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Asymptotic Behavior of Solutions of Generalized Nonlinear α-difference Equation of Second Order

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Abstract In this paper, the authors discuss the asymptotic behavior of solutions of the generalized nonlinear α-difference equation

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) + f(k)F(u(k)) = g(k), \tag{0.1}$$

$k \in [a,\infty)$, where the functions p, f, F and g are defined in their domain of definition and $\alpha > 1$, ℓ is positive real. Further, $uF(u) > 0$ for $u \neq 0$, $p(k) > 0$ for all $k \in [a,\infty)$ for some $a \in [0,\infty)$ and for all $0 \le j < \ell$, $R_{a+j,k} \to \infty$, where $R_{t+j,k} = \sum_{r=0}^{\frac{k-\ell-t-j}{\ell}} \frac{1}{p(t+j+r\ell)}$, $t \in [a,\infty)$ and $k \in \mathbb{N}_\ell(t+j+\ell)$.

1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0,1,2,3,\dots\}$. Eventhough many authors ([1], [20]-[22]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \quad k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{1.1}$$

no significant progress has taken place on this line. But recently, E. Thandapani, M.M.S. Manuel and G.B.A. Xavier [7] considered the definition of Δ as given in (1.1) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined by (1.1) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [7], [15]-[19]) were obtained. By extending the study related to the sequences of

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complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike were obtained for difference equation involving Δ_ℓ. The results obtained using Δ_ℓ can be found in ([7]-[19]). Jerzy Popenda and B. Szmanda ([4], [5]) defined Δ as

$$\Delta_\alpha u(k) = u(k+1) - \alpha u(k) \tag{1.2}$$

and based on this definition they studied the qualitative properties of a particular difference equation and no one else has handled this operator. In this paper, we have generalized the definition of Δ_α given in (1.2) and defined and denoted it as

$$\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k) \tag{1.3}$$

where $\alpha > 1$ and $\ell \in [0,\infty)$ and by defining its inverse, several interesting results on number theory were obtained.

In [6], John R. Graef worked on Asymptotic behaviour of solutions of a second order nonlinear differential equation and Blazej Szmanda [3] obtained the discrete analogous of [6]. The case of any real ℓ and $\alpha = 1$, in (1) were analysed in detail by M.M.S. Manuel and D.S. Dilip *et al.* [17]. In this paper the theory is extended from Δ_ℓ to $\Delta_{\alpha(\ell)}$ for all real $k \in [a,\infty)$ and we discuss asymptotic behavior of solutions of generalized nonlinear α-difference equation (0.1) is discussed.

Throughout this paper, we make use the following notations.

(a) $\mathbb{N} = \{0,1,2,3,\dots\}$, $\mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,
(b) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$.
(c) $\lceil x \rceil$ upper integer part of x.

2. Preliminaries

In this section, we present some preliminaries which will be useful for future discussion.

Definition 2.1 ([7]). Let $u(k)$, $k \in [0,\infty)$ be real or complex valued function and $\ell \in (0,\infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) = u(k), \text{ then } \ v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{2.1}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$. In general $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

Definition 2.2. The inverse of the Generalized α-difference operator denoted by $\Delta_{\alpha(\ell)}^{-1}$ on $u(k)$ is defined as follows. If $\Delta_{\alpha(\ell)} v(k) = u(k)$, then

$$\Delta_{\alpha(\ell)}^{-1} u(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j. \tag{2.2}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.

Lemma 2.3 ([7]). *If the real valued function $u(k)$ is defined for all $k \in [a,\infty)$, then*

$$\Delta_\ell^{-1}u(k) = \sum_{r=1}^{[\frac{k-a}{\ell}]} u(k-r\ell) + c_j, \tag{2.3}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - [\frac{k-a}{\ell}]\ell$.

Corollary 2.4. *If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2,\infty)$ and $j = k - k_2 - [\frac{k-k_2}{\ell}]\ell$, then*

$$v(k) - v(k_2+j) = \sum_{r=0}^{[\frac{k-k_2-j-\ell}{\ell}]} u(k_2+j+r\ell).$$

Proof. The proof follows by Definition 2.1, Lemma 2.4 and $c_j = v(k_2+j)$. □

Definition 2.5. The solution $u(k)$ of (0.1) is called oscillatory if for any $k_1 \in [a,\infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2+\ell) \le 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory, then it is said to be nonoscillatory (i.e., $u(k)u(k+\ell) > 0$ for all $k \in [k_1,\infty)$).

3. Main Results

In this section we present conditions for the oscillation and nonoscillation of equation (0.1).

Lemma 3.1. *The relation between Δ_ℓ and $\Delta_{\alpha(\ell)}$ is given by*

$$\alpha^{\lceil\frac{k+\ell}{\ell}\rceil}\Delta_\ell\left(\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}}\right) = \Delta_{\alpha(\ell)}u(k).$$

Theorem 3.2. *Consider the generalized difference equation*

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) + f(k)F(u(k)) = 0 \tag{3.1}$$

and assume that in addition to the given hypotheses on the functions p, f and F, $|F(u)|$ is bounded away from zero if $|u|$ is bounded away from zero, $f(k) \ge 0$ for all $k \in [a,\infty)$ and $\sum_{r=0}^{\infty} \alpha^{-\lceil\frac{k_2+j+r\ell}{\ell}\rceil} f(k_1+j+r\ell) = \infty$, then (3.1) is oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (0.1) and suppose that $u(k) > 0$ eventually. From the given hypothesis, there exists a positive constant c such that $F(u(k)) \ge c$ for all $k \in [k_2,\infty)$.

On the other hand, from (0.1), we have

$$\Delta_\ell\left(\frac{p(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}}\Delta_{\alpha(\ell)}u(k)\right) + \alpha^{-\lceil\frac{k+\ell}{\ell}\rceil}cf(k) \le 0, \quad k \in [k_1,\infty). \tag{3.2}$$

By Definition 2.1 and Theorem 2.4 we obtain

$$p(k)\alpha\Delta_\ell u(k) \le -\frac{c}{\alpha}\sum_{r=0}^{\frac{k-j-k_2-\ell}{\ell}} \alpha^{-\lceil\frac{k_2+j+r\ell}{\ell}\rceil} f(k_2+j+r\ell) \to -\infty \text{ as } k \to \infty.$$

We then have $\Delta_\ell \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} = -1/\alpha p(k)$. Again by Definition 2.1 and Theorem 2.4, we have

$$\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} \leq - \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{1}{\alpha p(a+j+r\ell)},$$

where $k \in [k_2, \infty)$, where $j = k - k_2 - [\frac{k-k_2}{\ell}]\ell$, which tends to $-\infty$ as $k \to \infty$. This leads to a contradiction to our assumption that $u(k) > 0$ eventually. The case $u(k) < 0$ eventually can be treated similarly. □

Example 3.3. *For the generalized* α*-difference equation*

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) - 2\alpha^2 u(k)(2k+\ell) = 0,$$

and for $p(k) = k$, $f = (2k+\ell)\alpha^{\lceil \frac{k}{\ell} \rceil}$, $F(u(k)) = \frac{-2\alpha^2 k u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}$, *the conditions of Theorem* 3.2 *hold and hence all the solutions of the generalized* α*-difference equation is oscillatory. Infact* $u(k) = (-\alpha)^{\lceil \frac{k}{\ell} \rceil}$ *is one such solution.*

Theorem 3.4. *Suppose that the following conditions hold.*

(i) $f(k) \geq b > 0$ *for all* $k \in [a, \infty)$,

(ii) $|F(u)|$ *is bounded away from zero if* $|u|$ *is bounded away from zero, and*

(iii) *the function* $G(k) = \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \alpha^{-\lceil \frac{a+j+r\ell}{\ell} \rceil} g(a+j+r\ell)$ *is bounded on* $[a, \infty)$.

Then, for every nonoscillatory solution $u(k)$ *of* (0.1), $\lim_{k\to\infty} u(k) = 0$.

Proof. In system form, equation (0.1) is equivalent to

$$\Delta_\ell \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} = \alpha^{-\lceil \frac{k+\ell}{\ell} \rceil}(v(k) + G(k))/p(k), \tag{3.3}$$

$$\Delta_\ell \frac{v(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} = -\alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f(k) F(u(k)). \tag{3.4}$$

If $u(k)$ is a nonoscillatory solution of (0.1), then we can assume that $u(k) > 0$ eventually (the case $u(k) < 0$ can be similarly treated). First, we shall show that $\liminf_{k\to\infty} u(k) = 0$. If not, there exist $k_1 \geq a$ and a positive constant c_1 such that $F(u(k)) \geq c_1$ for all $k \in [k_1, \infty)$. From (3.4) it follows that

$$\begin{aligned} \frac{v(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}} - \frac{v(k_1)}{\alpha^{\lceil \frac{k_1}{\ell} \rceil}} &= - \sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+r\ell}{\ell} \rceil} f(k_1+j+r\ell) F(u(k_1+j+r\ell)) \\ &\leq -c_1 \sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+r\ell}{\ell} \rceil} f(k_1+j+r\ell) \end{aligned}$$

which tends to $-\infty$ as $k \to \infty$.

We then have

$$\Delta_{\ell}\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}} = \alpha^{-\lceil\frac{k+\ell}{\ell}\rceil}(v(k)+G(k))/p(k) \leq -1/p(k) \text{ for all } k\in[k_2,\infty),$$

$$\text{for some } \ k_2 \geq k_1.$$

This implies that

$$\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}} \leq \frac{u(k_2)}{\alpha^{\lceil\frac{k_2}{\ell}\rceil}} - \sum_{r=0}^{\frac{k-\ell-k_2-j}{\ell}} 1/p(k_2+j+r\ell)$$

which tends to $-\infty$ as $k\to\infty$. But, this contradicts the fact that $u(k)$ is eventually positive. From the above argument, we also have

$$\sum_{r=0}^{\infty}\alpha^{-\lceil\frac{k_1+j+r\ell}{\ell}\rceil}f(k_1+j+r\ell)F(u(k_1+j+r\ell))<\infty. \tag{3.5}$$

If $\limsup\limits_{k\to\infty} u(k)=\gamma>0$, then there exists a sequence $\{k_t\}\subseteq[0,\infty)$, such that $u(k_t)\to\gamma$ as $t\to\infty$. Hence, there is $t(0)(k_{t(0)}\geq a)$ such that $u(k_t)\geq\gamma/2$ and $F(u(k_t))\geq c_2$ for all $t\geq t(0)$, where c_2 is a positive constant. But, then we have

$$\begin{aligned}
&\sum_{r=0}^{\frac{k_t-k_{t(0)}-j}{\ell}} \alpha^{-\lceil\frac{k_{t(0)}+j+r\ell}{\ell}\rceil}f(k_{t(0)}+j+r\ell)F(u(k_{t(0)}+j+r\ell))\\
&\geq \sum_{r=0}^{\frac{t-k_{t(0)}-j}{\ell}} \alpha^{-\lceil\frac{k_{t(0)}+j+r\ell}{\ell}\rceil}f(k_{t(0)}+j+r\ell)F(u(k_{t(0)}+j+r\ell))\\
&\geq bc_1(t-t(0)+\ell)
\end{aligned}$$

which tends to ∞ as $t\to\infty$, so that

$$\sum_{r=0}^{\infty}\alpha^{-\lceil\frac{k_1+j+r\ell}{\ell}\rceil}f(k_1+j+r\ell)F(u(k_1+j+r\ell))=\infty$$

which contradicts (3.5). □

Example 3.5. *For the generalized α-difference equation*

$$\Delta_{\alpha(\ell)}\left(\frac{1}{k}\Delta_{\alpha(\ell)}u(k)\right)+\frac{(\alpha^2-1)ku(k+2\ell)}{(k+\ell)}=\frac{(\alpha^2-1)}{k\alpha^{\lceil\frac{k}{\ell}\rceil}},$$

and for $p(k)=\frac{1}{k}$, $F(u(k))=\frac{ku(k+2\ell)}{(k+\ell)}$, *the conditions of Theorem 3.4 hold and hence all nonoscillatory solutions of the generalized α-difference equation, satisfies* $\lim\limits_{k\to\infty}u(k)=0$.

Theorem 3.6. *In addition to the condition* (ii), *let*

(iv) $f(k)>0$ *for all* $k\in[a,\infty)$, *and* $\sum\limits_{r=0}^{\infty}\alpha^{-\lceil\frac{k_1+j+r\ell}{\ell}\rceil}f(k_1+j+r\ell)=\infty$, *and*

(v) $\lim\limits_{k\to\infty} g(k)/f(k)=0$.

Then, for every nonoscillatory solution $u(k)$ *of* (0.1), $\liminf\limits_{k\to\infty}|u(k)|=0$.

Proof. Let $u(k)$ be a nonoscillatory solution of (0.1), say, $u(k) > 0$ for all $k \in [k_1, \infty)$, where $k_1 \geq a$. Then, $u(k)$ is also a nonoscillatory solution of

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) + [f(k) - g(k)/F(u(k))]F(u(k)) = 0, \quad k \in [k_1, \infty).$$

Suppose that $\liminf_{k\to\infty} u(k) > 0$, then by the hypotheses, there exists a positive constant c such that $F(u(k)) \geq c$ for all $k \in [k_1, \infty)$. Thus, by (v) there exists a $k_2 \geq k_1$ such that $g(k)/(f(k)F(u(k))) < 1/2$ for all $k \in [k_2, \infty)$. This implies that

$$f(k) - \frac{g(k)}{F(u(k))} = f(k)\left[1 - \frac{g(k)}{(f(k)F(u(k)))}\right] \geq \frac{1}{2}f(k), \quad k \in [k_2, \infty).$$

So from (iv) we get

$$\sum_{r=0}^{\infty} \alpha^{-\lceil \frac{k_1+j+r\ell}{\ell} \rceil}\left[f(k_1+j+r\ell) - \frac{g(k_1+j+r\ell)}{F(u(k_1+j+r\ell))}\right] = \infty.$$

But, then by Theorem 3.2, $u(k)$ must be oscillatory. This contradiction completes the proof. □

Example 3.7. *For the generalized α-difference equation*

$$\Delta_{\alpha(\ell)}\left(\frac{1}{k}\Delta_{\alpha(\ell)}u(k)\right) + \frac{(1-\alpha^2)u(k)}{k} = \frac{k(1-\alpha^2)}{(k+\ell)\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}},$$

and for $p(k) = \frac{1}{k}$, $f = k\alpha^{\lceil \frac{k}{\ell} \rceil}$, $F(u(k)) = \frac{(1-\alpha^2)u(k)^2}{k^2}$, the conditions of Theorem 3.6 hold and hence all the nonoscillatory solutions of the generalized α-difference equation satisfies $\lim_{k\to\infty} |u(k)| = 0$. $u(k) = \frac{1}{\alpha^{\lceil \frac{k}{\ell} \rceil}}$ is one such solution.

Theorem 3.8. *In addition to the condition (iv) let*

(vi) *$F(u)$ is continuous at $u = 0$, and*

(vii) $\displaystyle \lim_{k\to\infty}\inf \frac{\sum_{r=0}^{\frac{k-t-j}{\ell}} g(t+j+r\ell)}{\sum_{r=0}^{\frac{k-t-j}{\ell}} f(t+j+r\ell)} \geq c > 0$ *for every $t \in [a, \infty)$.*

Then, no solution of (0.1) approaches zero.

Proof. Let $u(k)$ be a solution of (0.1) which approaches zero. Then, by the hypotheses on the function F there exists a $k_1 \geq a$ such that $F(u(k)) < c/4$ for all $k \in [k_1, \infty)$. Hence, from equation (0.1) we have

$$p(k+\ell)\alpha\Delta_\ell \frac{u(k+\ell)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} - \alpha p(k_1+j)\Delta_\ell \frac{u(k_1+j)}{\alpha^{\lceil \frac{k_1+j}{\ell} \rceil}}$$

$$\geq -\frac{c}{4}\sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+\ell+r\ell}{\ell} \rceil} f(k_1+j+r\ell) + \sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+\ell+r\ell}{\ell} \rceil} g(k_1+j+r\ell),$$

which by (vii) yields

$$\frac{\alpha p(k+\ell)\Delta_\ell \frac{u(k+\ell)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+\ell+r\ell}{\ell} \rceil} f(k_1+j+r\ell)} - \frac{\alpha p(k_1+j)\Delta_\ell \frac{u(k_1+j)}{\alpha^{\lceil \frac{k_1+j}{\ell} \rceil}}}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} \alpha^{-\lceil \frac{k_1+j+\ell+r\ell}{\ell} \rceil} f(k_1+j+r\ell)}$$

$$\geq -\frac{c}{4} + \frac{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} g(k_1+j+r\ell)}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell)} \geq -\frac{c}{4} + \frac{c}{2} = \frac{c}{4} > 0,$$

for all large k. Now, because of (iv) in the above inequality implies that $p(k)\Delta_\ell \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}$ which tends to ∞ as $k \to \infty$, which in turn leads to the contradictive conclusion that $u(k) \to \infty$ as $k \to \infty$. □

Example 3.9. *For the generalized α-difference equation*

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) + \alpha^2(\alpha-1)ku(k) = \alpha^3(\alpha-1)(k+\ell)\alpha^{2\lceil \frac{k}{\ell} \rceil},$$

and for $F(u(k)) = u(k)$, $f = k\alpha^2(\alpha-1)$, $g = \alpha^3(\alpha-1)(k+\ell)\alpha^{2\lceil \frac{k}{\ell} \rceil}$, all the conditions of Theorem 3.8 *hold and hence all the solutions of the generalized α-difference equation are unbounded. $u(k) = \alpha^{2\lceil \frac{k}{\ell} \rceil}$ is one such solution.*

Remark 3.10. If we replace conditions (iv) and (vii) by

(iv)$'$ $f(k) < 0$ for all $k \in [a,\infty)$, and $\sum_{r=0}^{\infty} \alpha^{-\lceil \frac{t+j+\ell+r\ell}{\ell} \rceil} f(t+j+r\ell) = -\infty$ and

(v)$'$ $\limsup_{k\to\infty} \sum_{r=0}^{\frac{k-t-j}{\ell}} g(t+j+r\ell) / \sum_{r=0}^{\frac{k-t-j}{\ell}} f(t+j+r\ell) \leq c < 0$ for every $t \in [a,\infty)$,

then the assertion of Theorem 3.8 holds.

Theorem 3.11. *Suppose that the following conditions hold.*

(viii) *$F(u)$ is locally bounded in $[0,\infty)$*

(ix) $\sum_{r=0}^{\infty} \alpha^{-\lceil \frac{t+j+r\ell}{\ell} \rceil} |f(t+j+r\ell)| < \infty$, $\sum_{r=0}^{\infty} \alpha^{-\lceil \frac{t+j+r\ell}{\ell} \rceil} g(t+j+r\ell) = \infty$.

Then, every solution of (0.1) *is unbounded.*

Proof. Let $u(k)$ be a bounded solution of (0.1), i.e. $|u(k)| < M$, where M is a positive constant. Then, by (viii) there exist constants L_1 and L_2 such that

$L_1 \le F(u(k)) \le L_2$. But then, from (0.1) and (ix), we obtain

$$\begin{aligned} &\alpha p(k+\ell)\Delta_\ell \frac{u(k+\ell)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} - \alpha p(a+j)\Delta_\ell \frac{u(a+j)}{\alpha^{\lceil \frac{a+j}{\ell} \rceil}} \\ &\ge \sum_{r=0}^{\frac{k-a-j}{\ell}} \alpha^{-\lceil \frac{a+j+\ell+r\ell}{\ell} \rceil} g(a+j+r\ell) - L_2 \sum_{r=0}^{\frac{k-a-j}{\ell}} \alpha^{-\lceil \frac{a+j+\ell+r\ell}{\ell} \rceil} f^+(a+j+r\ell) \\ &\quad - L_1 \sum_{r=0}^{\frac{k-a-j}{\ell}} \alpha^{-\lceil \frac{a+j+\ell+r\ell}{\ell} \rceil} f^-(a+j+r\ell) \end{aligned}$$

which tends to ∞, as $k \to \infty$. However, this leads to the fact that $u(k) \to \infty$. This contradiction completes the proof. □

Example 3.12. *For the generalized α-difference equation*

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) + \alpha^2(\alpha-1)ku(k) = \alpha^3(\alpha-1)(k+\ell)\alpha^{2\lceil \frac{k}{\ell} \rceil},$$

and for $F(u(k)) = u(k)$, $f = k\alpha^2(\alpha-1)$, $g = \alpha^3(\alpha-1)(k+\ell)\alpha^{2\lceil \frac{k}{\ell} \rceil}$, all the conditions of Theorem 3.11 hold and hence all the solutions of generalized α-difference equation are unbounded. Infact $u(k) = \alpha^{2\lceil \frac{k}{\ell} \rceil}$ is one such solution.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.

[2] R.P. Agarwal, Young-Ho Kim and S.K. Sen, New retarded discrete inequalities with applications, *International Journal of Difference Equations* **1**(4) (2009), 1–19.

[3] Blazej Szmanda, Note on the behaviour of solutions of a second order nonlinear difference equation, *Atti Della Accademia Nazionale Dei Lincei* **69**(8) (1980), 120–125.

[4] Jerzy Popenda and Blazej Szmanda, On the oscillation of solutions of certain difference equations, *Demonstratio Mathematica* **XVII**(1) (1984), 153–164.

[5] Jerzy Popenda, Oscillation and nonoscillation theorems for second-order difference equations, *Journal of Mathematical Analysis and Applications* **123**(1) (1987), 34–38.

[6] John R. Graef and P.W. Spikes, Asymptotic behaviour of solutions of a second order nonlinear differential equation, *Journal of Differential Equation* **17**(2) (1975), 461–476.

[7] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Theory of generalized difference operator and its applications, *Far East Journal of Mathematical Sciences* **20**(2) (2006), 163–171.

[8] M. Maria Susai Manuel and G. Britto Antony Xavier, Generalized Difference calculus of sequences of real and complex numbers, *International Journal of Computational and Numerical Analysis and Applications* **6**(4) (2004), 401–415.

[9] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences* **23**(3) (2006), 295–304.

[10] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, Generalized bernoulli polynomials through weighted pochhammer symbols, *Far East Journal of Applied Mathematics* **26**(3) (2007), 321–333.

[11] M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, On the solutions and applications of some class of generalized difference equations, *Far East Journal of Applied Mathematics* **28**(2) (2007), 223–241.

[12] M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, *International Journal of Pure and Applied Mathematics* **33**(3) (2006), 333–343.

[13] M. Maria Susai Manuel and G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain class of generalized difference equations, *International Journal of Differential Equations and Applications* **10**(4) (2007), 423–433.

[14] M. Maria Susai Manuel, A. George Maria Selvam and G. Britto Antony Xavier, Regular sink and source in terms of solutions of certain class of generalized difference equations, *Far East Journal of Applied Mathematics* **28**(3) (2007), 441–454.

[15] M. Maria Susai Manuel, G. Britto Antony Xavier and V. Chandrasekar, Generalized difference operator of the second kind and its application to number theory, *International Journal of Pure and Applied Mathematics* **47**(1) (2008), 127–140.

[16] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar and R. Pugalarasu, On generalized difference operator of third kind and its applications to number theory, *International Journal of Pure and Applied Mathematics* **53**(1) (2009), 69–82.

[17] M. Maria Susai Manuel, G. Britto Antony Xavier, D.S. Dilip and G. Dominic Babu, Asymptotic behaviour of solutions of generalized nonlinear difference equations of second order, communicated to *Journal of Difference Equation*.

[18] R. Pugalarasu, M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, Theory of Generalized Difference operator of n-th kind and its applications in number theory (Part I), *International Journal of Pure and Applied Mathematics* **64**(1) (2010), 103–120.

[19] R. Pugalarasu, M. Maria Susai Manuel, V. Chandrasekar and G. Britto Antony Xavier, Theory of generalized difference operator of n-th kind and its applications in number theory (Part II), *International Journal of Pure and Applied Mathematics* **64**(1) (2010), 121–132.

[20] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York, 1990.

[21] Saber N. Elaydi, *An Introduction to Difference Equations*, 2nd edition, Springer, 1999.

[22] Walter G. Kelley and Allan C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, Inc., 1991.

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