Remark on a Theorem of Prolla

Zoran D. Mitrović

Abstract. In this paper, we obtained the form of the best approximation theorem of Prolla. Our result extends some results in literature.

1. Introduction and Preliminaries


Theorem 1.1 (Ky Fan, [2]). Let \( C \) be a nonempty, compact, convex subset of a normed linear space \( X \). Then for any continuous mapping \( f \) from \( C \) to \( X \), exists a point \( x_0 \in C \) with
\[
\|x_0 - f(x_0)\| = \inf_{x \in C} \|x - f(x)\|.
\]

This result has been generalized to other spaces \( X \) and other types of maps, see for example [4], [5], [7]. Prolla [6] and Carbone [1] obtained a form of the best approximation theorem of Ky Fan using almost affine and almost quasi-convex maps in normed vector spaces.

Definition 1.2. Let \( X \) be a normed space and \( C \) a nonempty convex subset of \( X \).

(i) A map \( g : C \to X \) is almost affine if for all \( x, y \in C \) and \( u \in C \)
\[
\|g(\lambda x + (1 - \lambda)y) - u\| \leq \lambda \|g(x) - u\| + (1 - \lambda)\|g(y) - u\|
\]
for each \( \lambda \) with \( 0 < \lambda < 1 \).

(ii) A map \( g : C \to X \) is almost quasi-convex if for all \( x, y \in C \) and \( u \in C \)
\[
\|g(\lambda x + (1 - \lambda)y) - u\| \leq \max\{\|g(x) - u\|, \|g(y) - u\|\}
\]
for each \( \lambda \) with \( 0 < \lambda < 1 \).

2000 Mathematics Subject Classification. 47H10.
Key words and phrases. KKM map; Best approximations.
Theorem 1.3 (J.B. Prolla, [6]). Let $X$ be a normed linear space, $C$ a nonempty convex compact subset of $X$, $f : C \to X$ a continuous map and $g : C \to C$ is a continuous, almost affine, onto map. Then there exists a point $x_0 \in C$, such that

$$
\|g(x_0) - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.
$$

Carbone [1] obtain the version of Theorem 1.3 using almost quasi-convex maps.

We extended results of Prolla [6] and Carbone [1].

For a nonempty subset $A$ of $X$, let $\text{co}A$ denote the convex hull of $A$.

Definition 1.4. Let $C$ be a nonempty subset of a topological vector space $X$. A map $H : C \to 2^X$ is called KKM map if for every finite set $\{x_1, \ldots, x_n\} \subset C$, we have

$$
\text{co} \{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} H(x_k).
$$

The following extension of the classical KKM principle is due to Ky Fan [3].

Theorem 1.5. [3] Let $X$ be a topological vector space, $K$ be a nonempty subset of $X$ and $H : K \to 2^K$ a map with closed values and KKM map. If $H(x)$ is compact for at least one $x \in K$ then $\bigcap_{x \in X} H(x) \neq \emptyset$.

2. Main Result

From Theorem 1.5 we obtain the following best approximation theorem in normed spaces.

Theorem 2.1. Let $X$ be a normed linear space, $C$ a nonempty convex compact subset of $X$, $f : C \to X$ and $g : C \to C$ continuous maps. If there exists an almost quasi-convex onto map $h : C \to C$ such that

$$
\|g(x) - f(x)\| \leq \|h(x) - f(x)\| \quad \text{for each } x \in C,
$$

then there exists a point $x_0 \in C$ such that

$$
\|g(x_0) - f(x_0)\| = \inf_{x \in C} \|x - f(x_0)\|.
$$

Proof. Let for every $y \in C$, $H : C \to 2^C$ be defined by

$$
H(y) = \{x \in C : \|g(x) - f(x)\| \leq \|h(y) - f(x)\|\}.
$$

We have that $H(y)$ is nonempty for all $y \in C$, because $y \in H(y)$ for all $y \in C$. Since $f$ and $g$ are continuous maps, then $H(y)$ is closed for all $y \in C$. Now, we show that for each finite set $\{x_1, \ldots, x_n\} \subset C$,

$$
\text{co} \{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} H(x_k).
$$

(2.2)
Suppose that
\[ \text{co} \{x_1, \ldots, x_n\} \not\subseteq \bigcup_{k=1}^{n} H(x_k) \] for some \( \{x_1, \ldots, x_n\} \subset C \).

Then there exists \( y_0 \in \text{co} \{x_1, \ldots, x_n\} \) such that \( y_0 \not\in H(x_k) \) for each \( k \in \{1, \ldots, n\} \).

So, we have
\[ \|g(y_0) - f(y_0)\| > \|h(x_k) - f(y_0)\| \quad \text{for each } k \in \{1, \ldots, n\}. \]

Therefore,
\[ \|g(y_0) - f(y_0)\| > \max_{k} \|h(x_k) - f(y_0)\| \geq \|h(y_0) - f(y_0)\|. \]

This is a contradiction with condition (2.1). Hence, condition (2.2) is true for each finite \( \{x_1, \ldots, x_n\} \subset C \) and a map \( H \) is KKM map. Now, from Theorem 1.5 it follows that there exists \( y_0 \in C \) such that
\[ y_0 \in \bigcap_{y \in C} H(y). \]

Therefore,
\[ \|g(y_0) - f(y_0)\| = \inf_{x \in C} \|x - f(y_0)\|. \]

Example 2.2. Let \( C = [0, 1] \) and define maps \( f, g, h : C \to C \) by
\[
\begin{align*}
f(x) &= 0, \\
h(x) &= x, \\
g(x) &= \begin{cases} 
  x, & x \in [0, \frac{1}{4}) \\
  -x + \frac{1}{2}, & x \in \left[\frac{1}{4}, \frac{1}{2}\right) \\
  2x - 1, & x \in \left[\frac{1}{2}, 1\right]. 
\end{cases}
\end{align*}
\]

Then map \( g \) is not almost quasi-convex and results of J.B. Prolla and A. Carbone are not applicable. Note that the maps \( f, g \) and \( h \) satisfy all hypotheses of Theorem 2.1.

References


Zoran D. Mitrović, *Faculty of Electrical Engineering, University of Banja Luka, 78000 Banja Luka, Patre 5, Bosnia and Herzegovina.*

E-mail: zmitrovic@etfbl.net

Received May 19, 2011
Accepted August 18, 2011