On \textit{fgspr}-Closed and \textit{fgspr}-Open Mappings

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\begin{abstract}
The purpose of this paper is to introduce a new type of fuzzy generalized mappings namely \textit{fgspr}-closed mappings, \textit{fgspr}-open mappings, \textit{fgspr}\textsuperscript{*}-closed mappings and \textit{fgspr}\textsuperscript{*}-open mappings in fuzzy topological spaces and study some of their properties.
\end{abstract}

\textbf{Keywords.} \textit{fgspr}-closed map; \textit{fgspr}-open map; \textit{fgspr}\textsuperscript{*}-closed map; \textit{fgspr}\textsuperscript{*}-open map

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\section{1. Introduction}

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh \cite{17}. Subsequently, several authors have applied various basic concepts from general topology to fuzzy sets and developed the theory of fuzzy topological spaces. Fuzzy topology was introduced by Chang \cite{6}. Azad \cite{11} introduced fuzzy semi continuity in 1981. Balasubramanian and Sundaram
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In this paper, \(fgspr\)-closed mappings, \(fgspr\)-open mappings, \(fgspr^{*}\)-closed mappings and \(fgspr^{*}\)-open mappings are introduced and some of their properties are studied.

2. Preliminaries

Let \(X\), \(Y\) and \(Z\) be fuzzy sets. Throughout this paper, \((X,\tau)\), \((Y,\sigma)\) and \((Z,\eta)\) (or simply \(X\), \(Y\) and \(Z\)) mean fuzzy topological spaces on which no separation axioms are assumed unless explicitly stated. Let \(f: (X,\tau) \rightarrow (Y,\sigma)\) be mapping from a fuzzy topological space \(X\) to fuzzy topological space \(Y\). Let us recall the following definitions which we shall require later.

**Definition 1.** A fuzzy set \(\lambda\) in a fuzzy topological space \((X,\tau)\) is called

(i) a fuzzy preopen set [5] if \(\lambda \leq \text{int}(\text{cl}(\lambda))\) and a fuzzy preclosed set if \(\text{cl}(\text{int}(\lambda)) \leq \lambda\).

(ii) a fuzzy semi-preopen set [14] if \(\lambda \leq \text{cl}(\text{int}(\text{cl}(\lambda)))\) and a fuzzy semi-preclosed set if \(\text{int}(\text{cl}(\text{int}(\lambda))) \leq \lambda\).

(iii) a fuzzy regular open set [1] if \(\text{int}(\text{cl}(\lambda)) = \lambda\) and a fuzzy regular closed set if \(\text{cl}(\text{int}(\lambda)) = \lambda\).

**Definition 2.** A fuzzy set \(\lambda\) in a fuzzy topological space \((X,\tau)\) is called

(i) a fuzzy generalized closed set (briefly, \(fg\)-closed) [2] if \(\text{cl}(\lambda) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu\) is a fuzzy open set in \(X\).

(ii) a fuzzy generalized pre closed set (briefly, \(fgp\)-closed) [7] if \(\text{pcl}(\lambda) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu\) is a fuzzy open set in \(X\).

(iii) a fuzzy generalized semi-pre closed set (briefly, \(fgsp\)-closed) [11] if \(\text{spcl}(\lambda) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu\) is a fuzzy open set in \(X\).

(iv) a fuzzy generalized preregular closed set (briefly, \(fgpr\)-closed) [15] if \(\text{pcl}(\lambda) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu\) is a fuzzy regular open set in \(X\).

(v) a fuzzy generalized semi preregular closed set (briefly, \(fgspr\)-closed) [12] if \(\text{spcl}(\lambda) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu\) is a fuzzy regular open set in \(X\).
**Definition 3.** Let $X$, $Y$ be two fuzzy topological spaces. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) a fuzzy generalized continuous (briefly, $fg$-continuous) if $f^{-1}(\lambda)$ is a fuzzy generalized open (fuzzy generalized closed) set in $X$, for every fuzzy open (fuzzy closed) set $\lambda$ in $Y$.

(ii) a fuzzy generalized semi preregular continuous (briefly, $fgspr$-continuous) if $f^{-1}(\lambda)$ is a fuzzy generalized semi preregular open (fuzzy generalized semi preregular closed) set in $X$, for every fuzzy open (fuzzy closed) set $\lambda$ in $Y$.

(iii) a fuzzy generalized semi preregular irresolute (briefly, $fgspr$-irresolute) if $f^{-1}(\lambda)$ is a fuzzy generalized semi preregular open (fuzzy generalized semi preregular closed) set in $X$, for every fuzzy generalized semi preregular open (fuzzy generalized semi preregular closed) set $\lambda$ in $Y$.

**Definition 4.** Let $X$, $Y$ be two fuzzy topological spaces. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) a fuzzy closed mapping (briefly, $f$-closed) if $f(\lambda)$ is a fuzzy closed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

(ii) a fuzzy preclosed mapping (briefly, $fp$-closed) if $f(\lambda)$ is a fuzzy preclosed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

(iii) a fuzzy $sp$-closed mapping (briefly, $fsp$-closed) if $f(\lambda)$ is a fuzzy $sp$-closed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

(iv) a fuzzy $gp$-closed mapping (briefly, $fgp$-closed) if $f(\lambda)$ is a fuzzy $gp$-closed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

(v) a fuzzy $gsp$-closed mapping (briefly, $fgsp$-closed) if $f(\lambda)$ is a fuzzy $gsp$-closed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

(vi) a fuzzy $gpr$-closed mapping (briefly, $fgpr$-closed) if $f(\lambda)$ is a fuzzy $gpr$-closed set in $Y$, for every fuzzy closed set $\lambda$ in $X$.

The corresponding open mappings are defined in the similar manner.

**Definition 5.** A fuzzy topological space $(X, \tau)$ is said to be

(i) a fuzzy $T_{1/2}$ space if every $fg$-closed is fuzzy closed.

(ii) a fuzzy semi preregular $T_{1/2}$ space if every $fgspr$-closed is fuzzy semi preclosed.

(iii) a fuzzy semi preregular $T^*_{1/2}$ space if every $fgspr$-closed is fuzzy closed.

### 3. $fgspr$-Closed Mappings

In this section, some properties of fuzzy generalized semi preregular closed mappings are studied.

**Definition 6.** Let $X$ and $Y$ be two fuzzy topological spaces. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be fuzzy generalized semi preregular closed (briefly, $fgspr$-closed) if the image of every fuzzy closed set in $X$ is a $fgspr$-closed set in $Y$. 
**Theorem 10.** Every \( f \)-closed map is a \( \text{fgspr} \)-closed map.

**Proof.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( f \)-closed map. Let \( \lambda \) be any fuzzy closed set in \( X \). Then \( f(\lambda) \) is a fuzzy closed set in \( Y \), as \( f \) is a \( f \)-closed map. Therefore, \( f(\lambda) \) is a \( \text{fgspr} \)-closed set in \( Y \), since every fuzzy closed set is a \( \text{fgspr} \)-closed set. Hence \( f \) is a \( \text{fgspr} \)-closed map.

The following example shows that the converse of the above theorem is not true.

**Example 7.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a,0.5),(b,0.4),(c,0.7)) \), \( \lambda_2 = ((a,0.8),(b,1),(c,0.4)) \) and \( \lambda_3 = ((a,0.5),(b,0.6),(c,0.3)) \). Let \( \tau = \{0,\lambda_1,1\} \) and \( \sigma = \{0,\lambda_2,1\} \). Define the mapping \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fuzzy closed set in \( X \) is \( \lambda_3 \) and \( f(\lambda_3) \) is a \( \text{fgspr} \)-closed set in \( Y \). Hence \( f \) is a \( \text{fgspr} \)-closed map.

**Theorem 8.** Every \( f \)-closed map is a \( \text{fgspr} \)-closed map.

**Proof.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( f \)-closed map. Let \( \lambda \) be any fuzzy closed set in \( X \). Then \( f(\lambda) \) is a fuzzy closed set in \( Y \), as \( f \) is a \( f \)-closed map. Therefore, \( f(\lambda) \) is a \( \text{fgspr} \)-closed set in \( Y \), since every fuzzy closed set is a \( \text{fgspr} \)-closed set. Hence \( f \) is a \( \text{fgspr} \)-closed map.

The following example shows that the converse of the above theorem is not true.

**Example 9.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a,0.5),(b,0.2),(c,0.6)) \), \( \lambda_2 = ((a,0.7),(b,1),(c,0.5)) \) and \( \lambda_3 = ((a,0.5),(b,0.8),(c,0.4)) \). Let \( \tau = \{0,\lambda_1,1\} \) and \( \sigma = \{0,\lambda_2,1\} \). Define the mapping \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fuzzy closed set in \( X \) is \( \lambda_3 \) and \( f(\lambda_3) \) is not a fuzzy closed set in \( Y \) but a \( \text{fgspr} \)-closed set in \( Y \). Hence \( f \) is a \( \text{fgspr} \)-closed map.

**Theorem 10.** Every fuzzy pre-closed (\( \text{fgp} \)-closed, \( \text{fsp} \)-closed, \( \text{fgspr} \)-closed and \( \text{fgpr} \)-closed) map is \( \text{fgspr} \)-closed.

**Proof.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a fuzzy pre-closed (\( \text{fgp} \)-closed, \( \text{fsp} \)-closed, \( \text{fgspr} \)-closed and \( \text{fgpr} \)-closed) map. Let \( \lambda \) be a fuzzy closed set in \( X \). Then \( f(\lambda) \) is a fuzzy closed set in \( Y \), as \( f \) is a fuzzy pre-closed (\( \text{fgp} \)-closed, \( \text{fsp} \)-closed, \( \text{fgspr} \)-closed and \( \text{fgpr} \)-closed) map. Therefore \( f(\lambda) \) is a \( \text{fgspr} \)-closed set in \( Y \), since every fuzzy pre-closed (\( \text{fgp} \)-closed, \( \text{fsp} \)-closed, \( \text{fgspr} \)-closed and \( \text{fgpr} \)-closed) set is a \( \text{fgspr} \)-closed set. Hence \( f \) is a \( \text{fgspr} \)-closed map.

The following examples show that the converse of the above theorems are not true.

**Example 11.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a,0.5),(b,0.2),(c,0.6)) \), \( \lambda_2 = ((a,0.7),(b,0),(c,0.4)) \) and \( \lambda_3 = ((a,0.5),(b,0.8),(c,0.4)) \). Let \( \tau = \{0,\lambda_1,1\} \) and \( \sigma = \{0,\lambda_2,1\} \). Define the mapping \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fuzzy closed set in \( X \) is \( \lambda_3 \) and \( f(\lambda_3) \) is not a fuzzy preclosed and a fuzzy semi preclosed set in \( Y \) but a \( \text{fgspr} \)-closed set in \( Y \). Hence \( f \) is a \( \text{fgspr} \)-closed map but not a fuzzy preclosed map and a fuzzy semi preclosed map.

**Example 12.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a,0.8),(b,0.7),(c,0.2)) \), \( \lambda_2 = ((a,0.2),(b,0.3),(c,0.4)) \), \( \lambda_3 = ((a,0.3),(b,0.5),(c,0.4)) \) and \( \lambda_4 = ((a,0.1),(b,0.3),(c,0.2)) \). Let \( \tau = \{0,\lambda_1,1\} \) and \( \sigma = \{0,\lambda_3,\lambda_4,1\} \). Define the mapping \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = a \), \( f(b) = b \) and \( f(c) = c \). Then the only fuzzy closed set in \( X \) is \( \lambda_2 \) and \( f(\lambda_2) \) is not a fuzzy \( \text{gp} \)-closed set and a fuzzy \( \text{gpr} \)-closed set in \( Y \) but a \( \text{fgspr} \)-closed set in \( Y \). Hence \( f \) is a \( \text{fgspr} \)-closed map but not a fuzzy \( \text{gp} \)-closed map and a fuzzy \( \text{gpr} \)-closed map.
Example 13. Let $X = \{a, b, c\}$, $Y = \{a, b, c\}$ and consider the fuzzy sets $\lambda_1 = ((a, 0.8), (b, 0.6), (c, 0.8))$, $\lambda_2 = ((a, 0.2), (b, 0.4), (c, 0.2))$ and $\lambda_3 = ((a, 0.3), (b, 0.5), (c, 0.4))$. Let $\tau = \{0, \lambda_1, 1\}$ and $\sigma = \{0, \lambda_2, \lambda_3, 1\}$. Define the mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Then the only fuzzy closed set in $X$ is $\lambda_2$ and $f(\lambda_2)$ is not a fuzzy gsp-closed set in $Y$ but a fgsp-closed set in $Y$. Hence $f$ is a fgsp-closed map but not a fuzzy gsp-closed map.

Remark 14. From the above results we get the following diagram:

<table>
<thead>
<tr>
<th>f-closed map</th>
<th>fgsp-closed map</th>
<th>fgpr-closed map</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\searrow)</td>
<td>(\downarrow)</td>
<td>(\nearrow)</td>
</tr>
<tr>
<td>fgpr-closed map</td>
<td>(\Uparrow)</td>
<td>(\nwarrow)</td>
</tr>
<tr>
<td>fp-closed map</td>
<td>fsp-closed map</td>
<td>fgsp-closed map</td>
</tr>
</tbody>
</table>

where $A \rightarrow B$ represents $A$ implies $B$ but not converse. The above diagram shows the relationships of fgsp-closed with some other existing fuzzy mappings.

The following theorem state under what conditions the reverse implications hold good.

Theorem 15. If $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map and $Y$ is fuzzy semi preregular $T_{1/2}$ space then $f$ is a fsp-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map. Let $\lambda$ be a fuzzy closed set in $X$. Then $f(\lambda)$ is a fgsp-closed set in $Y$ as $f$ is a fgsp-closed map. Since $Y$ is fuzzy semi preregular $T_{1/2}$ space, $f(\lambda)$ is a fsp-closed set in $Y$. Hence $f$ is a fsp-closed map.

Theorem 16. If $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map and $Y$ is fuzzy semi preregular $T_{1/2}$ space then $f$ is a fgsp-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map. Let $\lambda$ be a fuzzy closed set in $X$. Then $f(\lambda)$ is a fgsp-closed set in $Y$ as $f$ is a fgsp-closed map. Since $Y$ is fuzzy semi preregular $T_{1/2}$ space, $f(\lambda)$ is a fsp-closed set in $Y$. Every fsp-closed set is a fgsp-closed set. Therefore $f(\lambda)$ is a fgsp-closed set in $Y$. Hence $f$ is a fgsp-closed map.

Theorem 17. If $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map and $Y$ is fuzzy semi preregular $T_{1/2}^*$ space then $f$ is a f-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map. Let $\lambda$ be a fuzzy closed set in $X$. Then $f(\lambda)$ is a fgsp-closed set in $Y$ as $f$ is a fgsp-closed map. Since $Y$ is fuzzy semi preregular $T_{1/2}^*$ space, $f(\lambda)$ is a fuzzy closed set in $Y$. Hence $f$ is a f-closed map.

Theorem 18. If $f : (X, \tau) \to (Y, \sigma)$ is a fgsp-closed map and $Y$ is fuzzy semi preregular $T_{1/2}^*$ space then $f$ is a fp-closed map.
Theorem 21. If a function $f : (X, \tau) \to (Y, \sigma)$ is a $fgspr$-closed map and $Y$ is fuzzy semi preregular $T_{1/2}$ space, then $f$ is a $fgspr$-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $fgspr$-closed map. Let $\lambda$ be a fuzzy closed set in $X$. Then $f(\lambda)$ is a $fgspr$-closed set in $Y$. Since $Y$ is fuzzy semi preregular $T_{1/2}$ space, $f(\lambda)$ is a fuzzy closed set in $Y$. Therefore $f(\lambda)$ is a $fgpr$-closed set in $Y$. Hence $f$ is a $fgpr$-closed map.

Theorem 20. If $f : (X, \tau) \to (Y, \sigma)$ is a $fgspr$-closed map and $Y$ is fuzzy semi preregular $T_{1/2}$ space then $f$ is a $fgpr$-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $fgspr$-closed map. Let $\lambda$ be a fuzzy closed set in $X$. Then $f(\lambda)$ is a $fgspr$-closed set in $Y$. Since $Y$ is fuzzy semi preregular $T_{1/2}$ space, $f(\lambda)$ is a fuzzy closed set in $Y$. Therefore $f(\lambda)$ is a $fgpr$-closed set in $Y$. Hence $f$ is a $fgpr$-closed map.

Theorem 19. If $f : (X, \tau) \to (Y, \sigma)$ is a $fgspr$-closed map and $Y$ is fuzzy semi preregular $T_{1/2}$ space then $f$ is a $fgpr$-closed map.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $fgspr$-closed map. Suppose $f$ is a $fgspr$-closed map. If $\lambda$ is a fuzzy set in $X$, then $cl(\lambda)$ is a fuzzy closed set in $X$. Since $f(\lambda) \leq f(cl(\lambda))$. This implies that $fgspr-cl(f(\lambda)) \leq f(gspr-cl(f(cl(\lambda)))) = f(cl(\lambda))$. Therefore $fgspr-cl(f(\lambda)) \leq f(cl(\lambda))$.}

Theorem 22. A map $f : (X, \tau) \to (Y, \sigma)$ is $fgspr$-closed iff for each fuzzy set $\lambda$ of $Y$ and for each fuzzy open set $\mu$ such that $f^{-1}(\lambda) \leq \mu$, there is a $fgspr$-open set $\gamma$ of $Y$ such that $\lambda \leq \gamma$ and $f^{-1}(\gamma) \leq \mu$.

Proof. Suppose $f$ is a $fgspr$-closed map. Let $\lambda$ be a fuzzy set in $Y$ and $\mu$ be a fuzzy open set in $X$ such that $f^{-1}(\lambda) \leq \mu$. Then, $1 - \mu$ is a fuzzy closed set in $X$. Then $f(1 - \mu)$ is a $fgspr$-closed set in $Y$ since $f$ is a $fgspr$-closed map. So, $1 - f(1 - \mu)$ is a $fgpr$-open set in $Y$. Thus, choose $\gamma = 1 - f(1 - \mu)$ is a $fgspr$-open set in $Y$ such that $\lambda \leq \gamma$ and $f^{-1}(\gamma) \leq \mu$.

Conversely, suppose that $\alpha$ is a fuzzy closed set in $X$. Then $1 - \alpha$ is a fuzzy open set in $X$ and $f^{-1}(1 - f(\alpha)) \leq 1 - \alpha$. Then there exists a $fgspr$-open set $\gamma$ of $Y$ such that $1 - f(\alpha) \leq \gamma$ and $f^{-1}(\gamma) \leq 1 - \alpha$ and so $\alpha \leq f^{-1}(\gamma)$. Hence $1 - \gamma \leq f(\alpha) \leq f(1 - f^{-1}(\gamma)) \leq 1 - \gamma$. This implies that $f(\alpha) = 1 - \gamma$ since $1 - \gamma$ is a $fgspr$-closed set. $f(\alpha)$ is a $fgspr$-closed set and thus $f$ is a $fgspr$-closed map.
Theorem 23. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) be onto, \( \text{fgspr-irresolute} \) and fuzzy closed map. If \( (X, \tau) \) is fuzzy semi preregular \( T_{1/2}^* \) space, then \( (Y, \sigma) \) is also fuzzy semi preregular \( T_{1/2}^* \) space.

Proof. Let \( \lambda \) be a \( \text{fgspr} \)-closed set in \( Y \). Since \( f \) is \( \text{fgspr} \)-irresolute, then \( f^{-1}(\lambda) \) is a \( \text{fgspr} \)-closed set in \( X \). As \( (X, \tau) \) is fuzzy semi preregular \( T_{1/2}^* \) space, \( f^{-1}(\lambda) \) is a fuzzy closed set in \( X \). Again \( f \) is a fuzzy closed map, \( f(f^{-1}(\lambda)) \) is a fuzzy closed set in \( Y \). Since \( f \) is onto, \( f(f^{-1}(\lambda)) = \lambda \). Thus \( \lambda \) is a fuzzy closed set in \( Y \). Hence \( (Y, \sigma) \) is fuzzy semi preregular \( T_{1/2}^* \) space. \( \square \)

Theorem 24. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a \( f \)-closed map and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map.

Proof. Let \( \lambda \) be a fuzzy closed set in \( X \). Then \( f(\lambda) \) is a \( f \)-closed set in \( Y \), since \( f \) is a \( f \)-closed map in \( Y \). \( g(f(\lambda)) \) is a \( \text{fgspr} \)-closed set in \( Z \) as \( g \) is a \( \text{fgspr} \)-closed map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a \( \text{fgspr} \)-closed set in \( Z \). Hence \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map. \( \square \)

Theorem 25. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are \( \text{fgspr} \)-closed maps and \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map.

Proof. Let \( \lambda \) be a fuzzy closed set in \( X \). Then \( f(\lambda) \) is a \( \text{fgspr} \)-closed set in \( Y \), since \( f \) is a \( \text{fgspr} \)-closed map in \( Y \). As \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space, \( f(\lambda) \) is a fuzzy closed set in \( Y \). \( g(f(\lambda)) \) is a \( \text{fgspr} \)-closed set in \( Z \) as \( g \) is a \( \text{fgspr} \)-closed map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a \( \text{fgspr} \)-closed set in \( Z \). Hence \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map. \( \square \)

Theorem 26. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map.

(i) If \( f \) is \( f \)-continuous and surjective, then \( g \) is a \( \text{fgspr} \)-closed map.

(ii) If \( g \) is \( \text{fgspr} \)-irresolute and injective, then \( f \) is a \( \text{fgspr} \)-closed map

Proof. (i) Let \( \lambda \) be a fuzzy closed set in \( Y \). Then \( f^{-1}(\lambda) \) is a \( f \)-closed set in \( X \), since \( f \) is \( f \)-continuous. As \( g \circ f \) is a \( \text{fgspr} \)-closed map, \( g \circ f(f^{-1}(\lambda)) = g(\lambda) \) is a \( \text{fgspr} \)-closed set in \( Z \). Thus \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is a \( \text{fgspr} \)-closed map.

(ii) Let \( \mu \) be a fuzzy closed set in \( X \). Then \( g \circ f(\mu) \) is a \( \text{fgspr} \)-closed set in \( Z \), since \( g \circ f \) is a \( \text{fgspr} \)-closed map. As \( g \) is \( \text{fgspr} \)-irresolute, \( g^{-1}(g \circ f(\mu)) \) is a \( \text{fgspr} \)-closed set in \( Y \). Since \( g \) is injective, \( g^{-1}(g \circ f(\mu)) = f(\mu) \) is a \( \text{fgspr} \)-closed set in \( Y \). Therefore \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a \( \text{fgspr} \)-closed map. \( \square \)

4. \text{fgspr}-Open Mappings

In this section, some properties of fuzzy generalized semi preregular open mappings are studied.

Definition 27. Let \( X \) and \( Y \) be two fuzzy topological spaces. A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be fuzzy generalized semi preregular open (briefly, \( \text{fgspr} \)-open) if the image of every fuzzy open set in \( X \) is a \( \text{fgspr} \)-open set in \( Y \).
**Theorem 32.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a, 0.3), (b, 0.5), (c, 0.6)) \), \( \lambda_2 = ((a, 0.8), (b, 0.6), (c, 0.5)) \) and \( \lambda_3 = ((a, 0.2), (b, 0.4), (c, 0.5)) \). Let \( \tau = (0, \lambda_1, 1) \) and \( \sigma = (0, \lambda_2, 1) \). Define the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fuzzy open set in \( X \) is \( \lambda_3 \) and \( f(\lambda_3) \) is a \( \text{fgspr-int} \)-open set in \( Y \). Hence \( f \) is a \( \text{fgspr-open} \) map.

**Theorem 29.** Every \( f \)-open map is a \( \text{fgspr-open} \) map.

*Proof.* Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( f \)-open map. Let \( \lambda \) be any fuzzy open set in \( X \). Then \( f(\lambda) \) is a fuzzy open set in \( Y \), as \( f \) is a \( f \)-open map. Therefore \( f(\lambda) \) is a \( \text{fgspr-open} \) set in \( Y \), since every fuzzy open set is a \( \text{fgspr-open} \) set. Hence \( f \) is a \( \text{fgspr-open} \) map. \( \square \)

The following example shows that the converse of the above theorem is not true.

**Example 30.** Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = ((a, 0.4), (b, 0.5), (c, 0.2)) \), \( \lambda_2 = ((a, 0.9), (b, 0.7), (c, 0.8)) \) and \( \lambda_3 = ((a, 0.1), (b, 0.3), (c, 0.2)) \). Let \( \tau = (0, \lambda_1, 1) \) and \( \sigma = (0, \lambda_2, 1) \). Define the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fuzzy open set in \( X \) is \( \lambda_3 \) and \( f(\lambda_3) \) is not a fuzzy open set in \( Y \) but a \( \text{fgspr-open} \) set in \( Y \). Hence \( f \) is a \( \text{fgspr-open} \) map but not a fuzzy open map.

**Theorem 31.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \text{fgspr-open} \) map and \( Y \) is fuzzy semi preregular \( T_{1/2} \) space then \( f \) is a \( \text{fsp-open} \) map.

*Proof.* Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \text{fgspr-open} \) map. Let \( \lambda \) be a fuzzy open set in \( X \). Then \( f(\lambda) \) is a \( \text{fgspr-open} \) set in \( Y \) as \( f \) is a \( \text{fgspr-open} \) map. Since \( Y \) is fuzzy semi preregular \( T_{1/2} \) space, \( f(\lambda) \) is a \( \text{fsp-open} \) set in \( Y \). Hence \( f \) is a \( \text{fsp-open} \) map. \( \square \)

**Theorem 32.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \text{fgspr-open} \) map and \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space then \( f \) is a \( f \)-open map.

*Proof.* Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \text{fgspr-open} \) map. Let \( \lambda \) be a fuzzy open set in \( X \). Then \( f(\lambda) \) is a \( \text{fgspr-open} \) set in \( Y \) as \( f \) is a \( \text{fgspr-open} \) map. Since \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space, \( f(\lambda) \) is a fuzzy open set in \( Y \). Hence \( f \) is a \( f \)-open map. \( \square \)

**Theorem 33.** If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a \( \text{fgspr-open} \) map, then for each fuzzy set \( \lambda \) in \( X \), \( \text{fgspr-int}(f(\lambda)) \supseteq f(\text{int}(\lambda)) \).

*Proof.* Suppose \( f \) is a \( \text{fgspr-open} \) map. If \( \lambda \) is a fuzzy set in \( X \), then \( \text{int}(\lambda) \) is a fuzzy open set in \( X \). \( f(\text{int}(\lambda)) \) is a \( \text{fgspr-open} \) set in \( Y \). Since \( f(\lambda) \supseteq f(\text{int}(\lambda)) \). This implies that \( \text{fgspr-int}(f(\lambda)) \supseteq f(\text{fgspr-int}(f(\text{int}(\lambda)))) = f(\text{int}(\lambda)) \), as \( f(\text{int}(\lambda)) \) is a \( \text{fgspr-open} \) set in \( Y \). That is \( \text{fgspr-int}(f(\lambda)) \supseteq f(\text{int}(\lambda)) \). \( \square \)

**Theorem 34.** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \text{fgspr-open} \) iff for each fuzzy set \( \lambda \) of \( Y \) and for each fuzzy closed set \( \mu \) such that \( f^{-1}(\lambda) \subseteq \mu \), there is a \( \text{fgspr-closed} \) set \( \gamma \) of \( Y \) such that \( \lambda \subseteq \gamma \) and \( f^{-1}(\gamma) \subseteq \mu \).
Theorem 35. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) be onto, fgspr-irresolute and fuzzy open map. If \( (X, \tau) \) is fuzzy semi preregular \( T_{1/2}^* \) space, then \( (Y, \sigma) \) is also fuzzy semi preregular \( T_{1/2}^* \) space.

Proof. Let \( \lambda \) be a fgspr-open set in \( Y \). Since \( f \) is fgspr-irresolute, then \( f^{-1}(\lambda) \) is a fgspr-open set in \( Y \). As \( (X, \tau) \) is fuzzy semi preregular \( T_{1/2}^* \) space, \( f^{-1}(\lambda) \) is a fuzzy open set in \( X \). Again \( f \) is a fuzzy open map, \( f(f^{-1}(\lambda)) \) is a fuzzy open set in \( Y \). Since \( f \) is onto, \( f(f^{-1}(\lambda)) = \lambda \). Thus \( \lambda \) is a fuzzy open set in \( Y \). Hence \( (Y, \sigma) \) is fuzzy semi preregular \( T_{1/2}^* \) space.

\( \square \)

Theorem 36. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a \( f \)-open map and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is a fgspr-open map then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a fgspr-open map.

Proof. Let \( \lambda \) be a fuzzy open set in \( X \). Then \( f(\lambda) \) is a \( f \)-open set in \( Y \), since \( f \) is a \( f \)-open map in \( Y \). \( g(f(\lambda)) \) is a fgspr-open set in \( Z \) as \( g \) is a fgspr-open map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a fgspr-open set in \( Z \). Hence \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a fgspr-open map.

\( \square \)

Theorem 37. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) are fgspr-open maps and \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a fgspr-open map.

Proof. Let \( \lambda \) be a fuzzy open set in \( X \). Then \( f(\lambda) \) is a fgspr-open set in \( Y \), since \( f \) is a fgspr-open map in \( Y \). As \( Y \) is fuzzy semi preregular \( T_{1/2}^* \) space, \( f(\lambda) \) is a fuzzy open set in \( Y \). \( g(f(\lambda)) \) is a fgspr-open set in \( Z \) as \( g \) is a fgspr-open map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a fgspr-open set in \( Z \). Hence \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a fgspr-open map.

\( \square \)

Theorem 38. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is a fgspr-open map.

(i) If \( f \) is \( f \)-continuous and surjective, then \( g \) is a fgspr-open map.

(ii) If \( g \) is fgspr-irresolute and injective, then \( f \) is a fgspr-open map.

Proof. (i) Let \( \lambda \) be a fuzzy open set in \( Y \). Then \( f^{-1}(\lambda) \) is a \( f \)-open set in \( X \), since \( f \) is \( f \)-continuous. As \( g \circ f \) is a fgspr-open map, \( g \circ f(f^{-1}(\lambda)) = g(\lambda) \) is a fgspr-open set in \( Z \). Thus \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is a fgspr-open map.
Theorem 43. Let \( Y \) be a fuzzy open set in \( X \). Then \( g \circ f(\mu) \) is a fgspr-open set in \( Z \), since \( g \circ f \) is a fgspr-open map. As \( g \) is fgspr-irresolute, \( g^{-1}(g \circ f)(\mu) \) is a fgspr-open set in \( Y \). Since \( g \) is injective, \( g^{-1}(g \circ f)(\mu) = f(\mu) \) is a fgspr-open set in \( Y \). Therefore \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a fgspr-open map.

\[ \square \]

5. fgspr\(^\ast\)-Closed Mappings and fgspr\(^\ast\)-Open Mappings

In this section, some properties of fgspr\(^\ast\)-closed mappings and fgspr\(^\ast\)-open mappings are studied.

Definition 39. Let \( X \) and \( Y \) be two fuzzy topological spaces. A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be fuzzy generalized semi preregular\(^\ast\) closed (briefly, fgspr\(^\ast\)-closed) if the image of every fgspr\(^\ast\)-closed set in \( X \) is a fgspr\(^\ast\)-closed set in \( Y \).

Example 40. Let \( X = \{a, b, c\} \), \( Y = \{a, b, c\} \) and consider the fuzzy sets \( \lambda_1 = \{(a,0),(b,1),(c,0)\} \), \( \lambda_2 = \{(a,0),(b,1),(c,1)\} \), \( \lambda_3 = \{(a,1),(b,0),(c,0)\} \), \( \lambda_4 = \{(a,1),(b,0),(c,1)\} \), \( \lambda_5 = \{(a,1),(b,1),(c,0)\} \) and \( \lambda_6 = \{(a,0),(b,0),(c,1)\} \). Let \( \tau = \{0, \lambda_1, \lambda_2, 1\} \) and \( \sigma = \{0, \lambda_5, 1\} \). Define the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only fgspr\(^\ast\)-closed sets in \( X \) are \( \lambda_3 \), \( \lambda_4 \) and \( \lambda_6 \) and \( f(\lambda_3), f(\lambda_4) \) and \( f(\lambda_6) \) are fgspr\(^\ast\)-closed sets in \( Y \). Hence \( f \) is a fgspr\(^\ast\)-closed map.

Definition 41. Let \( X \) and \( Y \) be two fuzzy topological spaces. A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be fuzzy generalized semi preregular\(^\ast\) open (briefly, fgspr\(^\ast\)-open) if the image of every fgspr\(^\ast\)-open set in \( X \) is a fgspr\(^\ast\)-open set in \( Y \).

Example 42. Let \( X = \{a, b, c\} = Y \) and consider the fuzzy sets \( \lambda_1 = \{(a,1),(b,0),(c,0)\} \), \( \lambda_2 = \{(a,1),(b,1),(c,0)\} \), \( \lambda_3 = \{(a,1),(b,0),(c,1)\} \), \( \lambda_4 = \{(a,0),(b,1),(c,1)\} \) and \( \lambda_5 = \{(a,0),(b,0),(c,1)\} \). Let \( \tau = \{0, \lambda_1, \lambda_2, 1\} \) and \( \sigma = \{0, \lambda_3, 1\} \). Define the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(c) = a \) and \( f(b) = b \). Then the only fgspr\(^\ast\)-open sets in \( X \) are \( \lambda_1 \), \( \lambda_2 \), \( \lambda_3 \) and \( f(\lambda_1) \), \( f(\lambda_2) \) and \( f(\lambda_3) \) are fgspr\(^\ast\)-open sets in \( Y \). Hence \( f \) is a fgspr\(^\ast\)-open map.

Theorem 43. Every fgspr\(^\ast\)-closed (fgspr\(^\ast\)-open) map is a fgspr\(^\ast\)-closed (fgspr\(^\ast\)-open) map.

Proof. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a fgspr\(^\ast\)-closed map. Let \( \lambda \) be a fuzzy closed set in \( X \). Then \( \lambda \) is a fgspr\(^\ast\)-closed set in \( X \), since every fuzzy closed set is a fgspr\(^\ast\)-closed set. Therefore \( f(\lambda) \) is a fgspr\(^\ast\)-closed set in \( Y \), as \( f \) is a fgspr\(^\ast\)-closed map. Hence \( f \) is a fgspr\(^\ast\)-closed map. \( \square \)

The following example shows that the converse of the above theorem is not true.

Example 44. Let \( X = \{a, b, c\} = Y \) and consider the fuzzy sets \( \lambda_1 = \{(a,0),(b,1),(c,0)\} \), \( \lambda_2 = \{(a,0),(b,1),(c,1)\} \), \( \lambda_3 = \{(a,1),(b,0),(c,0)\} \), \( \lambda_4 = \{(a,1),(b,0),(c,1)\} \) and \( \lambda_5 = \{(a,1),(b,1),(c,0)\} \) and \( \lambda_6 = \{(a,0),(b,0),(c,1)\} \). Let \( \tau = \{0, \lambda_5, 1\} \) and \( \sigma = \{0, \lambda_1, \lambda_2, 1\} \). Define the mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = f(b) = a \) and \( f(c) = c \). Then the only \( f \)-closed set in \( X \) is \( \lambda_6 \) and \( f(\lambda_6) \) is a fgspr\(^\ast\)-closed sets in \( Y \). Hence \( f \) is a fgspr\(^\ast\)-closed map. But \( \lambda_1 \) and \( \lambda_2 \) are fgspr\(^\ast\)-closed sets in \( X \) and \( f(\lambda_1) \) and \( f(\lambda_2) \) are not fgspr\(^\ast\)-closed sets in \( Y \). Hence \( f \) is not a fgspr\(^\ast\)-closed map.
Theorem 45. If \( f : (X, \tau) \to (Y, \sigma) \) is a fgspr-closed (fgspr-open) map and \( g : (Y, \sigma) \to (Z, \eta) \) is a fgspr*-closed (fgspr*-open) map then \( g \circ f : (X, \tau) \to (Z, \eta) \) is a fgspr*-closed (fgspr*-open) map.

Proof. Let \( \lambda \) be a fuzzy closed set in \( X \). Then \( f(\lambda) \) is a fgspr-closed set in \( Y \), since \( f \) is a fgspr*-closed map in \( Y \). \( g(f(\lambda)) \) is a fgspr-closed set in \( Z \) as \( g \) is a fgspr*-closed map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a fgspr-closed set in \( Z \). Hence \( g \circ f : (X, \tau) \to (Z, \eta) \) is a fgspr*-closed map.

Theorem 46. Composition of two fgspr*-closed (fgspr*-open) mappings is fgspr*-closed (fgspr*-open).

(i.e.) If \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) is a fgspr*-closed (fgspr*-open) mappings then \( g \circ f : (X, \tau) \to (Z, \eta) \) is a fgspr*-closed (fgspr*-open) map.

Proof. Let \( \lambda \) be a fgspr-closed set in \( X \). Then \( f(\lambda) \) is a fgspr-closed set in \( Y \), since \( f \) is a fgspr*-closed map in \( Y \). \( g(f(\lambda)) \) is a fgspr-closed set in \( Z \) as \( g \) is a fgspr*-closed map. That is \( g \circ f(\lambda) = g(f(\lambda)) \) is a fgspr-closed set in \( Z \). Hence \( g \circ f : (X, \tau) \to (Z, \eta) \) is a fgspr*-closed map.

Theorem 47. Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two maps such that \( g \circ f : (X, \tau) \to (Z, \eta) \) is a fgspr*-closed (fgspr*-open) map.

(i) If \( f \) is fgspr-irresolute and surjective, then \( g \) is a fgspr*-closed (fgspr*-open) map.

(ii) If \( g \) is fgspr-irresolute and injective, then \( f \) is a fgspr*-closed (fgspr*-open) map.

Proof. (i) Let \( \lambda \) be a fgspr-closed set in \( Y \). Then \( f^{-1}(\lambda) \) is a fgspr-closed set in \( X \), since \( f \) is fgspr-irresolute. As \( g \circ f \) is a fgspr*-closed map, \( g \circ f(f^{-1}(\lambda)) = g(\lambda) \) is a fgspr-closed set in \( Z \). Thus \( g : (Y, \sigma) \to (Z, \eta) \) is a fgspr*-closed map.

(ii) Let \( \mu \) be a fgspr-closed set in \( X \). Then \( g \circ f(\mu) \) is a fgspr-closed set in \( Z \), since \( g \circ f \) is a fgspr*-closed map. As \( g \) is fgspr-irresolute, \( g^{-1}(g \circ f)(\mu) \) is a fgspr-closed set in \( Y \). Since \( g \) is injective, \( g^{-1}(g \circ f)(\mu) = f(\mu) \) is a fgspr-closed set in \( Y \). Therefore \( f : (X, \tau) \to (Y, \sigma) \) is a fgspr*-closed map.

6. Conclusion

It is an interesting exercise to work on fgspr-closed mappings and fgspr-open mappings with some other existing fuzzy mappings. Composition of these mappings have also been studied. Similarly, other forms of fgspr-closed sets and fgspr-open sets can be used to define fgspr*-closed mappings and fgspr*-open mappings. This new concept and its properties will be useful for future research in this field.

Competing Interests

The authors declare that they have no competing interests.
Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References


