On $L^1$-approximation of Trigonometric Series

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1. Introduction

In a recent paper S.P. Zhou [4] defined the notion of Logarithm Rest Bounded Variation Sequences (LRBVS$_N$) which plays central role in his paper. He established, among others, necessary and sufficient condition for $L^1$-convergence of the series

$$\sum_{n=1}^{\infty} a_n \sin nx$$

assuming that $a := \{a_n\} \in$ LRBVS$_N$, but without the prior condition that the sum function of (1.1) is integrable.

The notions and notations to be used in this paper are collected in Section 2.

Next, in a paper to be appearing in Acta Math. Hungar., R.J. Le and S.P. Zhou [1] proved some theorems studying the order of approximation by the partial sums of series (1.1) also maintaining that $a \in$ LRBVS$_N$.

As one of the referees of the paper [1], we analized why the logarithm sequences play the crucial role in $L^1$-convergence of sine series. After collecting the cardinal properties of the sequence $\{\log n\}$, we could show that if a sequence has three essential properties of the sequence $\{\log n\}$, then all of the relevant results of Zhou hold for this sequence, too.

These sequences have been called Log-Type Sequences, in symbol LTS. By means of LTS two further classes of sequences have been defined, the Log-Type

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Rest Bounded Sequences (LTRBVS) and the $\gamma$ Log-Type Rest Bounded Sequences ($\gamma$LTRBVS), which satisfy the following embedding relations:

\begin{equation}
\text{LRBVS}_N \subset \text{LTRBVS} \subset \gamma\text{LTRBVS}.
\end{equation}

The embedding relations (1.2) have offered to extend Zhou’s theorems. In [3] we established four theorems being analogies of Zhou’s theorems.

The aim of the present paper is similar to that of [3], to extend the theorems of Le and Zhou from the class LRBVS$_N$ to the classes LTRBVS or $\gamma$LTRBVS.

2. Notions and Notations

Let $L^2_{2\pi}$ be the space of all real or complex integrable functions $f(x)$ of period $2\pi$ endowed with norm

$$
\|f\| := \int_{-\pi}^{\pi} |f(x)|\,dx.
$$

For those $x$ where the trigonometric series converges, write

\begin{align}
(2.1) \quad f(x) &= \sum_{n=1}^{\infty} a_n \sin nx, \\
(2.2) \quad g(x) &= \sum_{n=1}^{\infty} a_n \cos nx,
\end{align}

and

\begin{equation}
(2.3) \quad h(x) := \sum_{n=-\infty}^{\infty} c_n e^{inx}.
\end{equation}

As usual, let $s_n(f, x)$ and $s_n(g, x)$ be $n$-th partial sums of (2.1) and (2.2), respectively, furthermore denote

\begin{equation}
(2.4) \quad s_n(h, x) := \sum_{k=-n}^{n} c_k e^{ikx}.
\end{equation}

Next we recall some definitions of generalization of decreasing monotonicity related to our topic.

A sequence $\mathbf{a} := \{a_n\}$ of positive numbers will be called Almost Monotone Sequence, briefly $\mathbf{a} \in \text{AMS}$, if $a_n \leq K(\mathbf{a})a_m$ for all $n \geq m$, where $K(\mathbf{a})$ is a positive constant.

Let $\gamma := \{\gamma_n\}$ be a given positive sequence. A null-sequence $\mathbf{a} := \{a_n\}$ ($a_n \to 0$) of real or complex numbers satisfying the inequalities

$$
\sum_{n=m}^{\infty} |\Delta a_n| \leq K(\mathbf{a})\gamma_m \quad (\Delta a_n := a_n - a_{n+1}), \ m = 1, 2, \ldots
$$

is said to be a sequence of $\gamma$ rest bounded variation, in symbol, $\mathbf{a} \in \gamma\text{RBVS}$.

If $\gamma_n \equiv |a_n|$, then $\gamma\text{RBVS}$ reduces to RBVS, that is, to a rest bounded variation sequence.
We emphasize that if \( a \in \gamma \text{RBVS} \) it may have infinitely many zeros and negative terms, but this is not the case if \( a \in \text{RBVS} \), see e.g. [2].

A real or complex bounded sequence \( c := \{c_n\} \) is named \text{Logarithm Rest Bounded Variation Sequence}, \( c \in \text{LRBVS}_N \), if \( N \) is a positive integer and the sequence \( \{c_n \log^{-N} n\} \) belongs to \( \gamma \text{RBVS} \), where \( \gamma_n := |c_n| \log^{-N} n \), see e.g. [1].

We shall also use the notation \( L \ll R \) at inequalities if there exists a positive constant \( K \) such that \( L \leq KR \) holds, not necessarily the same at each occurrence.

A positive nondecreasing sequence \( \alpha := \{\alpha_n\} \) will be called \text{Log-Type Sequence}, briefly LTS, if it satisfies the conditions:

\[
\alpha_n \to \infty, \\
\alpha_n^2 \ll \alpha_n, \\
\text{and} \\
|\Delta \alpha_n| \ll \frac{\alpha_n}{n \log n}.
\]

By means of Log-Type Sequence we defined the following two classes of sequences, in [3] only for positive \( \{a_n\} \).

Let \( \gamma := \{\gamma_n\} \) be a given positive sequence. If \( a := \{a_n\} \in \text{LTS} \) and \( \{\frac{a_n}{\alpha_n}\} \in \gamma \text{RBVS} \), then the sequence \( a := \{a_n\} \) will be called \( \gamma \text{ Log-Type Rest Bounded Variation Sequence} \), in symbol, \( a \in \gamma \text{LTRBVS} \).

If \( \gamma_n = \frac{|a_n|}{a_n} \), then the sequence \( a \) will be said simply \( \text{Log-Type Rest Bounded Variation Sequence} \), and denote by \( \text{LTRBVS} \).

In other words, \( a \in \text{LTRBVS} \), if \( a \in \text{LTS} \) and \( \{\frac{a_n}{\alpha_n}\} \in \text{RBVS} \).

3. Theorems

First we recall the main results of Le and Zhou [1], utilizing the notations of (2.1), \( i = 1, 2, 3 \).

Theorem A. Let a nonnegative sequence \( \{a_n\} \in \text{LRBVS}_N \), \( \{\psi_n\} \) a decreasing sequence tending to zero with

\[
\psi_n \ll \psi_{2n}.
\]

Then

\[
\|f - s_n(f)\| \ll \psi_n
\]

if and only if

\[
a_n \log n \ll \psi_n \quad \text{and} \quad \sum_{k=n}^\infty \frac{a_k}{k} \ll \psi_n.
\]

Theorem B. Let \( \{c_n\} \in \text{LRBVS}_N \) and \( \{\psi_n\} \) a decreasing null-sequence. If

\[
|c_n| \log n \ll \psi_n \quad \text{and} \quad \sum_{k=n}^\infty \frac{|c_k|}{k} \ll \psi_n
\]
and one of the following conditions

\[
\sum_{k=n+1}^{\infty} |\Delta c_k - \Delta c_{-k}| \log k \ll \psi_n
\]

or

\[
\sum_{k=n+1}^{\infty} |\Delta c_k + \Delta c_{-k}| \log k \ll \psi_n
\]
is satisfied, then

\[
\|h - s_n(h)\| \ll \psi_n
\]
holds.

**Corollary.** If a nonnegative sequence \(\{a_n\} \in \text{LRBVS}_N\), and \(\{\psi_n\}\) is a decreasing null-sequence, then (3.3) implies that

\[
\|f - s_n(f)\| + \|g - s_n(g)\| \ll \psi_n
\]
holds.

As a sample result proved in [3] and being used in the proof of our first theorem reads as follows.

**Theorem C.** Let \(a \in \text{LTRBVS}\), then the assertions

\[
\lim_{n \to \infty} \|f - s_n(f)\| = 0
\]
and

\[
\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty
\]
are equivalent.

We remark that if \(a_n = (\log n)^N\), then Theorem C includes Theorem 2 of [4].

We intend to prove the following theorems:

**Theorem 1.** Let a nonnegative sequence \(a \in \text{LTRBVS}\) and \(\{\psi_n\}\) be a decreasing null-sequence with (3.1). Then the assertions (3.2) and (3.3) are equivalent.

It is plain that if \(a_n = (\log n)^N\), then Theorem 1 reduces to Theorem A.

The implication (3.3) \(\Rightarrow\) (3.2) has a further generalization.

**Theorem 2.** Let \(\gamma := \\{\gamma_n\} \in \text{AMS}\) and a nonnegative sequence \(a \in \gamma \text{LTRBVS}\), furthermore \(\{\psi_n\}\) be a decreasing null-sequence. If

\[
\alpha_n \gamma_n \log n \ll \psi_n \quad \text{and} \quad \sum_{k=n}^{\infty} \frac{\alpha_k \gamma_k}{k} \ll \psi_n
\]
then (3.2) holds.
Theorem 3. Both Theorem B and Corollary can be improved such that the condition \( \{c_n\}(\{a_n\}) \in LRBVS_N \) is replaced by the assumption \( \{c_n\}(\{a_n\}) \in \gamma LTRBVS \), where \( \gamma_n := \frac{|c_n|}{a_n} \), respectively.

4. Proofs of the Theorems

Proof of Theorem 1. Principally our proof follows the proof of Theorem A. First we prove the sufficiency of the assumptions of (3.3). By Theorem C, condition (3.10) implies that \( \|f - s_n(f)\| \) tends to zero, consequently we only have to verify that (3.2) also holds.

By Abel's transformation

\[
\begin{align*}
(4.1) \quad f(x) - s_n(f, x) &= \sum_{k=n+1}^{\infty} a_k \sin kx \\
&= \sum_{k=n+1}^{\infty} \frac{a_k}{a_k} \sin kx \\
&= -\frac{a_{n+1}}{\alpha_{n+1}} \sum_{k=1}^{n} a_k \sin kx + \sum_{k=n+1}^{\infty} \Delta \frac{a_k}{a_k} \sum_{\nu=1}^{k} a_\nu \sin \nu x \\
&=: I_1(x) + I_2(x).
\end{align*}
\]

Since

\[
\sum_{k=1}^{n} a_k \sin kx = \sum_{k=1}^{n-1} \Delta a_k \sum_{\nu=1}^{k} \sin \nu x + a_n \sum_{k=1}^{n} \sin kx,
\]

thus

\[
\begin{align*}
\int_{0}^{\pi} \left| \sum_{k=1}^{n} a_k \sin kx \right| dx &\ll \sum_{k=1}^{n-1} |\Delta a_k| \int_{0}^{\pi} \left| \sum_{\nu=1}^{k} \sin \nu x \right| dx \\
&+ a_n \int_{0}^{\pi} \left| \sum_{k=1}^{n} \sin kx \right| dx \\
&\ll \left( \sum_{k=1}^{n-1} |\Delta a_k| \log k + a_n \log n \right) \\
&\ll a_n \log n.
\end{align*}
\]

Hence

\[
(4.2) \quad I_1 := \int_{0}^{\pi} |I_1(x)| dx \ll \frac{a_{n+1}}{a_n} \alpha_n \log n \ll a_{n+1} \log n
\]
and

\[ I_2 := \int_0^\pi |I_2(x)| \, dx \]

\[ \ll \sum_{k=n+1}^\infty \left| \Delta \frac{a_k}{a_k^'} \right| \int_0^\pi \left| \sum_{v=1}^k a_v \sin v x \right| \, dx \]

\[ \ll \sum_{k=n+1}^\infty \left| \Delta \frac{a_k}{a_k^'} \right| a_k \log k. \]

Denote

\[ R_n := \sum_{k=n}^\infty \left| \Delta \frac{a_k}{a_k^'} \right|, \quad n \geq 1. \]

Then

\[ I_2 \ll \sum_{k=n+1}^\infty (R_k - R_{k+1}) a_k \log k \]

\[ \ll \sum_{k=n+1}^\infty R_{k+1} (a_{k+1} \log(k+1) - a_k \log k) - R_{n+1} a_{n+1} \log(n+1). \]

Next, using the conditions \( \{ \frac{a_k}{a_n} \} \in \text{RBVS} \) and (2.7), we get

\[ I_2 \ll \sum_{k=n+1}^\infty \frac{a_{k+1}}{a_k^'} \left( |\Delta a_k| \log(k+1) - \frac{a_k}{v} \right) + a_{n+1} \log(n+1) \]

\[ \ll \sum_{k=n+1}^\infty \frac{a_{k+1}}{a_k^'} \left( \frac{a_k}{k} \right) + a_{n+1} \log(n+1). \]

Collecting the estimations (4.1)–(4.4), and using the assumptions (3.3), the implication (3.3) \( \Rightarrow \) (3.2) is proved.

In order to prove the necessity of (3.3) we define the following function:

\[ \phi_n(x) := \sum_{k=1}^n \left( \frac{\sin(n+k)x}{k} - \frac{\sin(n-k)x}{k} \right) = 2 \cos nx \sum_{k=1}^n \frac{\sin kx}{k}, \]

and utilize the well-known inequality

\[ \left| \sum_{k=1}^n \frac{\sin kx}{k} \right| \ll 1. \]

Since, by (3.2), we have that

\[ \sum_{k=1}^n \frac{a_{n+k}}{k} = \int_0^{2\pi} (f(x) - s_n(f, x)) \phi_n(x) \, dx \ll \|f - s_n(f)\| \ll \psi_n. \]

Furthermore, by \( \{ \frac{a_k}{a_n} \} \in \text{RBVS} \), for all \( 1 \leq k \leq n \)

\[ \frac{a_{2n+1}}{a_{2n+1}^'} \ll \frac{a_{2n}}{a_{2n}^'} \ll \frac{a_{n+k}}{a_{n+k}^'}, \]
and, by (2.5) and (2.6), \( a_{2n+1} \ll a_n \), thus (4.6) implies that
\[
(4.7) \quad a_{2n+1} \ll a_{2n} \ll a_{n+k}, \quad 1 \leq k \leq n.
\]

Now, using (4.5) and (4.7), we get
\[
(4.8) \quad a_{2n+1} \log(2n+1) \ll a_{2n} \log 2n \ll a_n \sum_{k=1}^{n-1} \frac{a_{n+k}}{k} \ll \psi_n,
\]
whence, by (3.1),
\[
(4.9) \quad a_n \log n \ll \psi_n
\]
also holds.

Finally we show that
\[
(4.10) \quad \sum_{k=n}^{\infty} \frac{a_k}{k} \ll \psi_n
\]
also comes from (3.2).

Since
\[
(4.11) \quad 2 \sum_{k=\lfloor (n+1)/2 \rfloor}^{\infty} \frac{a_{2k+1}}{2k+1} = \int_0^n (f(x) - s_N(f, x))dx \ll \|f - s_N(f)\| \ll \psi_n,
\]
thus, by virtue of (3.1), (4.7) and (4.11), we also verified (4.10).

This completes the proof. \( \square \)

**Proof of Theorem 2.** The proof proceeds on the line of Theorem 1 up to the estimation given in (4.3). Next, we utilize the new assumption \( \{ a_n \} \in \gamma \text{RBVS} \) instead of \( \{ a_n \} \in \text{RBVS} \), which implies that \( \frac{a_n}{a_n} \ll R_n \ll \gamma_n \), whence
\[
(4.12) \quad a_n \ll a_n \gamma_n, \quad n = 1, 2, \ldots
\]
follows. Using these estimation at the end of (4.4), we obtain that
\[
(4.13) \quad I_2 \ll \sum_{k=n+1}^{\infty} \frac{a_{k+1} \gamma_{k+1}}{k} + a_{n+1} \gamma_{n+1} \log(n + 1).
\]
Hence, by (3.11),
\[
(4.14) \quad I_2 \ll \psi_n
\]
follows.

If we put the estimations (4.12) into (4.2), too, then, by (3.11), we get that
\[
(4.15) \quad I_1 \ll \psi_n
\]
also holds.

The last two estimations and (4.1) convey the assertion of Theorem 2, thus the proof is complete. \( \square \)
Proof of Theorem 3. The proof is a simple repetition of the proof of Theorem B, putting everywhere $\alpha_n$ in place of $(\log n)^N$, and using Theorem 1 instead of Theorem A.

We omit the details.

□

References


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*Obviously the author has read only the English translation of [4] as referee of [1].