Generalization of the Central Subgroup of the Nonabelian Tensor Square of a Crystallographic Group with Symmetric Point Group

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Abstract. The central subgroup of the nonabelian tensor square of a group $G$, denoted by $\nabla(G)$, is a crucial tool in exploring the properties of a group. It is a normal subgroup generated by the element $g \otimes g$, for all $g \in G$. In this paper, the central subgroup of the nonabelian tensor square of a crystallographic group with symmetric point group is constructed and generalized up to finite dimension.

Keywords. Central subgroup of the nonabelian tensor square; Crystallographic group

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1. Introduction

Crystallographic groups have many interesting properties. The main focus in this paper is a crystallographic group with symmetric point group, denoted by $B_2$. In [1], the consistent polycyclic presentation of $B_2$ of dimension four, $B_2(4)$ has been constructed as follows.

$$B_2(4) = \langle a, b, l_1, l_2, l_3, l_4 \mid a^2 = l_3, b^3 = l_3^3 l_4^{-2}, b^a = b^2 l_3^{-1} l_4^2, l_1^a = l_1, l_2^a = l_1 l_2^{-1}, l_3^a = l_3, l_4^a = l_4^{-1} \rangle$$

$$l_1^b = l_2^{-1}, l_2^b = l_1 l_2^{-1}, l_3^b = l_3, l_4^b = l_4, l_1^i = l_j, l_j^{-1} = l_j \text{ for } j > i, 1 \leq i, j \leq 4) \quad (1.1)$$

The central subgroup of the nonabelian tensor square of a group $G$, denoted by $\nabla(G)$ is a normal subgroup generated by the element $g \otimes g$, for all $g \in G$. $G \otimes G$ is a group generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations $gh \otimes k = (g^h \otimes k^h)(h \otimes k)$ and $g \otimes hk = (g \otimes k)(g^h \otimes k^h)$ for all $g, h, k \in G$ where $g^h = h^{-1}gh$ [2]. Lemma 1 shows the close relationship between $\nabla(G)$ and the abelianization of the group.

**Lemma 1 ([3]).** Let $G$ be a group whose abelianization is finitely generated by the independent set $x_iG'$, $i = 1, \ldots, n$. Then, $\nabla(G) = \{[x_i, x_j^p] | 1 \leq i < j \leq s\}$.

In [4], the central subgroup of the nonabelian tensor square of the group $B_2(4)$ has been computed. Thus, the aim of this paper is to generalize the central subgroup of the nonabelian tensor square of the group $B_2$ up to dimension $n$.

2. Preliminaries

In this section, some basic definitions and some structural results are presented.

**Definition 1 ([5], Polycyclic Presentation).** Let $F_n$ be a free group on generators $g_1, \ldots, g_n$ and $R$ be a set of relations of group $G$. The relations of a polycyclic presentation have the form $g_i^{e_i} = g_{i+1}^{r_{i+1}} \cdots g_n^{r_n}$ for $i \leq I$, $g_j^{1} = g_{i+1}^{r_{i+1}} \cdots g_n^{r_n}$ for $j \leq i$, $g_j g_k^{-1} = g_{i+1}^{r_{i+1}} \cdots g_n^{r_n}$ for $j < i$ and $k < I$ for some $I \subseteq \{1, \ldots, n\}$, $e_i \in N$ for $i \in I$ and $x_i, y_i, z_i, \bar{y}_i, \bar{z}_i, h, k \in \mathbb{Z}$ for all $i, j$ and $k$.

**Definition 2 ([5], Consistent Polycyclic Presentation).** Let $G$ be a group generated by $g_1, \ldots, g_n$. The consistency of the relation in $G$ can be determined using the consistency relations $g_k(g_j g_i) = (g_k g_j) g_i$ for $k > j > i$, $(g_i^{e_i})g_i = g_i^{e_i-1}$ for $j > i$, $j \in I$, $g_j(g_i^{e_i}) = (g_j g_i) g_i^{e_i-1}$ for $j > i$, $f = \inf e I$, $(g_i^{e_i})g_i = g_i(g_i^{e_i})$ for $i \in I$ and $g_j = (g_j g_i^{-1}) g_i$ for $j > i$, $i \in I$.

**Definition 3 ([6]).** Let $G$ be a group with presentation $GR$ and let $G^\varphi$ be an isomorphic copy of $G$ via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. The group $\varphi(G)$ is defined to be $\varphi(G) = G, G^\varphi R, R^\varphi, [g, h^\varphi] = [g^\varphi, (\varphi(h))] = \varphi([g, h])$, for all $x, g, h \in G$.

**Lemma 2 ([7]).** Let $G$ be any crystallographic group of dimension $n$ with point group $P$. Let $B = G \times F_m$ where $F_m$ is a free abelian group of rank $m$. Then $B$ is a crystallographic group of dimension $n + m$ with point group $P$. 

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Theorem 1. The polycyclic presentation of \( l \) relations as given in Definition 2.

Lemma 4. The subgroup \( \text{V}(B_2(4)) \) is given as

\[
\text{V}(B_2(4)) = \langle [a,a^\varphi],[l_2,l_3^\varphi],[l_4,l_4^\varphi],[a,l_2^\varphi][l_2,a^\varphi],[a,l_4^\varphi][l_4,a^\varphi] \rangle \cong C_0 \times C_2 \times C_3.
\]

Lemma 5. Let \( G \) be a group with elements \( x \) and \( y \) such that \( [x,y]=1 \). Then,

(i) \( [x^n,(y^m)^\varphi][y^m,(x^n)^\varphi] = ([x,y^\varphi][y,(x^\varphi)])^nm \)

(ii) If \( g_1 \in G' \) or \( g_2 \in G' \), then \([g_1,g_2^\varphi]^{-1} = [g_2,g_2^\varphi]\).

3. Main Result

In this section, the central subgroup of the nonabelian tensor square of \( B_2 \) is generalized up to finite dimension. First, the generalized polycyclic presentation of \( B_2 \) is constructed as follows.

Lemma 5. The polycyclic presentation of \( B_2(n) \) is consistent where

\[
B_2(n) = \langle a,b,l_1,l_2,l_3,l_4 | a^2 = l_3, b^3 = l_3^2 l_4^{-2}, b^a = b^2 l_3^{-1} l_4^2, l_1^a = l_1, l_2^a = l_1 l_2^{-1}, l_3^a = l_3, l_4^a = l_4^{-1}, l_p^a = l_p, l_p^b = l_p, l_p^{l_1} = l_p, l_p^{l_2} = l_p, l_p^{l_3} = l_p \text{ for } 1 \leq i < j \leq n \rangle
\]

for \( 1 \leq i < j \leq n \) and \( 5 \leq p \leq n \) \hspace{1cm} (3.1)

Proof. By Lemma 2, \( B_2(n) = B_2(4) \times F_{n-4}^{ab} \) for \( n \leq 4 \) where \( B_2(4) \) has the presentation as in (1.1) and \( F_{n-4}^{ab} \) is free abelian of rank \( n-4 \) which is generated by \( l_5, l_6, \ldots, l_n \) and \( l_p \) commutes with all elements in \( B_2(n) \) for \( 5 \leq p \leq n \). Thus, \( l_p^a = l_p, l_p^b = l_p, l_p^{l_i} = l_p, l_p^{l_2} = l_p, l_p^{l_3} = l_p \) and \( l_p^{l_4} \) for \( 5 \leq p \leq n \). Therefore, \( B_2(n) \) has the polycyclic presentation as in (3.1) which satisfies all the relations as given in Definition 2.

Next, the generalization of the abelianization of the group \( B_2 \) is presented as follows.

Lemma 6. The abelianization of \( B_2(n) \),

\[
B_2(n)^{ab} = \langle a B_2(n)^\prime, l_2 B_2(n)^\prime, l_4 B_2(n)^\prime, l_p B_2(n)^\prime \rangle \cong C_0^{n-3} \times C_2 \times C_3 \text{ for } 5 \leq p \leq n.
\]

Proof. The abelianization of \( B_1(n)^{ab} \) is generated by \( a B_2(n)^\prime, b B_2(n)^\prime, l_2 B_2(n)^\prime, l_3 B_2(n)^\prime, l_4 B_2(n)^\prime \) and \( l_p B_2(n)^\prime \) for \( 5 \leq p \leq n \). By Lemma 3, the independent cosets are \( a B_2(n)^\prime, l_2 B_2(n)^\prime \) and \( l_4 B_2(n)^\prime \). Also, \( l_p B_1(n)^\prime \) is independent of other coset. Hence, it can be concluded that \( B_2(n)^{ab} = \langle a B_2(n)^\prime, l_2 B_2(n)^\prime, l_4 B_2(n)^\prime, l_p B_2(n)^\prime \rangle \). By Lemma 3, \( a B_2(n)^\prime \) is of infinite order, \( l_2 B_2(n)^\prime \)
is of order 3 and \( l_4 B_2(n) \) is of order 2. Besides, \( l_p B_2(n) \) is showed to have infinite order since there is no \( l_p \) in \( B_2(n) \) for any integer \( r \). Since \( 5 \leq p \leq n \), then there are \( n - 4 \) cosets in term of \( l_p B_2(n) \). Therefore, \( B_2(n)^{ab} \cong C_0 \times C_2 \times C_3 \times C_0^{n-4} = C_0^{n-3} \times C_2 \times C_3. \)

Then, the construction of \( \nabla(B_2(n)) \) is showed as in the following theorem.

**Theorem 2.** The subgroup \( \nabla(B_2(n)) \) is given as

\[
\nabla(B_2(n)) = \langle [a, a^p], [l_2, l_4^p], [l_4, l_4^p], [l_p, l_4^p], [a, l_4^p][l_2, a^p], [a, l_4^p][l_4, a^p], [a, l_4^p][l_p, a^p], [l_2, l_4^p][l_p, l_4^p][l_4, l_4^p], [l_p, l_4^p][l_4, l_4^p] \rangle
\]

\[
\cong C_0^{(n-3)(n-2)} \times C_2^{n-3} \times C_3^{n-2} \times C_4 \text{ for } 5 \leq p < q \leq n.
\]

**Proof.** By Lemma 2, \( B_1(n)^{ab} \) is generated by \( aB_2(n)^{ab}, l_2B_2(n)^{ab}, l_4B_2(n)^{ab}, \) and \( l_pB_2(n)^{ab} \) for \( 5 \leq p \leq n \). Thus, by Lemma 1, \( \nabla(B_1(n)) = \langle [a, a^p], [l_2, l_4^p], [l_4, l_4^p], [l_p, l_4^p], [a, l_4^p][l_2, a^p], [a, l_4^p][l_4, a^p], [a, l_4^p][l_p, a^p], [l_2, l_4^p][l_p, l_4^p], [l_4, l_4^p][l_p, l_4^p], [l_p, l_4^p][l_4, l_4^p], [l_p, l_4^p][l_4, l_4^p] \rangle \) for \( 5 \leq p < q \leq n \).

By Theorem 1, \( [a, a^p] \) has infinite order, \( [l_4, l_4^p] \) has order 4, \( [a, l_4^p] \) \( [l_4, a^p] \) has order 2, and both \( [a, l_2^p][l_2, a^p] \) and \( [l_2, l_2^p] \) have order 3. By Lemma 6(i), it can be concluded that \( [l_2, l_4^p][l_p, l_4^p] \) has order 3 since \( ([l_2, l_4^p][l_p, l_4^p])^3 = [l_2^3, l_4^p][l_p, l_4^p] = [l_2^3, l_4^p][l_2^3, l_4^p]^{-1} = 1 \). Similarly, \( [l_4, l_4^p][l_p, l_4^p] \) has order 2. Next, suppose that the order of \( (a, l_4^p) \) is finite, then \( (a, l_4^p) \) \( (a, l_4^p) \) is for any integers \( r \) and \( s \). Thus, \( [l_p, a^r] = [l_p, a^r]^{-1}. \) However, this is not true since there is no \( a^r \) \( l_4^p \) in \( B_2(n)^{ab} \). Therefore, \( [a, l_4^p][l_p, a^p] \) has infinite order. Using the similar argument, \( [l_p, l_4^p][l_4, l_4^p] \) and \( [l_p, l_4^p] \) also have infinite order.

Since \( 5 \leq p < q \leq n \), then there are \( n - 4 \) generators in terms of \( [l_p, l_4^p], [a, l_4^p][l_p, a^p], [l_2, l_4^p][l_p, l_4^p] \) and \( [l_4, l_4^p][l_p, l_4^p] \) and \( (n-5)(n-4) \) generators in term of \( [l_p, l_4^p][l_4, l_4^p] \). Hence, \( \nabla(B_2(n)) \cong C_0 \times C_3 \times C_4 \times C_0^{n-4} \times C_3 \times C_2 \times C_0^{n-4} \times C_3^{n-4} \times C_2 \times C_0^{(n-5)(n-4)} = C_0^{n-3} \times C_2 \times C_3^{n-2} \times C_4. \)

4. Conclusion

In this paper, the generalization of the central subgroup of the nonabelian tensor square of a crystallographic group with symmetric point group, \( B_2(n) \) is constructed up to finite dimension \( n \). Besides, the generalized polycyclic presentation and the generalized abelianization of the group are also presented.

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Competing Interests

The authors declare that they have no competing interests.
Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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