On Mixed Type Duality for Multiobjective Programming Containing Support Functions

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Abstract. A mixed type vector dual to a multiobjective programming problem containing support functions is formulated and various duality results are proved under generalized invexity conditions. Special cases are generated from our results.

1. Introduction

In [5], Husain et al. considered the following multiobjective programming containing support functions

\[
\text{(NP)} \quad \text{Minimize} \left( f^1(x) + S(x|C^1), \ldots, f^p(x) + S(x|C^p) \right) \\
\text{subject to} \quad g^j(x) + S(x|D^j) \leq 0, \quad j = 1, 2, \ldots, m.
\]

Where

(i) \( f^i : \mathbb{R}^n \to \mathbb{R} \) and \( g^j : \mathbb{R}^n \to \mathbb{R}, j = 1, 2, \ldots, m \) are differentiable functions and
(ii) \( S(\cdot|C^i), i = 1, 2, \ldots, p \) and \( S(\cdot|D^j), j = 1, 2, \ldots, m \) are support functions of a compact convex set \( C^i, i = 1, 2, \ldots, p \) and \( D^j, j = 1, 2, \ldots, m \) in \( \mathbb{R}^n \), to be defined later.

The following Wolfe type dual to the problem (NP) is presented [5]:

\[
\text{(WND)} \quad \text{Maximize} \left( f^1(u) + u^T z^1 + \sum_{j=1}^{m} y^j (g^j(u) + u^T w^j), \ldots, \\
\quad f^p(u) + u^T z^p + \sum_{j=1}^{m} y^j (g^j(u) + u^T w^j) \right)
\]

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subject to
\[ \sum_{i=1}^{p} \lambda_i \nabla (f^i(u) + u^T z^i) + \sum_{j=1}^{m} y^j \nabla (g^j(u) + u^T w^j) = 0, \]
\[ z^i \in C^i, \quad i = 1, 2, \ldots, p, \]
\[ w^j \in D^j, \quad j = 1, 2, \ldots, m, \]
\[ y \geq 0, \]
\[ \lambda > 0, \quad \sum_{i=1}^{p} \lambda^i = 1. \]
The problem (WND) is a dual to (NP) assuming that
\[ \sum_{i=1}^{p} \lambda^i (f^i(\cdot) + (\cdot)^T z^i) + \sum_{j=1}^{m} y^j (g^j(\cdot) + (\cdot)^T w^j) \]
is pseudoinvex with respect to \( \eta \). The authors in [5] further weakened the invexity required in Wolfe type by constructing the following Mond-Weir type vector dual.
The Mond-Weir vector type dual is the following to (NP):
(M-WNP) Maximize \((f^1(u) + u^T z^1, \ldots, f^p(u) + u^T z^p)\)
subject to
\[ \sum_{i=1}^{p} \lambda^i \nabla (f^i(u) + u^T z^i) + \sum_{j=1}^{m} y^j (g^j(u) + u^T w^j) = 0, \]
\[ z^i \in C^i, \quad i = 1, 2, \ldots, p, \]
\[ w^j \in D^j, \quad j = 1, 2, \ldots, m, \]
\[ \sum_{j=1}^{m} y^j (g^j(u) + u^T w^j) \geq 0, \]
\[ \lambda > 0, \quad y \geq 0. \]
Husain et al. [5] established usual duality theorems under the hypotheses that
\[ \sum_{i=1}^{p} \lambda^i \nabla (f^i(\cdot) + (\cdot)^T z^i) \]
is pseudoinvex and
\[ \sum_{j=1}^{m} y^j (g^j(\cdot) + (\cdot)^T w^j) \]
is quasi-invex with respect to the same \( \eta \).

In this paper, we propose in the spirit of Husain and Jabeen [4] and Xu [7], a mixed type dual to (NP) to combine the problems (WND) and (M-WNP) and establish various duality theorems under generalized invexity conditions. Special cases are discussed to show that our results extend some earlier results in the literature.

2. Pre-requisites

Before stating our multiobjective nonlinear problem, we mention the following conventions for vectors \( x \) and \( y \) in \( n \)-dimensional Euclidian space \( \mathbb{R}^n \) to be used...
throughout the analysis of this research.

\[ x < y \iff x_i < y_i, \quad i = 1, 2, \ldots, n. \]

\[ x \leq y \iff x_i \leq y_i, \quad i = 1, 2, \ldots, n. \]

\[ x \leq y \iff x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y \]

\( x \not\leq y, \) is the negation of \( x \leq y \)

For \( x, y \in \mathbb{R}, x \leq y \) and \( x < y \) have the usual meaning.

Before presenting our mixed type dual (Mix D), we mention some definitions of invexity and generalized invexity for easy reference.

**Definition 2.1 (Invexity).** The function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be invex with respect to \( \eta \) at \( \bar{x} \) if there exists a vector function \( \eta(x, \bar{x}) \in \mathbb{R}^n \), such that for all \( x \) and \( \bar{x} \in \mathbb{R}^n \)

\[ \phi(x) - \phi(\bar{x}) \geq \eta(x, \bar{x})^T \phi(\bar{x}). \]

**Definition 2.2 (Pseudoinvex).** The function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be pseudoinvex with respect to \( \eta \) at \( \bar{x} \) if there exists a vector function \( \eta(x, \bar{x}) \in \mathbb{R}^n \), such that for all \( x \) and \( \bar{x} \in \mathbb{R}^n \)

\[ \eta(x, \bar{x})^T \phi(\bar{x}) \geq 0 \]

implies

\[ \phi(x) \geq \phi(\bar{x}). \]

**Definition 2.3 (Quasi-invex).** The function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be quasi-invex with respect to \( \eta \) at \( \bar{x} \) if there exists a vector function \( \eta(x, \bar{x}) \in \mathbb{R}^n \), such that for all \( x \) and \( \bar{x} \in \mathbb{R}^n \)

\[ \phi(x) \leq \phi(\bar{x}) \]

implies

\[ \eta(x, \bar{x})^T \phi(\bar{x}) \leq 0. \]

**Definition 2.4 (Support function).** Let \( K \) be a compact set in \( \mathbb{R}^n \), then the support function of \( K \) is defined by

\[ S(x|K) = \max \{x^Tv \in K\}. \]

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis. The subdifferential of \( s(x|K) \) is given by

\[ \partial S(x|K) = \{z \in K | z^Tx = S(x|K)\}. \]

For a set \( K \), the normal cone to \( K \) at a point \( x \in K \) is defined by

\[ N_k(x) = \{y | y^T(z-x) \leq 0, \text{ for all } z \in K\}. \]

When \( K \) is a compact convex set, \( y \) is in \( N_k(x) \) if and only if \( S(y|K) = x^Ty \) i.e., \( x \) is a subdifferential of \( s \) at \( y \).
Definition 2.5 (Efficient solution). A feasible solution \( \bar{x} \) is efficient for (NP) if there exist no other feasible \( x \) for (VPE) such that for some \( i \in P = \{1, 2, \ldots, p\} \),
\[
f^i(x) + S(x|C^i) < f^i(\bar{x}) + S(\bar{x}|C^i)
\]
and
\[
f^j(x) + S(x|C^j) \leq f^j(\bar{x}) + S(\bar{x}|C^j) \quad \text{for all } j \in P, j \neq i.
\]

In order to prove the strong duality theorem we will invoke the following lemma due to Chankong and Haimes [1]. In the subsequent analysis we shall denote the set of feasible solutions of the problem (NP) by \( X \).

Lemma 2.6. A point \( \bar{x} \in X \) is an efficient for (NP), if and only if \( \bar{x} \in X \) solves the following problem:

\[
\text{(P}_k(\bar{x})\text{)} \quad \text{Minimize } f^k(x) + s(S|C^k)
\]
subject to
\[
f^i(x) + S(x|C^i) \leq f^i(\bar{x}) + S(\bar{x}|C^i) \quad \forall i \in P
\]
\[
g^j(x) + S(x|D^j) \leq 0, \quad j = 1, 2, \ldots, m.
\]

3. Mixed type duality

We formulate the following type dual (Mix D) to (NP):

\[
\text{(Mix D)} \quad \text{Maximize } \left(f^1(u) + u^Tz^1 + \sum_{j \in J^1} y^j(g^1(u) + u^Tw^j), \ldots, f^p(u) + u^Tz^p + \sum_{j \in J^p} y^j(g^p(u) + u^Tw^j)\right)
\]
subject to
\[
\begin{align}
\sum_{i=1}^p \lambda^i \nabla f^i(u) + u^Tz^i & + \sum_{j \in J^i} y^j \nabla g^i(u) + u^Tw^j &= 0, \\
\sum_{j \in J^i} y^j g^i(u) + u^Tw^j & \geq 0, \quad \alpha = 1, 2, \ldots, r, \\
z^i & \in C^i, \quad i = 1, 2, \ldots, p, \\
w^j & \in D^j, \quad j = 1, 2, \ldots, m, \\
y & \geq 0, \\
\lambda & \in \Lambda,
\end{align}
\]
where \( \Lambda = \left\{ \lambda \in \mathbb{R}^p \Big| \lambda > 0, \sum_{i=1}^p \lambda^i = 1 \right\} \).

where \( J^\alpha \subseteq M = \{1, 2, \ldots, m\}, \alpha = 0, 1, 2, \ldots, r \) with \( \bigcup_{\alpha=0}^r J^\alpha = M \) and \( J^\alpha \cap J^\beta = \phi, \) if \( \alpha \neq \beta \). If \( J^\alpha = M \), then (Mix D) becomes Wolfe type dual considered in [5], if \( J^\alpha = \phi \) and \( J^\alpha = M \) for some \( \alpha \in \{1, 2, \ldots, r\} \), then (Mix D) becomes Mond-Weir type dual considered in [5].
Theorem 3.1 (Weak duality). Let \( \bar{x} \) be feasible for (NP) and \((u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)\) be feasible for (Mix D). If for all feasible \((x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)\),
\[
\sum_{i=1}^{p} \lambda^i \nabla (f^i(x)) + (\cdot)^T z^i + \sum_{j \in J_a} y^j (g^j(x)) + (\cdot)^T w^j
\]
is pseudoinvex and \[\sum_{j \in J_a} y^j (g^j(x)) + (\cdot)^T w^j, \quad \alpha = 1, 2, \ldots, r\]
is quasi-invex with respect to \(\eta\), then the following cannot hold.
(7) \[f^i(x) + s(x|C^i) \leq f^i(u) + u^T z^i + \sum_{j \in J_a} y^j (g^j(u) + u^T w^j)\]
for all \(i \in \{1, \ldots, p\}\), and
(8) \[f^k(x) + s(x|C^k) < f^k(u) + u^T z^k + \sum_{j \in J_a} y^j (g^j(u) + u^T w^j)\]
for some \(k\).

Proof. Suppose that (7) and (8) hold. Then in view of \(\lambda > 0\) and \(\sum_{i=1}^{p} \lambda^i = 1\),
(7) and (8) together with \(x^T z^i \leq s(x|C^i), \quad i = 1, 2, \ldots, p\) and \(x^T w^j \leq s(x|D^j), \quad j = 1, 2, \ldots, m\) and the feasibility for (NP) and (Mix D) imply
\[
\sum_{i=1}^{p} \lambda^i (f^i(x) + x^T z^i) + \sum_{j \in J_a} y^j (g^j(x) + x^T w^j) < \sum_{i=1}^{p} \lambda^i (f^i(u) + u^T z^i) + \sum_{j \in J_a} y^j (g^j(u) + u^T w^j)
\]
This in view of the pseudoinvexity of
\[
\sum_{i=1}^{p} \lambda^i (f^i(\cdot) + (\cdot)^T z^i) + \sum_{j \in J_a} y^j (g^j(\cdot) + (\cdot)^T w^j)
\]
with respect to \(\eta\), implies,
(9) \[\eta^T (x, u) \left( \sum_{i=1}^{p} \lambda^i \nabla (f^i(u) + u^T z^i) + \sum_{j \in J_a} y^j \nabla (g^j(u) + u^T w^j) \right) < 0\]
Since \(\bar{x}\) is feasible for (VP), \((u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)\) is feasible for (Mix D), and \(x^T w^j \leq s(x|D^j), \quad j = 1, 2, \ldots, m\), we have
\[
\sum_{j \in J_a} y^j (g^j(x) + x^T w^j) \leq \sum_{j \in J_a} y^j (g^j(u) + u^T w^j), \quad \alpha = 1, 2, \ldots, r.
\]
This in view of quasi-invexity of \[\sum_{j \in J_a} y^j (g^j(\cdot) + (\cdot)^T w^j), \quad \alpha = 1, 2, \ldots, r\] with respect to \(\eta\), gives
\[\eta^T (x, u) \left( \sum_{j \in J_a} y^j \nabla (g^j(x) + x^T w^j) \right) \leq 0, \quad \alpha = 1, 2, \ldots, r
\]
Hence
(10) \[\eta^T (x, u) \nabla \left( \sum_{j \in M-J_a} y^j (g^j(x) + x^T w^j) \right) \leq 0, \quad \alpha = 1, 2, \ldots, r\]
Combining (9) and (10), we have
\[
\eta^T(x, u) \left( \sum_{i=1}^{p} \lambda_i^* \nabla(f^i(u) + u^T z^i) + \sum_{j=1}^{m} y^j \nabla(g^j(u) + u^T w^j) \right) < 0
\]
From the equality constraint of (Mix D), we have
\[
\eta^T(x, u) \left( \sum_{i=1}^{p} \lambda_i^* \nabla(f^i(u) + u^T z^i) + \sum_{j=1}^{m} y^j \nabla(g^j(u) + u^T w^j) \right) = 0
\]
The relation (12) contradicts (11). Hence the conclusion of the theorem is true.

**Theorem 3.2 (Strong duality).** Let \( \bar{x} \) be an efficient solution of (NP) and for at least one \( i, i \in \{1, 2, \ldots, p\} \), \( \bar{x} \) satisfies the regularity condition \( [3] \) for the problem \( (P_i(\bar{x})) \). Then there exist \( \lambda \in \mathbb{R}^p \) with \( \lambda^T = (\lambda^1, \ldots, \lambda^i, \ldots, \lambda^p) \), \( \bar{y} \in \mathbb{R}^m \) with \( \bar{y}^T = (\bar{y}^1, \ldots, \bar{y}^i, \ldots, \bar{y}^m) \), \( z^i \in \mathbb{R}^n \), \( i = \{1, 2, \ldots, p\} \) and \( w^j \in \mathbb{R}^n \), \( j = 1, 2, \ldots, m \) such that \( (x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda) \) is feasible for (Mix D) and the objectives of (NP) and (Mix D) are equal.

Further, if the hypotheses of Theorem 1 are satisfied, then \( (x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda) \) is an efficient solution of (Mix D).

**Proof.** Since \( \bar{x} \) is an efficient solution for \( (P_i(\bar{x})) \), this implies that there exists \( \xi \in \mathbb{R}^p \), \( v \in \mathbb{R}^m \) with \( \xi^T = (\xi^1, \ldots, \xi^i, \ldots, \xi^p) \) and \( z^i \in \mathbb{R}^n \), \( i = \{1, 2, \ldots, p\} \) such that
\[
\xi^T \nabla(f^i(x) + \bar{x}^T \bar{z}^i) + \sum_{i=1}^{p} \xi^i \nabla(f^i(x) + \bar{x}^T \bar{z}^i) + \sum_{j=1}^{m} y^j \nabla(g^j(x) + x^T w^j) = 0,
\]
\[
\sum_{j=1}^{m} \bar{v}^j \nabla(g^j(x) + x^T w^j) = 0,
\]
\[
\bar{x}^T \bar{z}^i = S(\bar{x} | C^i), \quad i = 1, 2, \ldots, p,
\]
\[
\bar{x}^T \bar{w}^j = S(\bar{x} | D^j), \quad j = 1, 2, \ldots, m,
\]
\[
z^i \in C^i, \quad i = 1, 2, \ldots, p,
\]
\[
w^j \in D^j, \quad j = 1, 2, \ldots, m,
\]
\[
\xi > 0, \quad v \geq 0
\]
Dividing (13), (14) and (19) by \( \sum_{i=1}^{p} \xi^i \neq 0 \), and putting \( \tilde{\lambda}^i = \frac{\bar{x}^i}{\sum_{i=1}^{p} \xi^i} \) and \( \tilde{y}^i = \frac{\bar{z}^i}{\sum_{i=1}^{p} \xi^i} \), we have
(20) \[ \sum_{i=1}^{p} \tilde{x}^i \partial f^i(\tilde{x}) + \tilde{x}^T \tilde{e}^i + \sum_{j=1}^{m} \tilde{y}^j \partial g^j(x) + \tilde{x}^T \tilde{w}^j = 0 \]

(21) \[ \sum_{j=1}^{m} \tilde{y}^j \partial g^j(x) + \tilde{x}^T \tilde{w}^j = 0 \]

(22) \[ \lambda > 0, \sum_{i=1}^{p} \lambda^i = 1, \bar{y} \succeq 0 \]

The equation (21) implies

(23) \[ \sum_{j \in J_0^p} \tilde{y}^j (g^j(x) + \tilde{x}^T \tilde{w}^j) = 0 \]

and

(24) \[ \sum_{j \in J_0^p} \tilde{y}^j (g^j(x) + \tilde{x}^T \tilde{w}^j) = 0, \alpha = 1, 2, \ldots, r \]

The relation (20), (22) and (24) imply that \((x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)\) is feasible for (Mix D).

\[ f^i(\bar{x}) + \tilde{x}^T \tilde{e}^i + \sum_{j \in J_0^p} \tilde{y}^j (g^j(x) + \tilde{x}^T \tilde{w}^j) = f^i(\bar{x}) + S(\bar{x}|C^i), \quad i = 1, 2, \ldots, p. \]

This implies the objective of the primal and dual problems are equal.

Further, in view of the assumptions Theorem 1, the efficiency of \(\bar{x}\) for (NP) is immediate.

**Theorem 3.3** *(Converse duality).* Let \((\bar{x}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda})\) be an efficient solution for (Mix D). Assume that

(A1) \( f \) and \( g \) are twice continuously differentiable,

(A2) \( \nabla f^i(\bar{x}) + \bar{e}^i + \sum_{j \in J_0^p} \bar{y}^j (\nabla g^j(x) + \bar{w}^j) \) are linearly independent,

(A3) \( \nabla^2 (\lambda^T f^i(\bar{x}) + y^T g(\bar{x})) \) is positive or negative definite.

Further, if the assumptions of Theorem 1 are met, then \(\bar{x}\) is an efficient solution.

**Proof:** Since \((\bar{x}, \bar{y}, \bar{z}^1, \ldots, \bar{z}^p, \bar{w}^1, \ldots, \bar{w}^m, \bar{\lambda})\) be an efficient solution of (Mix D), then there exist \(\tau \in R^p, \beta \in R^r, \gamma \in R \) for each \(\gamma\) constraints, \(\eta \in R^p\) with \(\eta^T = (\eta^1, \ldots, \eta^r, \ldots, \eta^p)\) and \(\mu \in R^m\) such that the following Fritz-John optimality conditions [2] are satisfied,

\[ -\sum_{i=1}^{p} \tau^i \left( \nabla (f^i(\bar{x}) + \bar{x}^T \bar{e}^i) + \sum_{j \in J_0^p} \bar{y}^j \nabla (g^j(x) + \bar{x}^T \bar{w}^j) \right) \]

\[ + \beta^T \nabla^2 (\lambda^T f(\bar{x}) + y^T g(\bar{x})) - \gamma \sum_{a=1}^{r} \sum_{j \in J_0^p} \bar{y}^j \nabla (g^j(x) + \bar{x}^T \bar{w}^j) = 0 \]

(25) \[ -(\tau^T e)(g^j(x) + \bar{x}^T \bar{w}^j + \beta^T \nabla (g^j(x) + \bar{x}^T \bar{w}^j)) - \mu^j = 0, \quad j \in J_0 \]

(26) \[ -(\tau^T e)(g^j(x) + \bar{x}^T \bar{w}^j + \beta^T \nabla (g^j(x) + \bar{x}^T \bar{w}^j)) - \mu^j = 0, \quad j \in J_0 \]
(27) \(-\gamma (g^i(\bar{x}) + \bar{x}^T \bar{w}^j + \beta^T \nabla (g^i(\bar{x}) + \bar{x}^T \bar{w}^j)) - \mu^j = 0, \quad j \in J_a, \quad \alpha = 1, \ldots, r\)

(28) \(\beta^T \nabla (f(\bar{x}) + \bar{x}^T \bar{z}^j) + \sum_{j \in J_s} \bar{y}^j \nabla g^i(\bar{x}) + \bar{w}^j) - \eta^j = 0\)

(29) \((\lambda^i \beta - \tau^i \bar{x}) \in N_{c^i}(\bar{z}^i), \quad i = 1, \ldots, p\)

(30) \((\beta - (\tau^T e) \bar{x}^j) \bar{y}^j \in N_{D^j}(\bar{w}^j), \quad j \in J_s\)

(31) \((\beta - \gamma \bar{x}) \bar{y}^j \in N_{D^j}(\bar{w}^j), \quad j \in J_a, \quad \alpha = i, \ldots, r\)

(32) \(\mu^T \bar{y} = 0\)

(33) \(\eta^T \lambda = 0\)

(34) \(\gamma \sum_{j \in J_s} \bar{y}^j \nabla (g^i(\bar{x}) + \bar{x}^T \bar{w}^j) = 0, \quad \alpha = 1, \ldots, r\)

(35) \((\tau, \mu, \eta, \gamma) \geq 0\)

(36) \((\tau, \beta, \mu, \eta, \gamma) \neq 0\)

Since \(\lambda > 0\), (33) implies \(\eta = 0\). Consequently (28) implies

(37) \(\left(\nabla (f^i(\bar{x}) + \bar{x}^T \bar{z}^i) + \sum_{j \in J_s} \bar{y}^j (\nabla g^i(\bar{x}) + \bar{w}^j)\right) \beta = 0\)

Using the equality constraint of (Mix D) in (25), we have

\[-\sum_{i=1}^p (\tau^i - \gamma \lambda^i) \left(\nabla f^i(\bar{x}) + \bar{z}^i + \sum_{j \in J_s} \bar{y}^j (\nabla g^i(\bar{x}) + \bar{w}^j)\right) + \beta^T \nabla f(\lambda^T f(\bar{x}) + \bar{y} g(\bar{x})) = 0\]

(38)

Postmultiplying (38) by \(\beta\) and then using (37), we have

\(\beta^T \nabla f(\lambda^T f(\bar{x}) + \bar{y} g(\bar{x})) = 0\)

This because of (\(A_3\)), yields

(39) \(\beta = 0\)

Using (39) along with (\(A_2\)), we have

(40) \(\tau^i - \gamma \lambda^i = 0, \quad i = 1, 2, \ldots, p\)

Suppose \(\gamma = 0\), then from (40) we have \(\tau = 0\). Consequently we have from (26) and (27), \(\mu = 0\).

Thus \((\tau, \beta, \mu, \eta, \gamma) = 0\), contradicting (36).

Hence \(\gamma > 0\) and \(\tau > 0\).

In view of (39), (29), (30) and (31) we have,

(41) \(\bar{x}^T \bar{z}^i = S(\bar{x}|C^i), \quad i = 1, 2, \ldots, p\)

(42) \(\bar{x}^T \bar{w}^j = S(\bar{x}|D^j), \quad j = 1, 2, \ldots, m\)
From (26) and (27) along with (42) and (35), we have
\[ g^j(x) + s(x|D^j) \leq 0, \quad j = 1, 2, \ldots, m \]
This implies the feasibility of \( \bar{x} \) for (VP).

From (26) and (32), we have
\[ \sum_{j \in J^\circ} \bar{y}^j(g^j(\bar{x}) + \bar{x}^T \bar{w}^j) = 0 \]
In view of this together with (41), we have
\[ f^i(\bar{x}) + \bar{x}^T \bar{z}^i + \sum_{j \in J^\circ} \bar{y}^j(g^j(\bar{x}) + \bar{x}^T \bar{w}^j) = f^i(\bar{x}) + S(\bar{x}|C^i), \quad i = 1, 2, \ldots, p \]
This establishes the equality of objective values of (NP).

This in view of the hypothesis of Theorem 1 gives the efficiency of \( \bar{x} \) for (NP).

4. Special cases

In this section, we specialize our problem (NP) and its mixed dual problems (Mix D). As discussed in [6] we may write \( S(x|C^i) = (x^T B^i x)^{1/2}, i = 1, \ldots, p \) and \( S(x|D^j) = (x^T E^j x)^{1/2}, j = 1, \ldots, m \) and the matrices \( B^i, i = 1, \ldots, p \) and \( E^j, j = 1, \ldots, m \) are positive semidefinite. Putting these in our problems, we have

\[ \text{(NP)}_1 \quad \text{Minimize} \quad f^1(x) + (x^T B^1 x)^{1/2}, \ldots, f^p(x) + (x^T B^p x)^{1/2} \]
subject to
\[ g^j(x) + (x^T E^j x)^{1/2} \leq 0, \quad j = 1, 2, \ldots, m \]

For the dual (Mix D) problem, we get

\[ \text{(Mix D)}_1 \quad \text{Maximize} \quad \left( f^1(u) + u^T B^1 z^1 + \sum_{j \in J^\circ} y^j(g^j(u) + u^T E^j w^j) \right) \]
\[ \left( f^p(u) + u^T B^p z^p + \sum_{j \in J^\circ} y^j(g^j(u) + u^T E^j w^j) \right) \]
subject to
\[ \sum_{i=1}^p \lambda^i (f^i(u) + u^T B^i z^i) + \sum_{j=1}^m y^j(g^j(u) + u^T D^j w^j) = 0, \]
\[ \sum_{j \in J^\circ} y^j(g^j(u) + u^T D^j w^j) \geq 0, \quad \alpha = 1, 2, \ldots, r, \]
\[ z^i B^i z \leq 1, \quad i = 1, 2, \ldots, p, \]
\[ (w^j)^T E^j w^j \leq 1, \quad j = 1, 2, \ldots, m, \]
\[ \lambda > 0, \quad y \geq 0. \]
References


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