Abstract. Let $G = (V(G), E(G))$ be a simple graph and an achromatic coloring of $G$ is a proper vertex coloring of $G$ in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of $G$ denoted by $\psi(G)$, is the greatest number of colors in an achromatic coloring of $G$. In this paper, we find out the achromatic number for Corona graph of Cycle with Path graphs on the same order $n$, Path with Cycle graphs on the same order $n$, Path with Complete graphs on the same order $n$, Path of order $n$ with Star graph on order $n+1$, Path with Wheel graphs on the same order $n$ and Ladder graph with Path graph on the same order $n$.

Keywords. Achromatic coloring; Achromatic number; Corona graph

MSC. 05C15

1. Introduction

Let $G$ be a finite undirected graph with no loops and multiple edges. A coloring of a graph $G$ is a partitioning of the vertex set $V$ into color classes. In graph theory, coloring of graphs are very extended areas of research. A coloring of a graph can be described by a function that
maps elements of a graph which are vertex-vertex coloring, edge-edge coloring or total coloring into some set of numbers which are called colors such that some property is satisfied. A proper coloring of the vertices of a graph \( G \) assigns a color to each vertex of \( G \) in such a way that no two adjacent vertices have the same color. The chromatic number \( \chi(G) \) is the minimum number of color required in any proper coloring of \( G \). An achromatic coloring \([2]\) of a graph is a proper vertex coloring such that each pair of color classes is adjacent by at least one edge. The achromatic number was defined and studied by Harary, Hedetniemi and Prins \([3]\). They shown that, for every complete \( n \)-coloring \( \tau \) of a graph \( G \), there exists a complete homomorphism \( \phi \) of \( G \) onto \( K_n \) and conversely. They considered the largest possible number of colors in an achromatic coloring is called the achromatic number and is denoted by \( \psi \). The greatest number of colors used in a complete coloring of \( G \) is the achromatic number \( \alpha(G) \) of \( G \). It is clear that \( \chi(G) \leq \alpha(G) \leq \psi(G) \).

Computing the achromatic number of a general graph was proved to be NP complete by Yannakakis and Gavril \([10]\) in 1980. In 1976, Hell and Miller \([9]\) who found the achromatic number of Paths and Cycles.

Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, etc.,

### 2. Preliminaries

Graph products are interesting and useful in many situations \([7]\). Let \( G_1 \) and \( G_2 \) be two graphs on disjoint sets of \( n_1 \) and \( n_2 \) vertices respectively. The Corona \( G_1 \circ G_2 \) of \( G_1 \) and \( G_2 \) is defined as the graph obtained by taking one copy of \( G_1 \) and \( n_1 \) copies of \( G_2 \) and then joining the \( i \)th vertex of \( G_1 \) to every vertex in the \( i \)th copy of \( G_2 \). The Corona of two graphs was first introduced by Frucht and Harary in 1970 \([8]\).

In this paper, we find the achromatic number for the Corona graph of Cycle with Path graph on the same order \( C_n \circ P_n \), Path with Cycle on the same order \( P_n \circ C_n \), Path with Complete graph on the same order \( P_n \circ K_n \), Path graph on order \( n \) with Star graph on the order \( n+1 \) say \( P_n \circ K_{1,n} \) and Path graph with Wheel graph on the same order \( P_n \circ W_n \) and Ladder graph with Path on the same order \( L_n \circ P_n \).

### 3. Achromatic Number on Corona Graphs

**Theorem 3.1.** For \( n \geq 5 \), the achromatic number of Corona of \( C_n \) with \( P_n \) is \( n + 4 \).

i.e., \( \psi(C_n \circ P_n) = n + 4 \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of Cycle graph \( C_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of Path graph \( P_n \).

i.e., \( V(C_n) = \{ v_i : 1 \leq i \leq n \} \) and \( V(P_n) = \{ u_1, u_2, \ldots, u_n \} \).

By the definition of Corona graph, each vertex of \( C_n \) is adjacent to every vertex of a copy of \( P_n \) i.e., every vertex of \( V(C_n) \) is adjacent to every vertex from the set \( V(P_n) \). Thus the Corona of
two graphs can be defined as

\[ V(C_n \circ P_n) = \{v_1, v_2, \ldots, v_n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}. \]

Let \( P_n^{(1)}, P_n^{(2)}, \ldots, P_n^{(n)} \) be the \( n \)-copies of the Path graph \( P_n \).

Now assign a proper vertex coloring as follows:

Consider the color class \( C = \{c_1, c_2, \ldots, c_{n+4}\} \).

- For \( 1 \leq i \leq n \), assign the color \( c_i \) to \( v_i \).
- For \( P_n^{(i)} \), \( i = 1, 2, 3, j = 2, 3, \ldots, n \), assign the color \( c_{n+j-1} \) to \( u_{ij} \).
- Assign the colors \( c_{i+2} \) to \( u_{ij} \) for \( i = 1, 2, 3 \).
- For \( V(P_n^{(i)}) \) for \( i = 5, 6, \ldots, n \), let us assign the colors to contribute some pairs as follows:
  - Assign \( c_{i+j+1} \) colors to all the vertices of \( V(P_n^{(i)}) \) for \( i = 5 \) and \( 1 \leq j \leq n-2 \).
  - Assign \( c_{i+j+1} \) colors to all the vertices of \( V(P_n^{(i)}) \) for \( i = 6 \) and \( 1 \leq j \leq n-3 \).
  - Assign \( c_{i+j+1} \) colors to all the vertices of \( V(P_n^{(i)}) \) for \( i = n \) and \( 1 \leq j \leq n-(n-3) \).

For the remaining vertices of \( V(P_n^{(i)}) \) for \( i = 5, 6, \ldots, n \), assign the colors \( c_1 \) and \( c_2 \) alternatively.

- In \( V(P_n^{(i)}) \) for \( i = 4 \) of every copy, assign the colors as follows:
  - Assign the colors \( c_{n+3}, c_{n+1}, c_{n+4} \) and \( c_{n+2} \) to \( u_{4j} \) \( (1 \leq j \leq 4) \) and assign the colors \( c_1 \) and \( c_2 \) alternatively to the remaining vertices of \( V_4 \) to make the coloring as achromatic.

Now the coloring makes the non-adjacency condition is possible. Thus by the procedure of achromatic coloring, the coloring accommodates all the pairs of the color class and hence it is maximal. An easy check shows that the above said coloring is achromatic.

Hence, \( \psi(C_n \circ P_n) = n + 4 \), for \( n \geq 5 \). \square

**Theorem 3.2.** For \( n \geq 5 \), the achromatic coloring of Corona of Path graph \( P_n \) with Cycle graph \( C_n \) is \( n + 4 \).

i.e., \( \psi(P_n \circ C_n) = n + 4 \).

**Proof.** The proof is same as Theorem 3.1. \square

**Theorem 3.3.** For \( n \geq 2 \), the achromatic coloring of Corona of \( P_n \) with \( K_n \) is \( 2n \).

i.e., \( \psi(P_n \circ K_n) = 2n \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of Path graph \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of Complete graph \( K_n \).

i.e., \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( V(K_n) = \{u_1, u_2, \ldots, u_n\} \).

Let \( V(P_n \circ K_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\} \).
By the definition of Corona graph, each vertex of \( P_n \) is adjacent to every vertex of a copy of \( K_n \) i.e., every vertex \( v_i \in V(P_n) \) is adjacent to every vertex from the set \( \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\} \). Thus the Corona of two graphs Path with Complete graphs is

\[
V(P_n \circ K_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}.
\]

Assign the following \( 2n \) coloring for \( P_n \circ K_n \) as achromatic:

Let \( K_{n}^{(1)}, K_{n}^{(2)}, \ldots, K_{n}^{(n)} \) be the \( n \)-copies of the Complete graph \( K_n \).

Consider the color class \( C = \{c_1, c_2, \ldots, c_{2n}\} \).

- For \( 1 \leq i \leq n \), assign the color \( c_i \) to \( v_i \).
- For \( 1 \leq i \leq n - 1 \), assign the color \( c_{i+2} \) to \( u_{ij} \).
- For \( K_{n}^{(i)} \), where \( 1 \leq i \leq n - 1 \), \( 2 \leq j \leq n - 1 \), assign the color \( c_{n+j} \).
- Assign the color \( c_i \) to all the vertices of \( V(K_{n}^{(n)}) \) for \( i = n \) and \( 1 \leq j \leq n - 1 \).
- Assign the color \( c_{n+1} \) to the missing vertex in \( V(K_{n}^{(i)}) \) for \( i = n \) and \( j = n \).

Thus any pair in the color class is adjacent by atleast one edge and by the very construction this coloring accommodates maximum number of pairs in the color class. Therefore, \( \psi(P_n \circ K_n) = 2n \), for \( n \geq 2 \).

**Theorem 3.4.** For \( n \geq 3 \), the achromatic coloring of Corona of \( P_n \) with \( K_{1,n} \) is \( 2n \).

i.e., \( \psi(P_n \circ K_{1,n}) = 2n \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the \( n \)-vertices of \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the \( n \)-vertices of \( K_{1,n} \).

i.e., \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( V(K_{1,n}) = \{u_i, u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\} \).

By the definition of corona graph, each vertex of \( P_n \) is adjacent to every copy of \( K_{1,n} \) i.e., every vertex \( v_i \in V(P_n) \) is adjacent to every vertex from the set \( \{u_i, u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\} \). Thus the Corona of two graphs is,

\[
V(P_n \circ K_{1,n}) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}.
\]

Let \( K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(n)} \) be the \( n \)-copies of the Star graph \( K_{1,n} \).

Now assign a proper vertex coloring as follows:

Consider the color class \( C = \{c_1, c_2, \ldots, c_{2n}\} \).

- Assign the color \( c_1 \) to \( v_1 \) and for \( 1 \leq i \leq n \), assign the color \( c_{1+2i} \) to \( u_{i+1} \).
- For \( 1 \leq i \leq n \), color the vertices \( u_i \) with color \( c_{2i} \).

To make the coloring as achromatic one, color the remaining vertices as follows:

- Color the vertices \( u_{ij} \) with color \( c_{ui+j} \) up to \( c_{2n} \), \( n \geq 3 \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).

Now this coloring will accommodates maximum number of pairs of the color class and an easy check shows that the above said coloring is achromatic. Hence \( \psi(P_n \circ K_{1,n}) = 2n \), for \( n \geq 3 \).

**Theorem 3.5.** For \( n \geq 3 \), the achromatic coloring of Corona of \( P_n \) with \( W_n \) is \( 2n \).

i.e., \( \psi(P_n \circ W_n) = 2n \).
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of Path graph $P_n$ and $u_1, u_2, \ldots, u_n$ be the vertices of Wheel graph $W_n$ with $u_n$ as the hub.

i.e., $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(W_n) = \{u_1, u_2, \ldots, u_n\}$.

By the definition of corona graph, each vertex of $P_n$ is adjacent to every vertex of a cop of $W_n$ i.e., every vertex of $V(P_n)$ is adjacent to every vertex from the set $V(W_n)$. Thus the Corona of two graphs Path with Wheel is, $V(P_n \circ W_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq \frac{n}{2}\}$.

Let $G_1 = P_n$ be a Path graph with $n$-vertices and $G_2 = W_n$ be a Wheel graph with $n$-vertices.

Now the corona $G = P_n \circ W_n$ is obtained by taking one copy of $P_n$ of order $n$ and $n$-copies of $W_n$ and then joining the $i$th vertex of $P_n$ to every vertex on the $i$th copy of $W_n$.

Let $W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(n)}$ be the $n$-copies of the Wheel graph $W_n$.

Now we consider the following two cases:

Case (i): when $n$ is odd
Consider the color class $C = \{c_1, c_2, \ldots, c_{2n}\}$.

- Assign the color $c_1$ to $v_1$ and assign the color $c_{1+2i}$ to the vertices $v_{i+1}$ for $1 \leq i < n$.
- Assign the color $c_{2i}$ to $u_{i1}$ for $1 \leq i \leq n$.
- Assign the color $c_{ij}$ to $u_{ij}$ and assign $c_{ui1+j-1}$ to $u_{ij}$ up to $c_{2n}$, $n \geq 3$ for $1 \leq i \leq n$, $2 \leq j \leq n$.
  - Color the remaining vertices of $u_{ij}$ with the color $c_i$ for $1 \leq i < n$.

This makes the above said coloring is achromatic one, since we have assigned colors to satisfy the definition of achromatic coloring.

Case (ii): when $n$ is even

- Assign the color $c_1$ to $v_1$ and assign the color $c_{1+2i}$ to the vertices $v_{i+1}$ for $1 \leq i < n$.
- Assign the color $c_{2i}$ to $u_{i1}$ for $1 \leq i \leq n$.
- For $1 \leq i \leq n/2$, $2 \leq j \leq n$, assign the color $c_{ui1+j-1}$ to $u_{ij}$ up to $c_{2n}$, $n \geq 3$ and for the remaining $u_{ij}$’s, assign the color $c_i$ for $1 \leq i \leq n$.

This coloring will accommodate all the missing pairs and an easy check shows that the above said coloring is achromatic. Therefore $\psi(P_n \circ K_{1,n}) = 2n$, for $n \geq 3$. □

Theorem 3.6. For $n \geq 6$, the achromatic coloring of corona of $L_n$ with $P_n$ is $2n + 2$.

i.e., $\psi(L_n \circ P_n) = 2n + 2$.

Proof. Let $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ be the vertices of the Ladder graph $L_n$ and $p_1, p_2, \ldots, p_n$ be the vertices of the Path graph $P_n$.

i.e., $V(L_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $V(P_n) = \{p_1, p_2, \ldots, p_n\}$.

Let $P_n^{(1)}, P_n^{(2)}, \ldots, P_n^{(n)}$ be the $n$-copies of the Path graph $P_n$ and $p_{ij}$ be the corresponding vertices of each $P_n^{(i)}$ where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. 

Now by the definition of corona graph, each vertex of $L_n$ is adjacent to every vertex in each copy of $P_n$ i.e., every vertex $v_i, u_i \in V(L_n)$, $i = 1, 2, \ldots, n$ is adjacent to every vertex in each copy of $P_n^{(i)}$, $i = 1, 2, \ldots, n$. i.e.,

$$V(L_n \circ P_n) = v_i \cup u_i \cup p_{ij}$$

$$= \{v_i; 1 \leq i \leq n\} \cup \{u_i; 1 \leq i \leq n\} \cup \{p_{ij}; 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}.$$ 

Assign the following $2n + 2$ coloring for $L_n \circ P_n$ as achromatic:

Consider the color class $C = \{c_1, c_2, \ldots, c_{2n+2}\}$.

- For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$ and for $1 \leq i \leq n$, assign the color $c_{n+i}$ to $u_n, u_{n-1}, \ldots, u_{n-(n-1)}$.

- Assign the color $c_{i+2}$ to $p_{i1}$, where $1 \leq i \leq n$ for all $v_i$ and $u_n, u_{n-1}, \ldots, u_{n-(n-1)}$.

- For $P_n^{(i)}$, $1 \leq i \leq n/2$, $j = 2, 3, \ldots, n-2$, assign the color $c_{i+j+2}$ and for $p_{i,n-1}$ and $p_{i,n}$, where $i = 1, 2, \ldots, 2n-1$, assign the colors $c_{2n+1}$ and $c_{2n+2}$.

- For all $p_{ij}$, assign the color $c_{i+j+1}$, where $n/2 < i \leq n-1$ and $2 \leq j \leq n-2$. But for $v_{n-i}$, $i = 1, 2, \ldots, n-1$.

- For $n < i \leq 2n-2$, $2 \leq j \leq 2n-i-1$, assign the color $c_{i+j+1}$ for some $p_{ij}$, for $i > n$ and for the remaining vertices of $p_{ij}$, assign the color $c_i$ where $1 \leq i \leq n$.

- For $i = 2n-1$,

  (i) Assign the colors $c_1, c_3, c_5, \ldots, c_{n-2}$, for $j = 3, 5, 7, \ldots, n$.

  (ii) Assign the colors $c_4, c_6, c_8, \ldots, c_{n-1}$, for $j \equiv 0 \text{ mod } 2$ but $j \neq 2$.

- For $i = 2n$,

  (i) Assign the color $c_{2n+2}$ for $j = 1$.

  (ii) Assign the color $c_j$ for $j \equiv 0 \text{ mod } 2$.

  (iii) Assign the color $c_{j+2}$, where $j = 3, 5, 7, \ldots, n+2$.

If we add one more color we will miss some more pairs. This will contradict the non adjacency condition. Hence $\psi(L_n \circ P_n) \leq 2n + 2$. This proves that this coloring is maximal and achromatic one. Therefore, $\psi(L_n \circ P_n) = 2n + 2$, for $n \geq 6$.

\section*{4. Conclusion}

In this paper, we have presented an achromatic coloring for corona graphs on several graphs which are Cycle with Path, Path and Cycle, Path with Complete, Path with Wheel and Ladder with Path on the same order and $n$ th order Path graph with $n + 1$ th order Star graph.

\section*{Competing Interests}

The authors declare that they have no competing interests.
Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References