Group Theoretical Study of Certain Generating Functions for Modified Jacobi Polynomials

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Abstract. The object of the present paper is to derive some generating functions with five parameters Lie-group for the modified Jacobi polynomial \( P_n^{(\alpha-n, \beta)}(x) \) by interpreting \( n, \alpha-n, \beta \) simultaneously by using the Weisner's group-theoretic method.

1. Introduction

The modified Jacobi polynomials \( P_n^{(\alpha-n, \beta)}(x) \), defined by

\[
P_n^{(\alpha-n, \beta)}(x) = \frac{(1 + \alpha - n)_n}{n!} 2F_1 \left[ \begin{array}{c} -n, \\ 1 + \alpha + \beta \\ 1 + \alpha - n \\ \end{array} ; \frac{1 - x}{2} \right]
\]

is the solution of following ordinary differential equation:

\[
(1 - x^2) \frac{d^2}{dx^2} P_n^{(\alpha-n, \beta)}(x) + \left[ \beta - \alpha + n - (2 + \alpha - \beta - n)x \right] \times \frac{d}{dx} P_n^{(\alpha-n, \beta)}(x) + n(1 + \alpha + \beta) P_n^{(\alpha-n, \beta)}(x) = 0.
\]

W. Miller’s (Jr.) Lie theoretic method is utilized of modified Jacobi polynomials \( P_n^{(\alpha-n, \beta)}(x) \) by making suitable interpretation to the index \( n \) in order to obtain new generating functions.

In 1986 Ghosh obtained some generating functions for \( P_n^{(\alpha-n, \beta)}(x) \) with the help of Weisner's method by given suitable interpretation to the index \( n \).

The object of the present paper investigation to apply Miller’s method to obtain some generating functions for modified Jacobi polynomial \( P_n^{(\alpha-n, \beta)}(x) \) by interpreting the index \( n \), with the help of Weisner’s group-theoretic method (Mcbride 1971).

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Here the following generating functions are derived for \( P_n^{(a-n,\beta)}(x) \) by finding a set of infinitesimal operators \( A_i(i = 1, 2, 3, 4, 5) \) constituting a Lie-algebra:

\[
p_n^{(a-n,\beta)} \left[ \frac{y x - t_1}{y - t_1} \right] = \sum_{p=0}^{n} \frac{1}{p!2^p} (\alpha + \beta + 1)_p P_n^{(a-n+p,\beta+p)}(x)(t_1)^p,
\]

\[
[1 - t_2 y(1 + x)]^{1+a}[1 + t_2 y(1 - x)]^{\beta} P_n^{(a-n,\beta)}[x + t_2 y(1 - x^2)]
\]

\[
= \sum_{k=0}^{\infty} \left( -2 \right)^k \frac{(-1)^k}{k!} \frac{n + 1}{(n + 1)_k} P_n^{(a-n-k,\beta-k)}(x)(t_2)^k,
\]

\[
(1 - t_3(1 + x))^{1+a}(1 + t_3(1 - x))^{\beta} P_n^{(a-n,\beta)} \left[ (yt_3(1 - yt_3(1 + x)))(x + yt_3(1 - x^2) + \frac{1}{w_1}(1 + yt_3(1 - x)) \right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{p=0}^{n-k} \left( -1 \right)^k \left( -1/w_1 \right)^p \frac{2^{k-p}(\alpha + \beta + 1)_p(n - p + 1)_k}{k!} P_n^{(a-n-p-k,\beta-p-k)}(x)(t_3)^{k-p}.
\]

2. Group theoretic method

Replacing \( d/\partial x \) by \( \partial/\partial x \), \( \alpha \) by \( y \frac{\partial}{\partial y} \), \( \beta \) by \( z \frac{\partial}{\partial z} \), \( n \) by \( t \frac{\partial}{\partial t} \) and \( P_n^{(a-n,\beta)}(x) \) by \( u(x,y,z,t) \) we get the following partial differential equation

\[
(1 - x^2) \frac{\partial^2 u}{\partial x^2} + z(1 - x) \frac{\partial^2 u}{\partial z \partial x} - y(1 + x) \frac{\partial^2 u}{\partial y \partial x} - t x \frac{\partial^2 u}{\partial t \partial x} + t y \frac{\partial^2 u}{\partial t \partial y} + t z \frac{\partial^2 u}{\partial t \partial z} + 2 x \frac{\partial u}{\partial x} + t \frac{\partial u}{\partial t} = 0.
\]

Thus \( u(x,y,z,t) = P_n^{(a-n,\beta)}(x)y^{a-\beta}z^\beta t^n \) is a solution of the differential equation (2.1).

Let us defined the infinitesimal operators \( A_i(i = 1, 2, 3, 4, 5) \)

\[
A_i = A_1^{(1)} \partial/\partial x + A_2^{(2)} \partial/\partial y + A_3^{(3)} \partial/\partial z + A_4^{(4)} \partial/\partial t + A_5^{(5)}.
\]

As follows

\[
\begin{align*}
A_1 &= y \partial/\partial y \\
A_2 &= z \partial/\partial z \\
A_3 &= t \partial/\partial t \\
A_4 &= (x - 1)e t^{-1} \partial/\partial x - z \partial/\partial t \\
A_5 &= (1 - x^2)e x^{-1} t \partial/\partial x - (x + 1)e x^{-1} t y \partial/\partial y - (x - 1) e^{-1} t^2 \partial/\partial t - (x + 1) e^{-1} t^2
\end{align*}
\]

\[
(2.3)
\]
which satisfy the following rules:

\[
\begin{align*}
A_1[p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n] &= (α-n)p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n \\
A_2[p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n] &= βp_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n \\
A_3[p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n] &= n p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n \\
A_4[p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n] &= -(α) p_{n-1}^{(α-n,β+1)}(x)y^{α-n}z^{β+1}t^{n-1} \\
A_5[p_n^{(α-n,β)}(x)y^{α-n}z^{β}t^n] &= -(n+1)p_{n+1}^{(α-n,β-1)}(x)y^{α-n}z^{β-1}t^{n+1}
\end{align*}
\]

(2.4)

3. Lie Algebra

Now we shall find the commutator relations by using commutator notation with

\[[A,B]u = (AB - BA)u\]

\[[A_1,A_2] = 0; \quad [A_2,A_3] = 0; \quad [A_3,A_4] = -A_4; \quad [A_4,A_5] = 2(A_1 + A_2)\]

\[[A_1,A_3] = 0; \quad [A_2,A_4] = 0; \quad [A_3,A_5] = A_5\]

\[[A_1,A_4] = 0; \quad [A_2,A_5] = -A_5;\]

\[[A_1,A_5] = 0;\]

So we see from the above commutator relations that set of operators \{1, A_i, i = 1, 2, 3, 4, 5\} generating a lie Algebra.

Now the partial differential operator \(L\), given by:

\[
L = (1 - x^2) \frac{∂^2 u}{∂x^2} + z(1 - x) \frac{∂^2 u}{∂z∂x} - y(1 + x) \frac{∂^2 u}{∂y∂x} - tx \frac{∂^2 u}{∂t∂x}
+ ty \frac{∂^2 u}{∂t∂y} + tz \frac{∂^2 u}{∂t∂z} + 2x \frac{∂u}{∂x} + t \frac{∂u}{∂t}
= 0.
\]

Which can be express as:

\[(x - 1)L = A_2 A_4 - 2A_3 (A_1 + A_3).\]  

(3.1)

It can be easy verified that the operator \(A_i\) \((i = 1, 2, 3, 4, 5)\) commute with \((x - 1)L\), i.e.

\[[ (x - 1)L, A_i] = 0.\]

(3.2)

The extended form of the group generated by \(A_i\) \((i = 1, 2, 3, 4, 5)\) are given by

\[
e^{nA_1}u(x, y, z, t) = u(x, e^{a_1}y, z, t),
\]

\[
e^{nA_2}u(x, y, z, t) = u(x, y, e^{a_2}z, t),
\]

\[
e^{nA_3}u(x, y, z, t) = u(x, y, z, e^{a_3}t),
\]

\[
e^{nA_4}u(x, y, z, t) = u \left( \frac{tx - a_4z}{t - a_4z}, y, z, t - a_4z \right),
\]

\[
e^{nA_5}u(x, y, z, t) = u(x, y, z, t).
\]
\[ e^{a_i A_i} u(x, y, z, t) = \left( \frac{z - t(1 + x)(a_5)}{z} \right) u \left[ \frac{xz + t(1 - x^2)a_5}{z}, \frac{yz - y t(1 + x)a_5}{z}, \right. \\
\left. \frac{z + t(1 - x)a_5}{z}, \frac{t z - t^2(1 + x)a_5}{z} \right], \]

where \( a_i \) \((i = 1, 2, 3, 4, 5)\) are constants.

Thus we have

\[ e^{a_i A_i} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} u(x, y, z, t) = \left( \frac{z - t(1 + x)a_5}{z} \right) u(\xi, \eta, \rho, \theta). \] (3.3)

Where

\[ \xi = \frac{(tz - t^2(1 + x)a_5)(xz + t(1 - x^2)a_5)}{tz^2 - t^2(1 + x)a_5 - a_4 z(z + t(1 - x)a_5)}, \]
\[ \eta = e^{a_1 - n} y \left( \frac{z - t(1 + x)(a_5 - n)}{z} \right), \]
\[ \rho = e^{a_2} \left( z + t(1 - x)a_5 \right), \]
\[ \theta = e^{a_3} \left( \frac{z - t(1 + x)a_5}{z} \right) \left[ t - \frac{a_4 z^2(z + t(1 - x)a_5)}{z - t(1 + x)a_5} \right]. \]

It may be interest to remark that, by virtue of the commutator relation given above

\[ \exp(a_5 A_5 + a_4 A_4 + a_3 A_3 + a_2 A_2 + a_1 A_1) \]
\[ \neq \exp(a_5 A_5) \exp(a_4 A_4) \exp(a_3 A_3) \exp(a_2 A_2) \exp(a_1 A_1). \] (3.4)

The relation (3.3) is obtained by using the operator mentioned in the right side of (3.4). Thus the order of \( A_i \) \((i = 1, 2, 3, 4, 5)\) can be change at case without uttering the effect in the left member of (3.4), while that can not be change in the right member of (3.4). So if we change the order of the operator mentioned in the right side of (3.4) the relation (3.3) will be changed.

4. Generating Functions

From the (2,1), \( u(x, y, z, t) = p(x) y^{(a-\beta)} z^\beta t^n \) is a solution of the systems.

\[ Lu = 0 \]
\[ (A_4 - \alpha - n)u = 0; \quad (A_2 - \beta)u = 0; \]
\[ Lu = 0 \]
\[ (A_3 - n)u = 0; \quad (A_1 + A_2 + A_3 - \beta - \alpha)u = 0; \]

From (3.2) we easily get

\[ S((x - 1)L)p(x) y^{(a-\beta)} z^\beta t^n = ((x - 1)L)S p(x) y^{(a-\beta)} z^\beta t^n = 0, \]

where

\[ S = e^{a_i A_i} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}. \]
There for the transformations \( S[P_n^{(a-n,\beta)}(x)y^{a-n}t^n] \) is annulled by \( L \).

By putting \( a_1 = a_2 = a_3 = 0 \) in (3.3) we get
\[
e^{a_n A_n} e^{a_n A_n} [P_n^{(a-n,\beta)}(x)y^{a-n}t^n] = y^{a-n} \left( \frac{z - t(1 + x)a_5}{z} \right)^{1+\alpha} \left( \frac{z + t(1 - x)a_5}{z - t(1 + x)a_5} \right) \left[ t - \frac{a_4z(z + t(1 - x)a_5)}{z - t(1 + x)a_5} \right]
\[
\times \frac{e^{(a-n,\beta)}}{z} \left[ tz - t^2(1 + x)a_5(xz + t(1 - x^2)a_5) - a_4z^2(z + t(1 - x)a_5) \right]
\times P_n^{(a-n,\beta)}(t^2 - t^2z(1 + x)a_5) - a_4z^2(z + t(1 - x)a_5)
\]
\[
= \sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{a_5^k (a_4)^p}{k!} (\alpha + \beta + 1)_p (-1)^k (2)^{(k-p)} (n - p + 1)_k
\times P_n^{(a-n+p-k,\beta-p-k)}(x) y^{a-n+p-k} t^{n-p+k}.
\]

(4.1)

If we change the order of \( e^{a_n A_n} e^{a_n A_n} \) we shall get the relation different from (4.1). But
\[
e^{a_n A_n} e^{a_n A_n} [P_n^{(a-n,\beta)}(x)y^{a-n}t^n]
= \sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{a_5^k (a_4)^p}{k!} (\alpha + \beta + 1)_p (-1)^k (2)^{(k-p)} (n - p + 1)_k
\times P_n^{(a-n+p-k,\beta-p-k)}(x) y^{a-n+p-k} t^{n-p+k}.
\]

(4.2)

Equating the result (4.1) and (4.2) we get
\[
y^{a-n} \left( \frac{z - t(1 + x)a_5}{z} \right)^{1+\alpha} \left( \frac{z + t(1 - x)a_5}{z - t(1 + x)a_5} \right) \left[ t - \frac{a_4z(z + t(1 - x)a_5)}{z - t(1 + x)a_5} \right]
\times \frac{e^{(a-n,\beta)}}{z} \left[ tz - t^2(1 + x)a_5(xz + t(1 - x^2)a_5) - a_4z^2(z + t(1 - x)a_5) \right]
\times P_n^{(a-n+p-k,\beta-p-k)}(x) y^{a-n+p-k} t^{n-p+k}.
\]

(4.3)

Now we shall consider the following cases:

**Case 1:** Let \( a_5 = 0 \), \( a_4 = 1 \) and writing \( t_1 = \frac{z}{x} \) in (4.3) we get
\[
P_n^{(a-n,\beta)} \left[ \frac{y x - t_1}{y - t_1} \right] = \sum_{\beta=0}^{n} \frac{1}{\beta!} \frac{1}{2^\beta} (\alpha + \beta + 1)_p P_n^{(a-n+p,\beta+p)}(x)(t_1)^p.
\]

(4.4)

Which is (1.3).

**Case 2:** Let \( a_5 = 1 \), \( a_4 = 0 \) and writing \( t_2 = \frac{z}{y} \) in (4.3) we get
\[
[1 - t_2 y(1 + x)]^{1+\alpha} [1 + t_2 y(1 - x)]^\beta P_n^{(a-n,\beta)}[x + t_2 y(1 - x^2)]
\]
\[
= \sum_{k=0}^{n} \frac{(-2)^k}{k!} (n + 1)_k P_n^{(a-n-k,\beta-k)}(x)(t_2)^k.
\]

(4.5)

Which is (1.4).
Case 3: Finally substituting $a_5 = 1$, $a_4 = \frac{-1}{w_1}$ and writing $\frac{x}{y^2} = t_3$ in (4.3). We get

$$[1 - t_3(1 + x)]^{1+a}[1 + t_3(1 - x)]^\beta \left[1 + \frac{1}{y t_3 w_1}(1 + y t_3(1 - x))\right]^n$$

$$\times p_n^{(\alpha-n,\beta)} \left[(y t_3(1 - y t_3(1 + x)))(x + y t_3(1 - x^2) + \frac{1}{w_1}(1 + y t_3(1 - x))\right]^n$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{n} \frac{(-1)^k}{k!} \frac{(-1/w_1)^p}{p!} 2^{k-p}(\alpha + \beta + 1)(n - p + 1)_k$$

$$\times p_n^{(\alpha-n+p-k,\beta+p-k)}(x)(t_3)^{k-p}. \quad (4.6)$$

Which is (1.5).

References


