in \mathcal{X}} J$ is an infimum of $\mathcal{X}$.

Proof. By Lemma 2.8, we have $\bigcap_{J \in \mathcal{X}} J \in I(S)$. Since $\bigcap_{J \in \mathcal{X}} J \subseteq J$ for all $J \in \mathcal{X}$, we have $\bigcap_{J \in \mathcal{X}} J$ is a lower bound of $\mathcal{X}$. Let $C$ be a lower bound of $\mathcal{X}$. Then $C \subseteq J$ for all $J \in \mathcal{X}$, so $C \subseteq \bigcap_{J \in \mathcal{X}} J$. Hence $\bigcap_{J \in \mathcal{X}} J$ is an infimum of $\mathcal{X}$. \hfill \Box

Lemma 2.13. Let $S$ be an additively commutative seminearring. If $e, f \in E^+$ and $r \in S$, then $e + f, er \in E^+.

Proof. Now, $(e + f) + (e + f) = (e + e) + (f + f) = e + f$ and $er + er = (e + e)r = er$. Hence $e + f, er \in E^+$. \hfill \Box
3. Main Results

In this section, we give some characterizations of $k$-ideals of seminearrings. Finally, we prove that the set of all right full $k$-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular.

**Theorem 3.1.** Let $S$ be an additively inverse seminearring. Then every right (left) $k$-ideal of $S$ is an additively inverse subseminearring of $S$.

**Proof.** Let $I$ be a right $k$-ideal of $S$. By Lemma 2.1, we have $I$ is a subseminearring of $S$. Let arbitrary $a \in I$. Since $S$ is an additively inverse seminearring, we obtain $a + a' + a = a$ and $a' + a + a' = a'$. Now, $a + (a' + a) = a + a' + a = a \in I$. Since $I$ is a right $k$-ideal of $S$, we have $a' + a \in I$. Again, $a' \in I$. Therefore $I$ is an additively inverse subseminearring of $S$. \qed

**Corollary 3.2.** Let $S$ be an additively inverse seminearring. Then every $k$-ideal of $S$ is an additively inverse subseminearring of $S$.

**Theorem 3.3.** Let $S$ be an additively inverse seminearring in which addition is commutative and $A$ a right ideal of $S$. Then

$$\bar{A} = \{a \in S \mid a + x \in A \text{ for all } x \in A\}$$

is a right $k$-ideal of $S$ such that $A \subseteq \bar{A}$.

**Proof.** Let $a, b \in \bar{A}$ and $r \in S$. Then $a + x, b + y \in A$ for some $x, y \in A$. Since $(a + b) + (x + y) = a + x + b + y \in A$ and $x + y \in A$, we have $a + b \in \bar{A}$. Since $ar + xr = (a + x)r \in A$ and $xr \in A$, we have $ar \in \bar{A}$. Hence $\bar{A}$ is a right ideal of $S$. Let $d \in S$ and $c \in \bar{A}$ be such that $c + d \in \bar{A}$. Then there exist $x, y \in A$ such that $c + x \in A$ and $c + d + y \in A$. Thus $d + (c + x + y) = (c + d + y) + x \in A$. Since $c + x + y \in A$, we have $d \in \bar{A}$. Therefore $\bar{A}$ is a right $k$-ideal of $S$. Let $a \in A$. Then $(a + a') + a = a \in A$, so $a + a' \in \bar{A}$. Suppose that $a \notin \bar{A}$. Since $a + a' \notin \bar{A}$, we get $a' \notin \bar{A}$. Since $a' + (a + a) = a + a' + a = a \in A$, we have $a' \notin \bar{A}$ that is a contradiction. Hence $a \in \bar{A}$ and so $A \subseteq \bar{A}$. \qed

**Corollary 3.4.** Let $S$ be an additively inverse seminearring in which addition is commutative and $A$ a right ideal of $S$. Then $\bar{A}$ is an additively inverse subseminearring of $S$ such that $A \subseteq \bar{A}$.

**Corollary 3.5.** Let $S$ be an additively inverse seminearring in which addition is commutative and $A$ a right ideal of $S$. Then $\bar{A} = A$ if and only if $A$ is a right $k$-ideal of $S$.

**Proof.** Assume that $\bar{A} = A$. Then, by Lemma 3.3, we have $\bar{A}$ is a right $k$-ideal of $S$. Hence $A$ is a right $k$-ideal of $S$.

Conversely, assume that $A$ is a right $k$-ideal of $S$. Then, by Lemma 3.3, we have $A \subseteq \bar{A}$. Let $x \in \bar{A}$. Then $x + y \in A$ for some $y \in A$. Since $A$ is a right $k$-ideal of $S$, we have $x \in A$. Thus $\bar{A} \subseteq A$, so $\bar{A} = A$. \qed
Lemma 3.6. Let \( S \) be an additively inverse seminearring in which addition is commutative, and \( A \) and \( B \) two right full \( k \)-ideals of \( S \). Then \( \overline{A \cap B} \) is a right full \( k \)-ideal of \( S \) such that \( A \subseteq \overline{A + B} \) and \( B \subseteq \overline{A + B} \).

Proof. By Lemma 2.3, we have \( A + B \) is a right ideal of \( S \). By Lemma 3.3, we have \( \overline{A + B} \) is a right \( k \)-ideal of \( S \) such that \( A + B \subseteq \overline{A + B} \). Since \( A \) and \( B \) are right full \( k \)-ideals of \( S \), we have \( E^+ \subseteq A \) and \( E^+ \subseteq B \). Now, let \( x \in E^+ \). Then \( x \in A \) and \( x \in B \), so \( x = x \in A + B \). Thus \( E^+ \subseteq A + B \subseteq \overline{A + B} \). Hence \( \overline{A + B} \) is a right full \( k \)-ideal of \( S \). Let \( a \in A \). Then \( a = a + a' + a \). We can show that \( a' + a \in E^+ \). Thus

\[
a = a + a' + a = a + (a' + a) \in A + E^+ \subseteq A + B \subseteq \overline{A + B}.
\]

Hence \( A \subseteq \overline{A + B} \). We can prove in a similar manner that \( B \subseteq \overline{A + B} \). This completes the proof. \( \square \)

Theorem 3.7. For an additively inverse seminearring \( S \) in which addition is commutative, \( I(S) \) is a complete lattice which is also modular.

Proof. By Lemma 2.12, we have \( I(S) \) is a partially ordered set under inclusion. Let \( A, B \in I(S) \). By Lemma 2.8, we have \( A \cap B \in I(S) \). By Lemma 3.6, we have

\[
A + B = \overline{A \cap B} \quad \text{and} \quad A \cap B = \overline{A + B}.
\]

Since \( A \cap B = A \cap B \subseteq A \) and \( A \cap B = A \cap B \subseteq B \), we have \( A \cap B \) is a lower bound of \( A \) and \( B \). Let \( C \in I(S) \) be such that \( C \subseteq A \) and \( C \subseteq B \). Then \( C \subseteq A \cap B = A \cap B \), so \( A \cap B \) is an infimum of \( A \) and \( B \). Since \( A \cap B = \overline{A + B} \) and \( \overline{A + B} \) is a right \( k \)-ideal of \( S \), we have \( A \subseteq \overline{A + B} = A \cap B \) and \( B \subseteq \overline{A + B} = A \cap B \). Thus \( A \cap B \) is an upper bound of \( A \) and \( B \). Let \( D \in I(S) \) be such that \( A \subseteq D \) and \( B \subseteq D \). Then \( A + B \subseteq D \). By Lemma 2.10, we have \( \overline{A + B} \subseteq \overline{D} \). By Corollary 3.5, we have \( \overline{D} = D \) and so \( \overline{A + B} \subseteq D \). Thus \( \overline{A + B} \) is a supremum of \( A \) and \( B \). Hence \( I(S) \) is a lattice. We shall show that \( I(S) \) is a modular lattice. Let \( A, B, C \in I(S) \) be such that \( A \cap B = A \cap C \) and \( A \cap B = A \cap C \) and \( B \subseteq C \). Now, let \( x \in C \). Then \( x \in A \cap C = A \cap B = A \cap B \). Thus there exists \( a + b \in A + B \) such that \( x + a + b \in A + B \), so \( x + a + b = a_1 + b_1 \) for some \( a_1 \in A \) and \( b_1 \in B \). This implies that \( x + a + a' + b = x + a + b + a' = a_1 + b_1 + a' \). Since \( x \in C, a + a' \in C \) and \( b \in B \subseteq C \), we have \( a_1 + b_1 + a' \in C \) but \( b_1 \in C \). Thus \( a_1 + a' \in C \). By Lemma 3.1, we have \( a_1 + a' \in A \) and so \( a_1 + a' \in A \cap C = A \cap B \). Thus \( a_1 + a' \in B \). Since \( x + a + b = a_1 + b_1 \), we have \( x + a + a' + b = a_1 + a' + b_1 \in B \). Since \( (a + a') + b \in B \) and \( B \) is a right \( k \)-ideal of \( S \), we have \( x \in B \) and so \( C \subseteq B \). Thus \( B = C \). Therefore \( I(S) \) is a modular lattice. By Lemma 2.12, we get that \( I(S) \) is complete. \( \square \)

In comparison our above results with results of \( k \)-ideals of semirings, we see that the set of all right full \( k \)-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular which is an analogous result of full \( k \)-ideals of semirings.
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References


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