Weak Integer Additive Set-Indexers of Certain Graph Products

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Abstract. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a graph is an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where $f(u) + f(v)$ is the sum set of $f(u)$ and $f(v)$. An IASI $f$ is said to be a weak IASI if $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ $\forall$ $uv \in E(G)$. In this paper, we study the admissibility of weak IASI by different products of two weak IASI graphs.

Keywords. Integer additive set-indexers; Mono-indexed elements of a graph; Weak integer additive set-indexers; Sparing number of a graph

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [12] and [11]. Unless mentioned otherwise, all graphs we consider here are simple, finite and have no isolated vertices.

Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. The sum set of two sets $A$ and $B$ is denoted by $A + B$ and is defined by $A + B = \{a + b : a \in A, b \in B\}$. If either $A$ or $B$ is countably infinite, then their sum set is also countably infinite. Hence, only finite sets are taken for this study. We denote the cardinality of a set $A$ by $|A|$. 
We call a set $B$ an integral multiple of another set $A$ if every element of $B$ is an integral multiple of the corresponding element of $A$. That is, for an integer $k > 1$, $B = kA \Rightarrow B = \{ka : a \in A\}$.

An integer additive set-indexer (IASI) of a graph $G$ is defined in ([7]) as an injective function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A graph that admits an IASI is called an integer additive set-indexed graph (IASI-graph).

Since the set-label of every edge of an IASI-graph $G$ is the sum set of the set-labels of its end vertices, no vertex of $G$ can have the empty set as its set-label. Hence, all sets we consider here are non-empty finite sets of non-negative integers.

The cardinality of an element of a graph $G$ is said to be the set-indexing number of that element. An IASI is said to be a $k$-uniform IASI if $f^+(e) = k$ for all edges $e$ of $G$. The following result provides the bounds for the set-indexing number of an edge in terms of the set-indexing numbers its end vertices.

Lemma 1.1 ([8]). If $f$ is an IASI on a graph $G$, then $\max(|f(u)|, |f(v)|) \leq f^+(uv) \leq |f(u)||f(v)|$, $\forall uv \in E(G)$.

An IASI $f$ of a given graph $G$ is called, in ([8]), a weak IASI of $G$ if $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A weak IASI $f$ is said to be weakly uniform IASI if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer $k$. A graph which admits a weak IASI may be called a weak IASI graph. The following lemma provides a necessary and sufficient condition for a given graph to admit a weak IASI.

Lemma 1.2 ([8]). An IASI $f$ of a given graph $G$ is a weak IASI of $G$ if and only if at least one end vertex of every edge of $G$ has a singleton set-label with respect to $f$.

An element (a vertex or an edge) of graph which has the set-indexing number 1 is called a mono-indexed element of that graph. The sparing number of a graph $G$ is defined as the minimum number of mono-indexed edges required for $G$ to admit a weak IASI. The sparing number of a graph $G$ is denoted by $\varphi(G)$.

The hereditary property of the existence of a weak IASI have been established in the following theorem.

Theorem 1.3 ([15]). Every subgraph of a weak IASI graph is also a weak IASI graph.

The following result provides a necessary and sufficient condition for a given graph $G$ to admit an IASI.

Theorem 1.4 ([15]). A graph $G$ admits a weak IASI if and only if $G$ is bipartite or it has at least one mono-indexed edge.

In view of the above theorem, it can be noted that the sparing number of a bipartite graph is 0.

The sparing number of certain standard graphs have been estimated as follows.

Theorem 1.5 ([15]). An odd cycle $C_n$ has a weak IASI if and only if it has at least one mono-indexed edge. That is, the sparing number of an odd cycle $C_n$ is 1.
Theorem 1.6 ([15]). Let $C_n$ be a cycle of length $n$ which admits a weak IASI, for a positive integer $n$. Then, $C_n$ has an odd number of mono-indexed edges when it is an odd cycle and has even number of mono-indexed edges, when it is an even cycle.

The following result explained the admissibility of a weak IASI by the union of two weak IASI graphs and hence estimated its sparing number.

Theorem 1.7 ([16]). The graph $G_1 \cup G_2$ admits a weak IASI if and only if both $G_1$ and $G_2$ are weak IASI graphs. Moreover, $\varphi(G_1 \cup G_2) = \varphi(G_1) + \varphi(G_2) - \varphi(G_1 \cap G_2)$.

The sparing number of a complete graph has been given in the following result.

Theorem 1.8 ([15]). The complete graph $K_n$ admits a weak IASI and it has at least $\frac{1}{2}(n-1)(n-2)$ mono-indexed edges. That is, $\varphi(K_n) = \frac{1}{2}(n-1)(n-2)$.

The theorem follows from the fact that at most one vertex of a complete graph can have a non-singleton set-label.

In this paper, we discuss the admissibility of weak IASI by certain products of given weak IASI-graphs.

2. Fundamental Products of Weak IASI Graphs

In different products of given graphs, we need to take several layers or copies of some or all given graphs and have to establish the adjacency between these copies, according to certain rules. Hence, it may not be possible to use the weak IASIs of given graphs to construct an induced weak IASI for a product of the given graphs. We have to define an IASI independently for a graph product.

We say that two copies of a graph are adjacent to each other in a graph product if there exist some edges between the vertices of those copies in the graph product.

In the first section, we discuss the admissibility of a weak IASI by the three fundamental products such as Cartesian product, direct product and strong product of two weak IASI graphs.

First, let us recall the definition of the Cartesian product of two graphs.

Definition 2.1 ([11]). Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be the two given graphs. The Cartesian product of $G_1$ and $G_2$, denoted by $G_1 \square G_2$, is the graph with vertex set $V_1 \times V_2$, such that two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V_1 \times V_2$ are adjacent in $G_1 \square G_2$ whenever $|u_1 = v_1$ and $u_2$ is adjacent to $v_2|$ or $|u_2 = v_2$ and $u_1$ is adjacent to $v_1|$. If $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$, then $|V(G_1 \square G_2)| = n_1n_2$ and $i = 1, 2$ and $|E(G_1 \square G_2)| = n_1m_2 + n_2m_1$.

The Cartesian product $G_1 \square G_2$ of two graphs $G_1$ and $G_2$ can be viewed as follows. Make $n_2$ copies of $G_1$. Denote these copies by $G_{11}$, which corresponds to the vertex $v_i$ of $G_2$. Now, join the corresponding vertices of two copies $G_{1i}$ and $G_{1j}$ if the corresponding vertices $v_i$ and $v_j$ are adjacent in $G_2$.

The Cartesian product of two bipartite graphs is also a bipartite graph. Also, the Cartesian products $G_1 \square G_2$ and $G_2 \square G_1$ of two graphs $G_1$ and $G_2$, are isomorphic graphs.

The existence of a weak IASI by the Cartesian product of two weak IASI-graphs is verified in the following theorem.
Theorem 2.2. Let $G_1$ and $G_2$ be two weak IASI graphs. Then, the Cartesian product $G_1 \Box G_2$ also admits a weak IASI.

Proof. Let $G_1$ and $G_2$ be two weak IASI graphs on $n_1$ and $n_2$ vertices respectively. We can view the product $G_1 \Box G_2$ as a union of $n_2$ copies of $G_1$ and a finite number of edges connecting the corresponding vertices of two copies $G_{1i}$ and $G_{1j}$ of $G_1$ according to the adjacency of the corresponding vertices $v_i$ and $v_j$ in $G_2$, where $1 \leq i \neq j \leq n_2$. Let us define a set-labeling function $f : V(G_1 \Box G_2) \rightarrow \mathcal{P}(\mathbb{N}_0)$, with respect to which, the vertices of $G_1 \Box G_2$ are assigned to distinct set-labels as explained below.

Let $u_{ij}$ be the $i$-th vertex of $G_{1j}$, the $j$-the copy of $G_1$. For odd values of $j$, label the vertices of $G_{1j}$ in such a way that the corresponding vertices of $G_{1j}$ have the same type of set-labels as that of $G_1$. That is, for odd $j$, let $u_{ij}$ have a singleton set-label or non-singleton set-label according to whether the corresponding vertex $u_i$ of $G_1$ has a singleton set-label or non-singleton set-label.

If $u_i$ be not an end vertex of a mono-indexed edge in $G_1$, then for even values of $j$, label the corresponding vertex $u_{ij}$ in such a way that $u_{ij}$ has a singleton set-label (or non-singleton set-label) according as the vertex $u_i$ of $G_1$ has non-singleton set-label (or singleton set-label). Also, label the vertices of $G_{1j}$ which are corresponding to the adjacent mono-indexed vertices of $G_1$, by singleton sets.

If $u_i u_j$ is a mono-indexed edge of $G_2$, then label the vertices of $G_{1i}$ and $G_{1j}$ such that no two corresponding vertices of $G_{1i}$ and $G_{1j}$ simultaneously have singleton set-labels or non-singleton set-labels.

We can see that no two adjacent vertices in $G_1 \Box G_2$ have non-singleton set-labels with respect to this labeling $f$. Therefore, $f$ is a weak IASI for the graph $G_1 \Box G_2$. \hfill \Box

If a graph $G$ is the Cartesian product of two graphs $G_1$ and $G_2$, then $G_1$ and $G_2$ are called the factors of $G$. A graph is said to be prime with respect to a given graph product if it is non-trivial and can not be represented as the product of two non trivial graphs.

The following result establishes the existence of an induced weak IASI for every factor of a non-prime graph that admits a weak IASI.

Theorem 2.3. Let $G$ is a non-prime graph which admits a weak IASI $f$. Then, every factor of $G$ also admits a weak IASI that is induced by $f$.

Proof. Let $G$ be a non-prime weak IASI graph with a weak IASI $f$ defined on it. If $G_1$ is a factor of $G$, then there is a subgraph $G_{1i}$ in $G$ which is isomorphic to $G_1$ such that $v_i$ is the vertex of $G_1$ corresponding to a vertex $v$ of $G_1$. Define a function $g : V(G_1) \rightarrow \mathcal{P}(\mathbb{N}_0)$ defined by $g(v) = f'(v_i)$ where $f' = f|_{G_{1i}}$, the restriction of $f$ to the subgraph $G_{1i}$. By Theorem 1.3, $f'$ is a weak IASI of $G_{1i}$ and hence $g$ is a weak IASI of $G_1$. \hfill \Box

Next, recall the definition of another graph product called the direct product of two graphs.

Definition 2.4. The direct product of two graphs $G_1$ and $G_2$, is the graph whose vertex set is $V(G_1) \times V(G_2)$ and for which the vertices $(u, v)$ and $(u', v')$ are adjacent if $uu' \in E(G_1)$ and $vv' \in E(G_2)$. The direct product of $G_1$ and $G_2$ is denoted by $G_1 \times G_2$.

Note that the direct product of two connected graphs can be a disconnected graph also. The direct product is also known as tensor product or cardinal product or cross product or categorical product or Kronecker product. The following theorem establishes the admissibility of weak IASI by the direct product of two weak IASI graphs.
Theorem 2.5. The direct product of two weak IASI graphs is also a weak IASI-graph.

Proof. Let \( G_1 \) and \( G_2 \) be two weak IASI graphs on \( n_1 \) and \( n_2 \) vertices, \( m_1 \) and \( m_2 \) edges respectively. Let \( V(G_1) = \{u_1, u_2, u_3, \ldots, u_{n_1}\} \) and \( V(G_2) = \{v_1, v_2, v_3, \ldots, v_{n_2}\} \). Make \( n_2 \) copies of \( V(G_1) \) and denote the \( j \)-th copy of \( V \) by \( V_j \), where \( 1 \leq j \leq n_2 \). Let \( V_j = \{u_{ij}: 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \), where \( u_{ij} \) represents a copy of \( u_i \) in \( G_1 \).

Since the vertex \( u_{ij} \) is adjacent to a vertex \( u_{rs} \) if \( u_i \) and \( u_r \) are adjacent in \( G_1 \) and \( u_j \) and \( u_s \) are adjacent in \( G_2 \), no two vertices in the copy \( V_j \) can be adjacent to each other in \( G_1 \times G_2 \). Hence, define a function \( f_j \) on the vertex set \( V_j \) such that it assigns the set-labels to the vertices of \( V_j \), which are integral multiples of the set-labels of the corresponding vertices of \( G_1 \). Clearly, no two adjacent edges in \( G_1 \times G_2 \) have non-singleton set-labels. Therefore, this labeling is a weak IASI on \( G_1 \times G_2 \).

Next, let us consider the following definition of the strong product of two graphs.

Definition 2.6 (\cite{11}). The strong product of two graphs \( G_1 \) and \( G_2 \) is the graph, denoted by \( G_1 \boxtimes G_2 \), whose vertex set is \( V(G_1) \times V(G_2) \), the vertices \((u,v)\) and \((u',v')\) are adjacent in \( G_1 \boxtimes G_2 \) if \( uu' \in E(G_1) \) and \( v = v' \) or \( u = u' \) and \( vv' \in E(G_2) \) or \( uu' \in E(G_1) \) and \( vv' \in E(G_2) \).

From this definition, we understand that \( E(G_1 \boxtimes G_2) = E(G_1 \square G_2) \cup E(G_1 \times G_2) \). Now, we prove the existence of weak IASI for the strong product of two weak IASI graphs in the following theorem.

Theorem 2.7. The strong product of two weak IASI graphs also admits a weak IASI.

Proof. Let \( G_1 \) and \( G_2 \) be two weak IASI graphs on \( n_1 \) and \( n_2 \) vertices with the corresponding weak IASIs \( f_1 \) and \( f_2 \) respectively. Let \( G = G_1 \boxtimes G_2 \). Then, \( G \) can be viewed as follows.

Take \( n_2 \) copies of \( G_1 \), denoted by \( G_{1i} \), for \( 1 \leq i \leq n_2 \). Let \( u_{ij} \) be the \( i \)-th vertex of the \( j \)-th copy of \( G_1 \), where \( 1 \leq i \leq n_1, 1 \leq j \leq n_2 \). If a copy \( G_{1j} \) is adjacent to another copy \( G_{1k} \) in \( G \), then the vertex \( u_{ij} \) will be adjacent to the vertices \( u_{ik}, u_{i+1,j}, u_{i-1,j} \), if they exist.

Let \( f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0) \) be an IASI defined on \( G \), which labels the vertices of \( G \) in the following way. Label the corresponding vertices of the first copy \( G_{11} \) of \( G_1 \) by the same set-labels of the vertices of \( G_1 \). Now, by Definition 2.6, a vertex of the copies of \( G_1 \) that are adjacent to \( G_{11} \) can have a non-singleton set label if and only if the corresponding vertex and its adjacent vertices in \( G_1 \) are mono-indexed. Let \( G_{1r} \) be the next copy of \( G_1 \) which is not adjacent to \( G_{11} \). Label the vertices of this copy by an integral multiple of the set-labels of the corresponding vertices of \( G_1 \) and label the vertices of adjacent copies of \( G_{1r} \) such that no vertex of \( G_{1r} \) has a non-singleton set-label unless the corresponding vertex and its adjacent vertices in \( G_{1r} \) are mono-indexed. Proceed in this way until all the vertices in \( G \) are set-labeled. Then, we have a set-labeling in which no two adjacent vertices of \( G \) have non-singleton set-labels. Hence, \( f \) is a weak IASI on \( G = G_1 \boxtimes G_2 \). This completes the proof.

3. Other Products of Weak IASI Graphs

In the previous section, we have discussed the admissibility of weak IASI by three fundamental products of weak IASI graphs. Now, we proceed to discuss the existence of weak IASI for certain other graph products.

Now, recall the definition of lexicographic product of two graphs.
Definition 3.1 ([13]). The lexicographic product or composition of two graphs \( G_1 \) and \( G_2 \) is the graph, denoted by \( G_1[G_2] \), is the graph whose vertex set \( V(G_1) \times V(G_2) \) and for two vertices \((u, v)\) and \((u', v')\) are adjacent if \([u]u' \in E(G_1)\) or \([u = u' \text{ and } vv' \in E(G_2)\]).

Admissibility of weak IASI by the lexicographic product of two weak IASI graphs is established in the following theorem.

**Theorem 3.2.** The lexicographic product of two weak IASI graphs admits a weak IASI.

**Proof.** Let \( G_1 \) and \( G_2 \) be two weak IASI graphs on \( n_1 \) and \( n_2 \) vertices respectively. The composition of \( G_1 \) and \( G_2 \) can be viewed as follows. Take \( n_1 \) copies of \( G_2 \), denoted by \( G_{2i}; 1 \leq i \leq n_1 \). Every vertex of a copy \( G_{2i} \) is adjacent to all vertices of another copy \( G_{2j} \) in \( G_1 \circ G_2 \) if the corresponding vertices \( v_i \) and \( v_j \) are adjacent in \( G_1 \).

Label the corresponding vertices of the first copy \( G_{21} \) of \( G_2 \) by the same set-labels of the vertices of \( G_2 \). Since every vertex of \( G_{21} \) is adjacent to all vertices of its adjacent copies, these vertices must be labeled by distinct singleton sets. Now, label the vertices of the next copy \( G_{2r} \) of \( G_2 \) which is not adjacent to \( G_{21} \) by an integral multiple of the set-labels of the corresponding vertices of \( G_2 \) and label the vertices of the adjacent copies of \( G_{2r} \) by singleton sets. Proceed in this way until all the vertices in \( G \) are set-labeled. This set-labeling is a weak IASI for \( G_1[G_2] \).

The next graph product we are going to discuss now is the corona of two weak IASI graphs.

**Definition 3.3 ([12]).** The corona of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \circ G_2 \), is the graph obtained by taking one copy of \( G_1 \) (which has \( p_1 \) vertices) and \( p_1 \) copies of \( G_2 \) and then joining the \( i \)-th point of \( G_1 \) to every point in the \( i \)-th copy of \( G_2 \). The number of vertices and edges in \( G_1 \circ G_2 \) are \( p_1(1 + p_2) \) and \( q_1 + p_1q_2 + p_1p_2 \) respectively, where \( p_i \) and \( q_i \) are the number of vertices and edges of the graph \( G_i, i = 1, 2 \).

The following theorem establishes a necessary and sufficient condition for the corona of two weak IASI graphs to admit a weak IASI.

**Theorem 3.4.** Let \( G_1 \) and \( G_2 \) be two weak IASI graphs on \( m \) and \( n \) vertices respectively. Then,

(i) \( G_1 \circ G_2 \) admits a weak IASI if and only if either \( G_1 \) is 1-uniform or it has \( r \) copies of \( G_2 \) that are 1-uniform, where \( r \) is the number of vertices in \( G_1 \) that are not mono-indexed.

(ii) \( G_2 \circ G_1 \) admits a weak IASI if and only if either \( G_2 \) is 1-uniform or it has \( l \) copies of \( G_2 \) that are 1-uniform, where \( l \) is the number of vertices in \( G_2 \) that are not mono-indexed.

**Proof.** Consider the corona \( G_1 \circ G_2 \). Let \( f \) be a weak IASI of \( G_1 \) and \( f_i \) be a weak IASI on the \( i \)-th copy \( G_{2i} \) of \( G_2 \) all whose vertices are connected to the \( i \)-th vertex of \( G_1 \). Define the function \( g \) on \( G_1 \circ G_2 \) by

\[
    g(v) = \begin{cases} 
    f(v) & \text{if } v \in G_1 \\
    f_i(v) & \text{if } v \in G_{2i}, 1 \leq i \leq m 
    \end{cases}
\]

Assume that \( G_1 \circ G_2 \) is a weak IASI graph. If \( G_1 \) is 1-uniform, then the proof is complete. If \( G_1 \) is not 1-uniform, then the vertex set \( V \) of \( G_1 \) can be divided into two disjoint sets \( V_1 \) and \( V_2 \), where \( V_1 \) is the set of all mono-indexed vertices in \( G_1 \) and \( V_2 \) is the set of all vertices that are not mono-indexed in \( G_1 \). Then, we observe that any copy \( G_{2i} \) of \( G_2 \) that are connected to
Theorem 3.5. Let $G_1$ be a weak IASI graph on $n_1$ vertices, $m_1$ edges and $r_1$ mono-indexed vertices and $G_2$ be a weak IASI graph on $n_2$ vertices, $m_2$ edges and $r_2$ mono-indexed vertices. Then, the sparing number of $G_1 \odot G_2$ is $r_1(1 + r_2) + (n_1 - r_1)m_2$ and the sparing number of $G_2 \odot G_1$ is $r_2(1 + r_1) + (n_2 - r_2)m_1$.

Proof. Since $G_1$ has $r_1$ mono-indexed vertices, $(n_1 - r_1)$ copies of $G_2$ must be 1-uniform in $G_1 \odot G_2$. In the remaining $r_1$ copies, label the vertices by the set-labels which are some integral multiples of the set-labels of the corresponding vertices of $G_2$ (in such a way that no two copies of $G_2$ have the same set-label). Hence, each of these copies contains the same number of mono-indexed edges as that of $G_2$. Therefore, the total number of mono-indexed edges in $G_1 \odot G_2$ is $r_1 + (n_1 - r_1)m_2 + r_1r_2 = r_1(1 + r_2) + (n_1 - r_1)m_2$.

Similarly, we can prove the other part also.

Theorem 3.6. (\cite{[9]}). The rooted product of a graph $G_1$ on $n_1$ vertices and rooted graph $G_2$ on $n_2$ vertices, denoted by $G_1 \odot G_2$, is defined as the graph obtained by taking $n_1$ copies of $G_2$, and for every vertex $v_i$ of $G_1$, identifying $v_i$ with the root node of the $i$-th copy of $G_2$.

The following theorem verifies the admissibility of weak IASI by the rooted product of two graphs.

Theorem 3.7. The rooted product of two weak IASI graphs is also a weak IASI graph.

Proof. Let $G_1$ and $G_2$ be the given graphs with the weak IASIs $f_1$ and $f_2$ defined on them respectively. Also let $V(G_1) = \{u_1, u_2, u_3, \ldots, u_{n_1}\}$ be the vertex set of $G_1$ and let $V(G_2) = \{v_1, v_2, v_3, \ldots, v_{n_2}\}$ be the vertex set of $G_2$. Let $G = G_1 \odot G_2$. Without loss of generality, let $v_1$ be the root vertex of $G_2$. Now, make $n_2$ copies of $G_2$, denoted by $G_{2r}$, $1 \leq r \leq n_1$, with $V(G_{2r}) = \{v_{1r}, v_{2r}, v_{3r}, \ldots, v_{n_{2r}}\}$.

Define a function $f : V(G) \rightarrow \mathcal{P}(\mathbb{N}_0)$ with the following conditions.

(1) For $1 \leq i \leq n_1$, define a function $f_{2r} : V(G_{2r}) \rightarrow \mathcal{P}(\mathbb{N}_0)$ such that $f_{2r}(v_{ir}) = r f_2(v_i)$, where $r f_2(v_i)$ is the set obtained by multiplying the elements of the set-label $f_2(v_i)$ by the integer $r$. 

We now proceed to determine the sparing number of the corona of two graphs.
(2) The vertex \( u'_r \) obtained by identifying the vertex \( u_r \) of \( G_1 \) and the root vertex \( v_{1r} \) of the \( r \)-th copy \( G_{2r} \) of \( G_2 \) has the same set-label of \( u_r \), unless \( u_r \) has a non-singleton set label and \( v_{1r} \) is mono-indexed. In this case, let \( u'_r \) assumes the same set-label of \( v_{1r} \).

Then, under \( f \), no two adjacent vertices of \( G \) have non-singleton set-labels. That is, \( f \) is a weak IASI on \( G = G_1 \circ G_2 \). This completes the proof.

4. Conclusion

In this paper, we have discussed the admissibility of weak IASI by the certain products of two graphs which admit weak IASIs. Some problems in this area are still open. In our present discussion we have not studied about the sparing number of the graph products, other than corona, of two arbitrary graphs \( G_1 \) and \( G_2 \). Uncertainty in the adjacency pattern of different graphs makes the study about the sparing number of the products of two arbitrary graphs a little complex. An investigation to determine the sparing number of different products of two arbitrary graphs in terms of their orders, sizes and the vertex degrees in each of them, seem to be promising. The admissibility of weak IASIs by certain other graph products is also worth studying.

Competing Interests

The authors declare that they have no competing interests.

Authors’ Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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