**Journal of Informatics and Mathematical Sciences**
Volume 4 (2012), Number 1, pp. 85–92

http://www.rgnpublications.com

# Modelling of the 3R Motion at Non-Parallel Planes

Senay Baydas and Bulent Karakas

**Abstract.** We construct two similar planar mechanisms which have different and non-parallel planes. We build up a new connection between these mentioned mechanisms in this paper. How the motion of a mechanism is carried to another plane without making a difference in mechanism algorithm and some necessary mathematical relationships are found out. Therefore, a mechanism structure can be transported from one of the intersecting planes to another planes without changing its mechanism algorithm. This mechanism structure is as finger motion and the most important result is this.

## 1. Introduction

In mechanism theory, there are two main actions: Rotation and translation. In the case of rotation, there are similar rotations at parallel planes, which are perpendicular to rotation axis. In the case of translation, parallel translations exist at all planes, which are passing the line carrying the translation vector. One of the algorithms which are used in expressing a mechanism is the Denavit-Hartenberg (D-H) representation [7, 8].

Figural form of rotation matrix is different when axis of rotation is an axis different from coordinate axes. Let **r** be any axis described by vector $\overrightarrow{r}$ passing through origin and different from coordinate axes. The rotation around **r** axis described by vector $\overrightarrow{r}$ is defined in horizontal planes which are perpendicular to **r**. When new coordinate system is $(O:UVW)$ and $\overrightarrow{r}=(r_x,r_y,r_z)$, $\|\overrightarrow{r}\|=1$, rotation matrix is known [2]*. Let $\alpha$ and $\beta$ be intersecting planes and intersecting angle $(\widehat{\alpha,\beta})=\phi$, $0<\phi<\frac{\pi}{2}$. Primarily, if the rotation at $(O:XYZ)$ is around z-axis and $\alpha$ and $\beta$ intersects along $y$-axis, the normal of $\alpha$ is $\overrightarrow{z}$ and the normal of $\beta$ is $z'$, here $z'=R_y(\phi)(z)$. Accepting $z'$ as **r**, rotation at $\beta$ can be defined. The analysis of $2R$ planar mechanism, not including our goal, is made in [1] and [3]. If the distance between the joints of an RR chain is allowed to vary, then we obtain the structure of a three degree-of-freedom planar manipulator. This variation can

2010 *Mathematics Subject Classification.* 53Z99, 62K25, 51-04.
*Key words and phrases.* Motion; Synchronized; Mechanism.

* We will use $R_r(\phi)$ instead of $R_{r,\phi}$ in our paper.

be introduced by a revolute joint to form a $3R$ open chain. The formulas for the RR chain can be used to analyze the $3R$ chains with minor modifications [5].

Firstly, the $3R$ mechanism will be constructed and analyzed at $\alpha$-plane as $M_\alpha$. Then the analogous $3R$ will be defined at $\beta$ as $M_\beta$ and subsequently relation between $M_\alpha$ and $M_\beta$ will be built.

## 2. Kinematic equations for $3R$ at the $\alpha$-plane $(M_\alpha)$

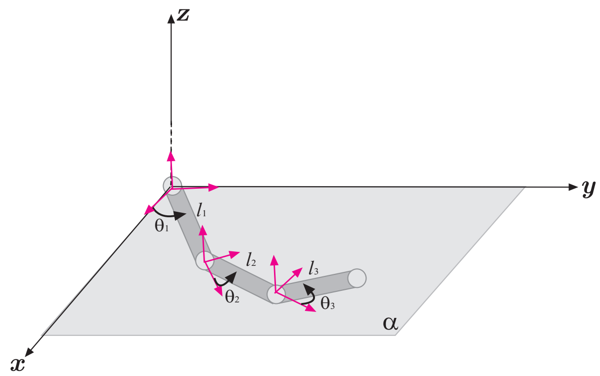


**Figure 1.** Location of mechanism at $\alpha$-plane

The design of a $3R$ mechanism which consists of three joints is seen at Figure 1 as [6] and [4]. Suppose that its inertial point of base frame is O origin point, rotation axis is z-axis, rotation plane is $xoy$-plane. In the parameters of the mechanism whose links are $l_1, l_2, l_3$ in length and rotation angles are $\theta_1, \theta_2, \theta_3$ we have shown the parameters according to D-H representation at Table 1.

**Table 1.** D-H parameters

| $\theta_i$ | $d_i$ | $a_i$ | $\alpha_i$ |
|---|---|---|---|
| $\theta_1$ | $-$ | $l_1$ | $-$ |
| $\theta_2$ | $-$ | $l_2$ | $-$ |
| $\theta_3$ | $-$ | $l_3$ | $-$ |

According to Table 1, ${}^{i}A_{i+1}$ transformation matrices belonging to mechanism can be written as thus:

$$
{}^{i-1}A_i = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & l_i C\theta_i \\ S\theta_i & C\theta_i & 0 & l_i S\theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$1 \le i \le 3$. ${}^0A_3$ transformation matrix of position of the end-effector is calculated from multiplying ${}^0A_1\ {}^1A_2\ {}^2A_3$ so we have[†]

$$
{}^0A_3 = \begin{bmatrix} C\theta_{123} & -S\theta_{123} & 0 & l_1C\theta_1 + l_2C\theta_{12} + l_3C\theta_{123} \\ S\theta_{123} & C\theta_{123} & 0 & l_1S\theta_1 + l_2S\theta_{12} + l_3S\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Also we know that

$$
T =^0 A_3 \tag{1}
$$

Now, using the inverse kinematic equations, we will find $\theta_i$ (for $i = 1,2,3$) and therefore $\theta_2$ is obtained as follows

$$
\theta_2 = \arccos\left(\frac{(p_x - l_3n_x)^2 + (p_y + l_3o_x)^2 - l_1^2 - l_2^2}{2l_1l_2}\right).
$$

From equation (2)

$$
{}^0A_1^{-1}T = {}^1A_2\ {}^2A_3 \tag{2}
$$

can be written. $\theta_1$ is attained as the following:

$$
\theta_1 = \arccos\left(\frac{A(l_1 + l_2C\theta_2) + Bl_2S\theta_2}{(p_x - l_3n_x)^2 + (p_y - l_3n_y)^2}\right)
$$

and $\theta_3$ is equated as

$$
\theta_3 = \arccos(n_xC\theta_1 + n_yS\theta_1) - \theta_2
$$

Furthermore, for both $\theta_1 = \pi/2$, $\theta_2 = \pi/3$, $\theta_3 = \pi/6$, $\gamma = \pi/4$ and $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $\theta_3 = \pi/2$, $\gamma = \pi/4$, these datum are analyzed.

## 3. Location of 3R mechanism at $\beta$ ($M_\beta$)

Let $\beta : Ax + By + Cz + D = 0$ be a given plane. For arbitrary $x_0, y_0$, $(x_0, y_0, z_0) \in \beta$, where $z_0 = \frac{1}{C}(-D - Ax_0 - By_0)$. Let $v = (v_1, v_2, v_3)$ be any vector, passing $(x_0, y_0, z_0)$ and perpendicular $N_\beta$. The normal vector of $\beta$ plane is

$$
N_\beta = \frac{1}{\sqrt{A^2 + B^2 + C^2}}(A, B, C).
$$

For a frame which presumes $(x_0, y_0, z_0)$ point as a start

$$
v \times N_\beta = u
$$

could be selected as

$$
(v \times N_\beta, v, N_\beta)
$$

[†]We used $\theta_{123}$ instead of $(\theta_1 + \theta_2 + \theta_3)$ for the sake of brevity.

for $\beta$. The matrix form of a revolute on $\beta$ plane, around an axis defined by $\overrightarrow{w} = N_\beta$ is

$$R_{w,\theta} = \begin{bmatrix} w_1^2(1-C\theta)+C\theta & w_1w_2(1-C\theta)-w_3S\theta & w_1w_3(1-C\theta)+w_2S\theta \\ w_1w_2(1-C\theta)+w_3S\theta & w_2^2(1-C\theta)+C\theta & w_2w_3(1-C\theta)-w_1S\theta \\ w_1w_3(1-C\theta)-w_2S\theta & w_2w_3(1-C\theta)+w_1S\theta & w_3^2(1-C\theta)+C\theta \end{bmatrix}.$$

Let us construct the simulation of the mechanism constructed in $\alpha$ as part of a planar mechanism in $\beta$ plane. By the simulation the equality of corresponding link lengths and link angles is meant. Let us apply the tasks undertaken by $x, y, z$ axes to $u, v, w$ axes respectively in the use of D-H representation based on the assumptions and demonstrations in Figure 2.

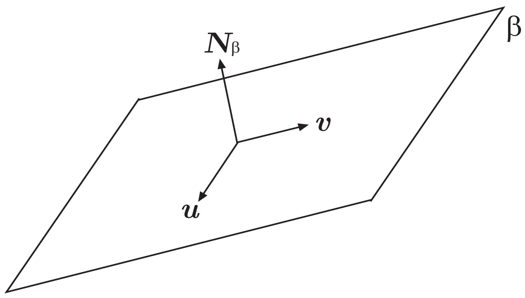


**Figure 2.** $u, v, N_\beta$

Hence, the D-H transformation matrices belonging to $M_\beta$ mechanism built in $\beta$ would be found as

$${}^0B_1 = \begin{bmatrix} w_1^2(1-C\theta_1)+C\theta_1 & w_1w_2(1-C\theta_1)-w_3S\theta_1 & w_1w_3(1-C\theta_1)+w_2S\theta_1 & l_1u_1C\theta_1+l_1v_1S\theta_1 \\ w_1w_2(1-C\theta_1)+w_3S\theta_1 & w_2^2(1-C\theta_1)+C\theta_1 & w_2w_3(1-C\theta_1)-w_1S\theta_1 & l_1u_2C\theta_1+l_1v_2S\theta_1 \\ w_1w_3(1-C\theta_1)-w_2S\theta_1 & w_2w_3(1-C\theta_1)+w_1S\theta_1 & w_3^2(1-C\theta_1)+C\theta_1 & l_1u_3C\theta_1+l_1v_3S\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1B_2 = \begin{bmatrix} w_1^2(1-C\theta_2)+C\theta_2 & w_1w_2(1-C\theta_2)-w_3S\theta_2 & w_1w_3(1-C\theta_2)+w_2S\theta_2 & l_2u_1C\theta_2+l_2v_1S\theta_2 \\ w_1w_2(1-C\theta_2)+w_3S\theta_2 & w_2^2(1-C\theta_2)+C\theta_2 & w_2w_3(1-C\theta_2)-w_1S\theta_2 & l_2u_2C\theta_2+l_2v_2S\theta_2 \\ w_1w_3(1-C\theta_2)-w_2S\theta_2 & w_2w_3(1-C\theta_2)+w_1S\theta_2 & w_3^2(1-C\theta_2)+C\theta_2 & l_2u_3C\theta_2+l_2v_3S\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2B_3 = \begin{bmatrix} w_1^2(1-C\theta_3)+C\theta_3 & w_1w_2(1-C\theta_3)-w_3S\theta_3 & w_1w_3(1-C\theta_3)+w_2S\theta_3 & l_3u_1C\theta_3+l_3v_1S\theta_3 \\ w_1w_2(1-C\theta_3)+w_3S\theta_3 & w_2^2(1-C\theta_3)+C\theta_3 & w_2w_3(1-C\theta_3)-w_1S\theta_3 & l_3u_2C\theta_3+l_3v_2S\theta_3 \\ w_1w_3(1-C\theta_3)-w_2S\theta_3 & w_2w_3(1-C\theta_3)+w_1S\theta_3 & w_3^2(1-C\theta_3)+C\theta_3 & l_3u_3C\theta_3+l_3v_3S\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and as a result, the motion matrix of the mechanism could be obtained by

$${}^0B_3 = {}^0B_1\ {}^1B_2\ {}^2B_3 \tag{3}$$

multiplicity.

An analysis of the mechanism defined by (9) equality could be made in $\beta$. However, constructing the mechanism in $\beta$ and making its analysis is not relevant with the design of a mechanism moving synchronically in $\alpha$ and $\beta$,which was the chief focus of concern in this study. A second method, which is much more relevant with our objective, is the construction of the mechanism in $\beta$ together with its co-motion structure, obtained through the transportation of the mechanism in $\alpha$ to the one constructed in $\beta$. The aim of the following section is this: The mechanism will be built in $\alpha$ and transported to $\beta$, and the motion matrix of the synchronized mechanism transported to $\beta$ will be shown as it is indicated in ${}^0B_3$.

$\alpha$ and $\beta$ are two planes which intersect, through one axis ($y$-axis) (non-coincident) and let the angle between $\alpha$ and $\beta$ be $\gamma$. $\beta$-plane can be taken as the rotating of $\alpha$-plane about $y$-axis an angle of $\gamma$ at Figure 3.

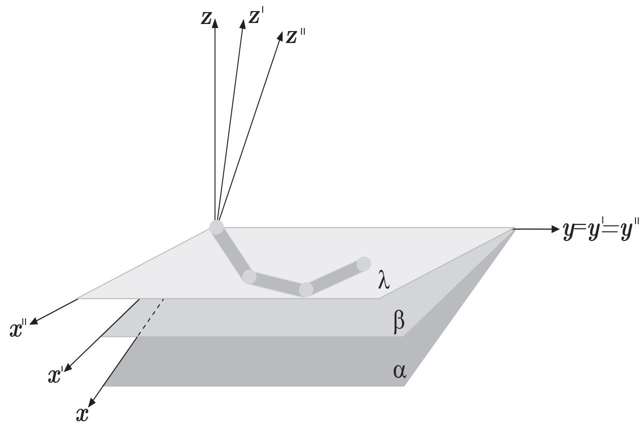


**Figure 3.** Location of mechanism at $\beta$-plane

So we can assume that the plane $\beta$ is an image of $\alpha$, that is

$$\beta = R_y(\gamma)(\alpha).$$

Since it will not eclipse the generality we can select $\alpha$ plane as $z = 0$ plane. Let plane of $\alpha : z = 0$ be given. In this case rotation matrix is $N_\alpha = (0, 0, 1)$ and thus $N_\beta$ is found as follows:

$$R_y(\gamma).N_\alpha = \begin{bmatrix} C\gamma & 0 & S\gamma \\ 0 & 1 & 0 \\ -S\gamma & 0 & C\gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} S\gamma \\ 0 \\ C\gamma \end{bmatrix} = N_\beta\,,$$

where $N_2$ is rotation axis of $\beta$-plane.

Thus, rotation matrix shown with $R_r(\phi)$ given in (1) is used instead of the rotation matrix about $z$-axis at $\alpha$. Rotation matrices at the mechanism are $R_{N_\beta}(\theta_i)$, (for $i = 1, 2, 3$).

${}^0B_3$, transformation matrix belonging to the 3$R$ mechanism at $\beta$, is

$$\begin{aligned} {}^0B_3 &= {}^0B_1\ {}^1B_2\ {}^2B_3 \\ &= \begin{bmatrix} R_{N_\beta}(\theta_1) & l_1'' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{N_\beta}(\theta_2) & l_2'' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{N_\beta}(\theta_3) & l_3'' \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{4}$$

we obtained ${}^0B_3$ as follows:

$$ {}^0B_3 = \begin{bmatrix} C^2\gamma C\theta_{123} + S^2\gamma & -C\gamma S\theta_{123} & S\gamma C\gamma(1 - C\theta_{123}) & C\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ C\gamma S\theta_{123} & C\theta_{123} & -S\gamma S\theta_{123} & l_1 S\theta_1 + l_2 S\theta_{12} + l_3 S\theta_{123} \\ S\gamma C\gamma(1 - C\theta_{123}) & S\gamma S\theta_{123} & S^2\gamma C\theta_{123} + C^2\gamma & -S\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5} $$

## 4. **Transition from $M_\alpha$ to $M_\beta$**

Suppose that $\alpha$-plane is $xoz$-plane. $\beta$-plane is occurred, rotating $\alpha$-plane about $y$-axis an angle of $\gamma$. Transformation matrix from $\alpha$ to $\beta$ is

$$ R_y(\gamma) = \begin{bmatrix} C\gamma & 0 & S\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -S\gamma & 0 & C\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

The mechanism at $xoz$-plane is at rotating plane which is rotated about $y$-axis an angle of $\gamma$ and transformation matrix of the mechanism is

$$ R_y(\gamma)\ {}^0A_3 \ = \begin{bmatrix} C\gamma C\theta_{123} & -C\gamma S\theta_{123} & S\gamma & C\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ S\theta_{123} & C\theta_{123} & 0 & l_1 S\theta_1 + l_2 S\theta_{12} + l_3 S\theta_{123} \\ -S\gamma C\theta_{123} & S\gamma S\theta_{123} & C\gamma & -S\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6} $$

It is clear that the position of the end-effector belonging to ${}^0B_3$ is the same as the position of the end-effector belonging to $R_y(\gamma)\ {}^0A_3$. The theorem which underpins our study and the preparation proof of which was made throughout Section 3 is as follows.

## 5. **Transition from $M_\beta$ to $M_\xi$**

$$ R_y(\xi) = \begin{bmatrix} C\xi & 0 & S\xi & 0 \\ 0 & 1 & 0 & 0 \\ -S\xi & 0 & C\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ R_y(\xi)(R_y(\gamma)\ {}^0A_3) = \begin{bmatrix} C\xi & 0 & S\xi & 0 \\ 0 & 1 & 0 & 0 \\ -S\xi & 0 & C\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C\gamma C\theta_{123} & -C\gamma S\theta_{123} & S\gamma & C\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ S\theta_{123} & C\theta_{123} & 0 & l_1 S\theta_1 + l_2 S\theta_{12} + l_3 S\theta_{123} \\ -S\gamma C\theta_{123} & S\gamma S\theta_{123} & C\gamma & -S\gamma\left(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

**Theorem 1.** *Let $\alpha$ and $\beta$ be intersecting planes. If $L$ is intersecting line and $\gamma$ is the angle between two planes, $Rot_L(\gamma)$ transports the mechanism at $\alpha$ to $\beta$ with scynhronized motion.*

***Proof.*** Since it will not eclipse the generalization $xoy$-plane can be taken as $\alpha$ plane. In this case the motion matrix of the mechanism built in $\beta$ is ${}^0B_3$ and the motion matrices of the mechanism carried from $\alpha$ to $\beta$ are respectively as follows:

$$
{}^0B_3 = \begin{bmatrix} C^2\gamma C\theta_{123} + S^2\gamma & -C\gamma S\theta_{123} & S\gamma C\gamma(1 - C\theta_{123}) & C\gamma(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}) \\ C\gamma S\theta_{123} & C\theta_{123} & -S\gamma S\theta_{123} & l_1 S\theta_1 + l_2 S\theta_{12} + l_3 S\theta_{123} \\ S\gamma C\gamma(1 - C\theta_{123}) & S\gamma S\theta_{123} & S^2\gamma C\theta_{123} + C^2\gamma & -S\gamma(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

and

$$
R_y(\gamma)\ {}^0A_3 = \begin{bmatrix} C\gamma C\theta_{123} & -C\gamma S\theta_{123} & S\gamma & C\gamma(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}) \\ S\theta_{123} & C\theta_{123} & 0 & l_1 S\theta_1 + l_2 S\theta_{12} + l_3 S\theta_{123} \\ -S\gamma C\theta_{123} & S\gamma S\theta_{123} & C\gamma & -S\gamma(l_1 C\theta_1 + l_2 C\theta_{12} + l_3 C\theta_{123}) \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The components belonging to the end effector in both matrix can be seen as equal and the equality complements the proof. $\square$

## 6. Conclusions

The planar mechanisms can be constructed based on the normal of planes, the rotations around a rotating axis and the translations in the determined direction. The simulation of a planar mechanism can be obtained and transported to another plane intersecting the plane of an other mechanism, and thus, $2 \times 3R$ space mechanism constituted of two synchronically moving $3R$ mechanism can be built. In our study, the construction of $2 \times 3R$ has been examined. The algorithm of this study is as thus:

Let $\alpha$ and $\beta$ be two planes intersected with the intersection angle of $\gamma$, $0 < \gamma < \frac{\pi}{2}$. Let the mechanism $M_\alpha$ be set up in $\alpha$. Let write the translation matrix from $\alpha$ to $\beta$ with $\gamma$ angle of rotation throughout the intersection line of the planes be written. Then we obtain the following result as thus: The end matrix of the mechanism at $\alpha$ is ${}^0A_3$ and the end matrix of mechanism at $\beta$ is $R_y(\gamma)\ {}^0A_3$.

This matrix will yield to the similar last effector components resembling ${}^0B_3$ obtained through being designed at $\beta$ plane by $M_\beta$. Thus, a planar mechanism can be transmitted in the way moving synchronically towards all planes intersecting with the mechanism's plane. In this study, it has been shown that the mechanism of $M_\alpha$ designed at $\alpha$ plane can be converted into $M_\beta$ mechanism through the rotation of $R_y(\gamma)$ without being redesigned at $\beta$. The mechanic systems obtained on a synchronic basis constitute the foundation of the mechanism called as finger motion.

Consequently, when $\alpha$ and $\beta$ planes are intersecting and non-coincident, the mechanism at $\alpha$ can be translated to $\beta$ with $R_y(\gamma)$ and to $\lambda$ with $R_y(\xi)$ as synchronized. Thus, $3 \times 3R$ space mechanism moving synchronically can be obtained. The result has been proved in Theorem 1.

## References

[1] S. Bulut and M.B. Terzioğlu, Joint angle variations analyses of the two link planar manipulator in welding by using inverse kinematics, *Robotica* **24** (2006), 355–363.

[2] K.S. Fu, R.C. Gonzalez and C.S.G. Lee, *Robotics*, Mc Graw-Hill International Editions, Singapore, 1987.

[3] J.Q. Gan, E. Oyama, E.M. Rosales and H. Hu, A complete analytical solution to the inverse kinematics of the Pioneer 2 robotic arm, *Robotica* **23** (2005), 123–129.

[4] E. Lee and C. Marvoidis, Solving the geometric design problem of spatial 3*R* robot manipulators using polynomial homotopy continuation, *Journal of Mechanical Design* **124** (2002), 652–661.

[5] J.M. McCarthy, *Geometric Design of Linkages*, Springer Verlag, New York, 2000.

[6] J. Nielsen and B. Roth, On the kinematic analysis of robotic mechanisms, *The International Journal of Robotics Research* 18 (12) (1999), 1147–1160.

[7] S.B. Niku, *Introduction to Robotics*, Prentice Hall, Jersey, 2001.

[8] R.J. Schilling, *Fundamentals of Robotics*, Prentice Hall, New Jersey, 1990.

Senay Baydas, *Department of Mathematics, Faculty of Science, Yuzuncu Yil University, Van, 65080, Turkey*.
*E-mail*: `senay.baydas@gmail.com`

Bulent Karakas, *Yuzuncu Yil University, Van, 65080, Turkey*.
*E-mail*: `bulentkarakas@gmail.com`

*Received* August 14, 2011
*Accepted* October 1, 2011