Fuzzy Idealization on $gb$-Continuity

V. Chitra, V. Indhumathi and C. Selvapriya*

Department of Mathematics, NGM College, Pollachi 642001, India

*Corresponding author: priyamath1729@gmail.com

Abstract. In this work, we introduce the notions of $f_{gb^{-}}I$-closed sets, $f_{gb^{-}}I$-continuous, $f_{gb^{-}}I$-irresolute, $f_{gb^{-}}I$-open functions and $f_{gb^{*}^{-}}I$-open functions in fuzzy ideal topological spaces and investigate some of their characterizations and properties.

Keywords. $f_{gb^{-}}I$-closed sets; $f_{gb^{-}}I$-neighbourhood; $f_{gbq^{-}}I$-neighbourhood; $f_{gb^{-}}I$-continuous functions; $f_{gb^{-}}I$-irresolute functions; $f_{gb^{-}}I$-closed functions; $f_{gb^{*}^{-}}I$-open functions; fuzzy $gb^{-}I$-$T_{1/2}$-space

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1. Introduction

Fuzziness is one of the important and useful concepts in the modern scientific studies. Zadeh [16] introduced the notion of fuzzy sets. Throughout, the development of fuzzy set theory, many interesting phenomena have been observed. Fuzzy topology was introduced by Chang [4]. An alternate definition of fuzzy topology was given by Lowen [7]. Vaidyanathaswamy [14] studied the topic of ideal topological spaces intensively. Mahmoud [9] and Sarkar [13] introduced
the fuzzy ideal topology. The concept of fgb-closed sets and fgb-continuous are introduced by Benchalli and Karnel [3]. In this paper, we introduced fgb-I-closed sets, fgb-I-continuous, fgb-I-irresolute, fgb-I-closed functions, fgb-I-open functions and fgb*-I-open functions in fuzzy ideal topological spaces.

2. Preliminaries

Let $X$ be a non-empty set. A family $\tau$ of fuzzy sets of $X$ is called a fuzzy topology [4] on $X$ if the null fuzzy set $\emptyset$ and the whole fuzzy set $X$ belongs to $\tau$ and $\tau$ is closed with respect to any union and finite intersection. If $\tau$ is a fuzzy topology on $X$, then the pair $(X, \tau)$ is called a fuzzy topological space. A fuzzy point in $X$ with support $x \in X$ and value $\alpha$ ($0 \leq \alpha \leq 1$) is denoted by $x_\alpha$. For a fuzzy set $A$ in $X$, $\text{Cl}(A)$, $\text{Int}(A)$ and $1 - A$ denote the closure, interior and complement of $A$. A fuzzy set $A$ in $(X, \tau)$ is said to be quasi-coincident [12] with a fuzzy set $B$, denoted by $A \equiv_{\text{q-nc}} B$, if there exists a point $x \in X$ such that $A(x) + B(x) > 1$ [5]. A fuzzy set $B$ in $(X, \tau)$ is called a q-neighbourhood [12] of a fuzzy point $x_\beta$ if there exists a fuzzy open set $U$ of $X$ such that $x_\beta q U \subseteq V$ [5]. A nonempty collection of fuzzy sets $I$ of a set $X$ is called a fuzzy ideal [6,11], if

1. $A \in I$ and $A \leq B$, then $A \lor B \in I$.
2. $A \in I$ and $B \in I$, then $A \lor B \in I$.

The triplet $(X, \tau, I)$ is called fuzzy ideal topological space with a fuzzy ideal $I$ and a fuzzy topology $\tau$.

For $(X, \tau, I)$, the fuzzy local function of $A \leq X$ with respect to $\tau$ and $I$ denoted by $A^*(\tau, I)$ (briefly $A^*$) and is defined $A^* = \bigvee\{x \in X : A \land U \not\subseteq I \text{ for every } U \in \tau\}$. While $A^*$ is the union of the fuzzy points such that if $U \in \tau$ and $E \in I$, then there is atleast one $y \in X$ for which $U(y) + A(y) - 1 > E(y)$ [8]. Fuzzy closure operator of a fuzzy set $A$ in $(X, \tau, I)$ is defined as $\text{Cl}^*(A) = A \lor A^*$. A subset $A$ in $(X, \tau, I)$ is called fuzzy ideal open [10] if $A \subseteq \text{Int}(A^*)$.

**Definition 2.1** ([2]). A fuzzy set $A$ in a fuzzy topological space $X$ is called

(i) fuzzy $b$-open set if $A \subseteq (\text{IntCl}(A) \lor \text{ClInt}(A))$.

(ii) fuzzy $b$-closed set if $(\text{IntCl}(A) \lor \text{Int}(A)) \subseteq A$.

**Definition 2.2** ([2]). Fuzzy $b$-closure and fuzzy $b$-interior of a fuzzy set $A$ is given by

(i) $\text{bCl}(A) = \text{Cl}\{B : B \text{ is a fuzzy } b\text{-closed set of } X \text{ and } B \geq A\}$.

(ii) $\text{bInt}(A) = \text{Int}\{C : C \text{ is a fuzzy } b\text{-open set of } X \text{ and } A \geq C\}$.

**Definition 2.3.** A fuzzy set $A$ in a fuzzy topological space $(X, \tau)$ is called

(i) a generalized closed (briefly $g$-closed) fuzzy set [1] if $\text{Cl}(A) \leq B$ whenever $A \leq B$ and $B$ is fuzzy open set in $(X, \tau)$.

(ii) a fuzzy generalized $b$-closed (briefly $fgb$-closed) set [2] if $\text{bCl}(A) \leq B$ whenever $A \leq B$ and $B$ is fuzzy open in $(X, \tau)$.
Definition 2.4 ([3]). A fuzzy topological space \((X, \tau)\) is called fuzzy \(gb\)-\(T_{1/2}\)-space (briefly \(fgb\)-\(T_{1/2}\)-space) if every \(fgb\)-closed set in \(X\) is fuzzy \(b\)-closed in \(X\).

Definition 2.5 ([1]). Let \(X, Y\) be two fuzzy topological spaces. A function \(f : (X, \tau) \to (Y, \sigma)\) is called a \(fg\)-continuous if \(f^{-1}(A)\) is \(g\)-closed fuzzy set in \(X\), for every closed fuzzy set \(A\) of \(Y\).

Definition 2.6 ([3]). A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be fuzzy generalized \(b\)-continuous (briefly \(fgb\)-continuous) if \(f^{-1}(A)\) is \(fgb\)-closed set in \(X\), for every fuzzy closed set \(A\) in \(Y\).

Definition 2.7 ([15]). A subset \(A\) of a fuzzy ideal topological space \((X, \tau, I)\) is said to be fuzzy \(b\)-\(I\)-open set if \(A \leq Cl^*(Int(A)) \lor Int(Cl^*(A))\).

Definition 2.8 ([11]). A function \(f : (X, \tau, I) \to (Y, \sigma)\) is called fuzzy-I-continuous if \(f^{-1}(V)\) is fuzzy \(I\)-open set for each \(V\) in \(Y\) is fuzzy open.

Definition 2.9 ([15]). A function \(f : (X, \tau, I) \to (Y, \sigma)\) is called fuzzy \(b\)-\(I\)-continuous if the inverse image of each fuzzy open set in \(Y\) is fuzzy \(b\)-\(I\)-open in \(X, \tau, I\).

Remark 2.1 ([2]). Every fuzzy closed set in a fuzzy topological space \((X, \tau)\) is fuzzy \(b\)-closed.

Theorem 2.1 ([15]). In a fuzzy ideal topological space \((X, \tau, I)\), the following statements are holds:
(a) every fuzzy closed set is \(fb\)-\(I\)-closed set.
(b) every fuzzy \(I\)-closed set is \(fb\)-\(I\)-closed set.
(c) every fuzzy \(b\)-\(I\)-closed set is \(fb\)-closed set.

3. Fuzzy \(gb\)-\(I\)-Closed Sets

Definition 3.1. A subset \(A\) of a fuzzy ideal topological space \((X, \tau, I)\) is said to be fuzzy generalized \(b\)-\(I\)-closed (briefly \(fgb\)-\(I\)-closed) if \(bCl(A) \leq B\), whenever \(A \leq B\) and \(B\) is a fuzzy \(I\)-open in \(X, \tau, I\).

Remark 3.1. A subset \(A\) of a fuzzy ideal topological space \((X, \tau, I)\) is called fuzzy generalized \(b\)-\(I\)-open (\(fgb\)-\(I\)-open) if its complement \(1 - A\) is \(fgb\)-\(I\)-closed.

Theorem 3.1. Every fuzzy closed set in \((X, \tau, I)\) is \(fgb\)-\(I\)-closed.

Proof. Assume \(A\) is a \(f\)-closed set in \(X\). Let \(A \leq B\), \(B\) is fuzzy \(I\)-open set in \(X\). By Remark 2.1, \(A\) is \(fb\)-closed, \(Cl(A) = bCl(A) = A \leq B\). Thus \(bCl(A) \leq B\). Hence \(A\) is \(fgb\)-\(I\)-closed. \(\Box\)

Theorem 3.2. Every fuzzy \(I\)-closed set in \((X, \tau, I)\) is \(fgb\)-\(I\)-closed.

Proof. Assume \(A\) is a fuzzy \(I\)-closed set in \(X\). Let \(A \leq B\), \(B\) is a fuzzy \(I\)-open set in \(X\). By Theorem 2.1 every fuzzy \(I\)-closed is \(fb\)-\(I\)-closed. Thus \(A\) is \(fb\)-\(I\)-closed. Again by Theorem 2.1,
every fb-I-closed set is fb-closed, \( Cl(A) = bCl(A) = A \leq B \). Thus \( bCl(A) \leq B \). Hence \( A \) is fgb-I-closed.

**Theorem 3.3.** Every fb-closed set in fuzzy ideal topological space \((X, \tau, I)\) is fgb-I-closed.

**Proof.** Assume \( A \) is fb-closed set in \( X \). Let \( A \leq B \), \( B \) is fuzzy I-open set in \( X \). Since \( A \) is fb-closed, we have \( bCl(A) = A \leq B \). Therefore \( bCl(A) \leq B \). Hence \( A \) is fgb-closed.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.1.** Let \( X = \{a, b\} \) and \( A, B, C \) be the fuzzy sets defined as follows:

\[
A(a) = 0.3, \ A(b) = 0.4, \\
B(a) = 0.4, \ B(b) = 0.5, \\
C(a) = 0.3, \ C(b) = 0.7.
\]

Let \( \tau = (0, 1, A) \) be the topology on \( X \) and \( I = \{0\} \) be the fuzzy ideal on \( X \). Then \( C \) is fgb-I-closed but not fb-closed in \( X \).

**Theorem 3.4.** A subset \( A \) of a fuzzy ideal topological space \((X, \tau, I)\) is called fgb-I-open if and only if \( B \leq bInt(A) \), whenever \( B \) is fuzzy I-closed set and \( B \leq A \).

**Proof.** Suppose \( A \) is a fgb-I-open set in \( X \). Then \( 1 - A \) is fgb-I-closed in \( X \). Let \( B \) be a fuzzy I-closed set in \( X \) such that \( B \leq A \). Then \( 1 - A \leq 1 - B \), \( 1 - B \) is fuzzy I-open set in \( X \).

Since \( 1 - A \) is fgb-I-closed, we have \( bCl(1 - A) \leq 1 - B \), implies \( 1 - bInt(A) \leq 1 - B \). Therefore, \( B \leq bInt(A) \).

Conversely, assume that \( B \leq bInt(A) \), whenever \( B \leq A \) and \( B \) is fuzzy I-closed set in \( X \). Then \( 1 - bInt(A) \leq 1 - B = C \), \( C \) is fuzzy I-open set in \( (X, \tau, I) \). Thus \( bCl(1 - A) \leq C \), implies \( 1 - A \) is fgb-I-closed. Hence \( A \) is fgb-I-open.

**Theorem 3.5.** If \( A \) is a fgb-I-closed set in \((X, \tau, I)\) and \( A \leq B \leq bCl(A) \), then \( B \) is a fgb-I-closed set in \((X, \tau, I)\).

**Proof.** Let \( C \) be fuzzy I-open in \( X \) such that \( B \leq C \). Suppose \( A \leq B \), then \( A \leq C \). Since \( A \) is a fgb-I-closed set in \( X \), it follows that \( bCl(A) \leq C \). Now, \( B \leq bCl(A) \) implies \( bCl(B) \leq bCl(bCl(A)) = bCl(A) \). Thus \( bCl(B) \leq C \). Hence \( B \) is a fgb-I-closed set in \( X \).

**Theorem 3.6.** If \( A \) is a fgb-I-open set in fuzzy ideal topological space \((X, \tau, I)\) and \( bInt(A) \leq B \leq A \) then \( B \) is fgb-I-open in \((X, \tau, I)\).

**Proof.** Let \( A \) be a fgb-open set and \( B \) be any fuzzy set in \( X \), such that \( bInt(A) \leq B \leq A \).

Then \( 1 - A \) is fgb-I-closed in \( X \) and \( 1 - A \leq 1 - B \leq bCl(1 - A) \). Then from Theorem 3.5, \( 1 - B \) is a fgb-I-closed set in \( X \). Hence \( B \) is a fgb-I-open set in \( X \).
3.1 Fuzzy Generalized \( b-I \)-Neighbourhood and Fuzzy Generalized \( bq-I \)-Neighbourhood

**Definition 3.2.** Let \( A \) be a subset of a fuzzy ideal topological space \((X, \tau, I)\) and \( x_p \) be a fuzzy point in \((X, \tau, I)\), then \( A \) is called fuzzy generalized \( b-I \)-neighbourhood (briefly \( fgb-I \)-neighbourhood) of \( x_p \) if and only if there exists a \( fgb-I \)-open set \( B \) in \((X, \tau, I)\) such that \( x_p \in B \leq A \).

**Definition 3.3.** Let \( A \) be a subset of a fuzzy ideal topological space \((X, \tau, I)\) and \( x_p \) be a fuzzy point in \((X, \tau, I)\), then \( A \) is called fuzzy generalized \( bq-I \)-neighbourhood (briefly \( fgbq-I \)-neighbourhood) of \( x_p \) if and only if there exists a \( fgb-I \)-open set \( B \) such that \( x_p \in B \leq A \).

**Theorem 3.7.** \( A \) is \( fgb-I \)-open set in \((X, \tau, I)\) if and only if for each fuzzy point \( x_p \in A \), \( A \) is a \( fgb-I \)-neighbourhood of \( x_p \).

**Proof.** Let \( A \) be a \( fgb-I \)-open set in \( X \). For each \( x_p \in A \), \( A \leq A \). Therefore \( A \) is a \( fgb-I \)-neighbourhood of \( x_p \). Conversely, let \( A \) be a \( fgb-I \)-neighbourhood of \( x_p \). Then there exists a \( fgb-I \)-open set \( B \) in \((X, \tau, I)\) such that \( x_p \in B \leq A \). Hence \( A \) is \( fgb-I \)-open set in \((X, \tau, I)\).

**Theorem 3.8.** If \( A \) and \( B \) are \( fgb-I \)-neighbourhoods of \( x_p \) then \( A \wedge B \) is also a \( fgb-I \)-neighbourhood of \( x_p \).

**Proof.** Let \( A \) and \( B \) be \( fgb-I \)-neighbourhoods of \( x_p \). Then there exist \( fgb-I \)-open sets \( L \) and \( M \) such that \( x_p \in L \wedge M \leq A \wedge B \). Thus \( A \wedge B \) is also a \( fgb-I \)-neighbourhood of \( x_p \).}

4. Fuzzy \( gb-I \)-Continuous Functions

**Definition 4.1.** A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is said to be fuzzy generalized \( b-I \)-continuous (briefly \( fgb-I \)-continuous), if \( f^{-1}(A) \) is \( fgb-I \)-closed set in \((X, \tau, I)\), for every fuzzy closed set \( A \) in \( Y \).

**Theorem 4.1.** \( f : (X, \tau, I) \rightarrow (Y, \tau) \) is \( fgb-I \)-continuous if and only if the inverse image of each fuzzy open set of \((Y, \sigma)\) is \( fgb-I \)-open set of \((X, \tau, I)\).

**Proof.** Suppose \( B \) is a \( f \)-open set of \( Y \). Then \( 1-B \) is \( f \)-closed in \( Y \). Since \( f : X \rightarrow Y \) is \( fgb-I \)-continuous \( f^{-1}(1-B) = 1-f^{-1}(B) \) is \( fgb-I \)-closed set of \( X \). Hence \( f^{-1}(B) \) is a \( fgb-I \)-open set of \( X \).

Conversely, let \( A \) be a fuzzy closed set in \( Y \). Then \( 1-A \) is \( f \)-open in \( Y \). By hypothesis, inverse image of \( 1-A \) in \( Y \) is a \( fgb-I \)-open set in \( X \). Thus \( f^{-1}(1-A) = 1-f^{-1}(A) \) is a \( fgb-I \)-open set in \( X \). Therefore \( f^{-1}(A) \) is a \( fgb-I \)-closed set in \( X \). Hence \( f \) is \( fgb-I \)-continuous.

**Theorem 4.2.** If \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is \( fgb-I \)-continuous then

(a) for each fuzzy point \( x_p \) of \( X \) and each \( A \in Y \) such that \( f(x_p) \cap A \), there exists a \( fgb-I \)-open set \( A \) of \( X \) such that \( x_p \in B \) and \( f(B) \leq A \).
Theorem 4.3. Every fuzzy-I-continuous function is fgb-I-continuous function.

Proof. Suppose \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is a fuzzy-I-continuous function. Assume \( A \) is a fuzzy open set in \( Y \). Since \( f \) is fuzzy-I-continuous, \( f^{-1}(A) \) is fuzzy-I-open in \( X \). Therefore, by Theorem 3.2, \( f^{-1}(A) \) is a fgb-I-open set in \( X \). Thus \( f \) is fgb-I-continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 4.1. Let \( X = Y = \{a, b\} \) and the fuzzy sets \( A \) and \( B \) be defined as follows:

\[
A(a) = 0.9, \ A(b) = 0.8, \ \ B(a) = 0.3, \ B(b) = 0.4.
\]

Let \( \tau = (0,1,A) \) and \( \sigma = (0,1,B) \) be the topologies on \( X \) and \( Y \) respectively and \( I = \{0\} \) be the fuzzy ideal on \( X \).

Let \( f : X \rightarrow Y \) be the identity function. Then \( f \) is fgb-I-continuous but not fuzzy-I-continuous.

Definition 4.2. A function \( f : (X, \tau, I) \rightarrow (Y, \sigma, I) \) is said to be fuzzy gb-I-irresolute (briefly fgb-I-irresolute), if \( f^{-1}(A) \) is fgb-I-closed set in \( (X, \tau, I) \), for every fgb-I-closed set \( A \) in \( (Y, \sigma, I) \).

Theorem 4.4. A function \( f : (X, \tau, I) \rightarrow (Y, \sigma, I) \) is fgb-I-irresolute if and only if the inverse image of every fgb-I-open set \( Y \) is fgb-I-open set in \( X \).

Proof. Let \( A \) be a fgb-I-open set in \( Y \). Then \( 1-A \) is fgb-I-closed in \( Y \). Since \( f \) is fgb-I-irresolute, \( f^{-1}(1-A) = 1 - f^{-1}(A) \) is a fgb-I-closed set in \( X \). Thus \( f^{-1}(A) \) is a fgb-I-open set in \( X \).

Conversely, assume that \( A \) is a fgb-I-closed set in \( Y \). Then \( 1-A \) is fgb-I-open in \( Y \). By hypothesis, \( f^{-1}(1-A) = 1 - f^{-1}(A) \) is a fgb-I-open set in \( X \). Thus \( f^{-1}(A) \) is a fgb-I-closed set in \( X \). Hence \( f \) is fgb-I-irresolute.

Theorem 4.5. Every fgb-I-irresolute function is fgb-I-continuous.

Proof. Suppose \( f : (X, \tau I) \rightarrow (Y, \sigma, I) \) is fgb-I-irresolute and let \( F \) be a fuzzy-I-closed set in \( Y \). Then \( F \) is a fgb-I-closed set in \( Y \), by Theorem 3.2. Since \( f \) is fgb-I-irresolute, \( f^{-1}(F) \) is a fgb-I-closed set in \( X \). Thus \( f \) is fgb-I-continuous.
The converse of the above theorem need not be true as seen from the following example.

**Example 4.2.** Let \( X = Y = [a, b] \), and the fuzzy sets \( A, B, C, D \) and \( E \) be defined as follows:

\[
A(a) = 0.9, \ A(b) = 0.9, \quad B(a) = 0.8, \ B(b) = 0.5, \quad C(a) = 0.6, \ C(b) = 0.5, \\
D(a) = 0.5, \ D(b) = 0.2, \quad E(a) = 0.5, \ E(b) = 0.6.
\]

Let \( \tau = \{0, 1, A, B, C, D\} \) and \( \sigma = \{0, 1, C\} \) be the topologies on \( X \) and \( Y \) respectively and \( I = \{0\} \) be the fuzzy ideal on \( X \) and \( Y \) respectively. Define \( f : X \to Y \) by \( f(a) = c, \ f(b) = a \) and \( f(c) = b \). Then \( f \) is \( \text{fgb-continuous} \) but not \( \text{fgb-irresolute} \).

**Theorem 4.6.** Let \( f : (X, \tau, I) \to (Y, \sigma, I) \) and \( g : (Y, \sigma, I) \to (Z, \gamma, I) \) be two functions. Then

1. \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-continuous} \) if \( f \) is \( \text{fgb-I-continuous} \) and \( g \) is \( f \)-I-continuous.

2. \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-irresolute} \) if \( f \) is \( \text{fgb-I-irresolute} \) and \( g \) is \( \text{fgb-I-irresolute} \) function.

3. \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-continuous} \) if \( f \) is \( \text{fgb-I-irresolute} \) and \( g \) is \( \text{fgb-I-continuous} \).

**Proof.**

1. Suppose \( B \) is a fuzzy \( I \)-closed set of \( Z \). Since \( g : (Y, \sigma, I) \to (Z, \gamma, I) \) is fuzzy-I-continuous, \( g(\gamma^{-1}(B)) \) is a fuzzy-I-closed set of \( Y \). Also since \( f : (X, \tau, I) \to (Y, \sigma, I) \) is \( \text{fgb-I-continuous} \), then from Definition 4.1, \( f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B) \) is a \( \text{fgb-I-closed} \) set of \( X \). Thus \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-continuous} \).

2. Let \( g : (Y, \sigma, I) \to (Z, \gamma I) \) be \( \text{fgb-I-irresolute} \) and \( B \) be a \( \text{fgb-I-closed} \) subset of \( Z \). Since \( g \) is \( \text{fgb-I-irresolute} \), \( g(\gamma^{-1}(B)) \) is a \( \text{fgb-I-closed} \) set of \( Y \). Also since \( f : (X, \tau, I) \to (Y, \sigma, I) \) is \( \text{fgb-I-irresolute} \), we have \( f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B) \) is a \( \text{fgb-I-closed} \) set of \( X \). Hence \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-irresolute} \).

3. Let \( B \) be fuzzy \( \text{b-I-closed} \) subset of \( Z \). Since \( g : (Y, \sigma, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-continuous} \), \( g^{-1}(B) \) is a \( \text{fgb-I-closed} \) set of \( Y \). Also \( f : (X, \tau, I) \to (Y, \sigma I) \) is \( \text{fgb-I-irresolute} \), \( g^{-1}(B) \) is \( \text{fgb-I-closed} \) in \( X \). Therefore \( f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B) \) is a \( \text{fgb-I-closed} \) set of \( X \). Hence \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( \text{fgb-I-continuous} \).

**Definition 4.3.** A fuzzy ideal topological space \( (X, \tau, I) \) is fuzzy \( \text{fgb-I-}T_{1/2} \)-space (briefly \( \text{fgb-I-}T_{1/2} \)-space) if every \( \text{fgb-I-closed} \) set in \( X \) is fuzzy \( \text{b-I-closed} \) in \( X \).

**Theorem 4.7.** A fuzzy ideal topological space \( (X, \tau, I) \) is \( \text{fgb-I-}T_{1/2} \)-space if and only if every fuzzy set in \( (X, \tau, I) \) is both fuzzy \( \text{b-I-open} \) and \( \text{fgb-I-open} \).

**Proof.** Suppose \( X \) is \( \text{fgb-I-}T_{1/2} \)-space and \( A \) is a \( \text{fgb-I-open} \) set in \( X \). Then \( 1 - A \) is a \( \text{fgb-I-closed} \) set in \( X \). Since \( X \) is \( \text{fgb-I-}T_{1/2} \)-space, every \( \text{fgb-I-closed} \) set in \( X \) is fuzzy \( \text{b-I-closed} \) in \( X \). Therefore \( 1 - A \) is a \( \text{b-I-closed} \) set and hence \( A \) is a \( \text{b-I-open} \) set in \( X \).

Conversely, assume that \( A \) is a \( \text{fgb-I-closed} \) set in \( X \). Then \( 1 - A \) is a \( \text{fgb-I-open} \) set. By
hypothesis, $1-A$ is fuzzy $b-I$-open. Therefore $A$ is fuzzy $b-I$-closed. Thus every $fgb-I$-closed set in $X$ is fuzzy $b-I$-closed. Hence $X$ is $fgb-I-T_{1/2}$-space.

**Definition 4.4.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $fb^*-I$-continuous if $f^{-1}(A)$ is fuzzy $b-I$-closed set in $(X, \tau, I)$, for every fuzzy $b$-closed set of $Y$.

**Theorem 4.8.** If $f : (X, \tau, I) \to (Y, \sigma)$ is $fb^*-I$-continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is $fgb$-continuous then $g \circ f : (X, \tau, I) \to (Z, \sigma)$ is $fb^*-I$-continuous if $Y$ is $fgb-T_{1/2}$-space.

**Proof.** Let $B$ be a fuzzy $b$-closed set of $Z$. Since $g : (Y, \sigma) \to (Z, \gamma)$ is $fgb$-continuous, $g^{-1}(B)$ is a $fgb$-closed set of $Y$. Since $Y$ is $fgb-T_{1/2}$-space, every $fgb$-closed set is fuzzy $b$-closed in $Y$. Thus $g^{-1}(B)$ is a fuzzy $b$-closed set of $Y$. Also since $f : (X, \tau, I) \to (Y, \sigma)$ is $fb^*-I$-continuous, $f^{-1}(g^{-1}(B))$ is a fuzzy $b-I$-closed set in $X$. Thus $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is a fuzzy $b-I$-closed set $X$. Hence $g \circ f : (X, \tau, I) \to (Z, \gamma)$ is $fb^*-I$-continuous.

**Theorem 4.9.** If $(X, \tau, I)$ is $fgb-I-T_{1/2}$-space and $f : (X, \tau, I) \to (Y, \sigma)$ is $fgb-I$-continuous, then $f$ is $fb-I$-continuous.

**Proof.** Let $B$ be a fuzzy closed set in $Y$. Since $f : (X, \tau, I) \to (Y, \sigma)$ is $fgb-I$-continuous, $f^{-1}(B)$ is a $fgb-I$-closed set in $X$. Also $X$ is $fgb-I-T_{1/2}$-space, every $fgb-I$-closed set is fuzzy $b-I$-closed. Thus $f^{-1}(B)$ is a fuzzy $b-I$-closed set in $X$. Hence $f : (X, \tau, I) \to (Y, \sigma)$ is $fb-I$-continuous.

**Definition 4.5.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is called $fb^*-I$-closed if $f(A)$ is fuzzy $b$-closed in $Y$ for every fuzzy $b-I$-closed set $A$ in $X$.

**Theorem 4.10.** Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be onto, $fgb-I$-irresolute and $fb^*-I$-closed. If $X$ is $fgb-I-T_{1/2}$-space, then $Y$ is $fgb-I-T_{1/2}$-space.

**Proof.** Let $A$ be a $fgb-I$-closed set in $Y$. Since $f : (X, \tau, I) \to (Y, \sigma, I)$ is $fgb-I$-irresolute, $f^{-1}(A)$ is a $fgb-I$-closed set in $X$. Also since $X$ is $fgb-I-T_{1/2}$-space, $f^{-1}(A)$ is fuzzy $b-I$-closed set in $X$. Again $f : (X, \tau, I) \to (Y, \sigma, I)$ is $fb^*-I$-closed, $f(f^{-1}(A))$ is fuzzy $b-I$-closed in $Y$. Since $f$ is onto, $f(f^{-1}(A)) = A$. Thus $A$ is fuzzy $b-I$-closed in $Y$. Hence $Y$ is $fgb-I-T_{1/2}$-space.

**Definition 4.6.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be fuzzy $gb-I$-open (briefly $fgb-I$-open) if the image of every fuzzy $I$-open set in $X$, is $fgb$-open set in $Y$.

**Definition 4.7.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be fuzzy $gb-I$-closed (briefly $fgb-I$-closed) if the image of every fuzzy $I$-closed set in $X$ is $fgb$-closed set in $Y$.

**Definition 4.8.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be fuzzy $gb^*-I$-open (briefly $fgb^*-I$-open) if the image of every $fgb-I$-open set in $X$, is $fgb$-open set in $Y$.
Definition 4.9. A function \( f : (X, \tau, I) \to (Y, \sigma) \) is said to be fuzzy \( gb^* \)-I-closed (briefly \( fgb^* \)-I-closed) if the image every \( fgb-I \)-closed set in \( X \) is \( fgb \)-closed set in \( Y \).

Remark 4.1. Every \( fgb^* \)-I-open function is \( fgb-I \)-open.

The converse of the above statement is not true as seen from the following example.

Example 4.3. Let \( X = (a, b) \), \( Y = (x, y) \) and the fuzzy sets \( A \), \( B \) are defined as follows:

\[
A(a) = 0.9, \quad A(b) = 0.7, \quad B(x) = 0.5, \quad B(y) = 0.4.
\]

Let \( \tau = \{0, 1, A\} \), \( \sigma = \{0, 1, B\} \) be the topologies on \( X \) and \( Y \) respectively and \( I = \{0\} \) be the fuzzy ideal on \( X \). Then the function \( f : (X, \tau, I) \to (Y, \sigma) \) defined by \( f(a) = x \) and \( f(b) = y \) is \( fgb-I \)-open but not \( fgb^* \)-I-open.

Theorem 4.11. A function \( f : (X, \tau, I) \to (Y, \sigma) \) is \( fgb-I \)-closed if and only if for each fuzzy set \( A \) of \( Y \) and for each fuzzy I-open set \( B \) such that \( f^{-1}(A) \leq B \), there is a \( fgb \)-open set \( C \) of \( Y \) such that \( A \leq C \) and \( f^{-1}(C) \leq B \).

**Proof.** Suppose that \( f \) is \( fgb-I \)-closed. Let \( A \) be a fuzzy set of \( Y \), and \( B \) be a fuzzy \( I \)-open set of \( X \), such that \( f^{-1}(A) \leq B \). Then \( C = 1 - f(1 - B) \) is a \( fgb \)-open set in \( Y \) such that \( A \leq C \) and \( f^{-1}(C) \leq B \).

Conversely, suppose that \( F \) is a fuzzy \( I \)-closed set of \( X \). Then \( f^{-1}(1 - f(F)) \leq 1 - F \), and \( 1 - F \) is a fuzzy \( I \)-open set. By hypothesis, there is a \( fgb \)-open set \( C \) of \( Y \) such that \( 1 - f(F) \leq C \) and \( f^{-1}(C) \leq 1 - F \). Therefore \( F \leq 1 - f^{-1}(C) \). Hence \( 1 - C \leq f(F) \leq f(1 - f^{-1}(F)) \leq 1 - C \), which implies \( f(F) = 1 - C \). Since \( 1 - C \) is a \( fgb \)-closed set, \( f(F) \) is a \( fgb \)-closed set and thus \( f \) is \( fgb-I \)-closed. \( \square \)

Theorem 4.12. If \( f : (X, \tau, I) \to (Y, \sigma) \) is bijective, fuzzy-\( I \)-continuous and \( fgb-I \)-closed and \( A \) is \( fgb-I \)-closed set in \( (X, \tau, I) \), then \( f(A) \) is a \( fgb-I \)-closed set in \( (Y, \sigma) \).

**Proof.** Let \( f(A) \leq B \), \( B \) is a fuzzy open set in \( Y \). Since \( f \) is fuzzy-\( I \)-continuous, \( f^{-1}(B) \) is an fuzzy \( I \)-open set containing \( A \). Also \( A \) is \( fgb-I \)-closed set, we have \( bCl(A) \leq f^{-1}(B) \). Since \( f \) is \( fgb-I \)-closed, \( f(bCl(A)) \) is a \( fgb \)-closed set contained in the fuzzy open set \( B \). It follows that, \( bCl(f(bCl(A))) \leq B \) and so \( bCl(f(A)) \leq B \). Thus \( f(A) \) is a \( fgb \)-closed set in \( Y \). \( \square \)

Theorem 4.13. If \( f : (X, \tau, I) \to (Y, \sigma) \) is \( fgb-I \)-closed and \( g : (Y, \sigma) \to (Z, \gamma) \) is \( fgb^* \)-closed, then \( g \circ f \) is \( fgb^* \)-I-closed.

**Proof.** Let \( A \) be a fuzzy \( I \)-closed set in \( X \). Then \( f(A) \) is \( fgb \)-closed in \( Y \) as \( f \) is \( fgb-I \)-closed.

Since \( g : (Y, \sigma) \to (Z, \gamma) \) is \( fgb^* \)-closed, \( g(f(A)) \) is \( fgb \)-closed in \( Z \). Thus \( g(f(A)) = (g \circ f)(A) \) is \( fgb \)-closed in \( Z \). Therefore \( g \circ f \) is \( fgb^* \)-I-closed. \( \square \)

Theorem 4.14. Let \( f : (X, \tau, I) \to (Y, \sigma, I) \), \( g : (Y, \sigma, I) \to (Z, \gamma, I) \) be two functions such that \( g \circ f : (X, \tau, I) \to (Z, \gamma, I) \) is \( fgb^* \)-I-closed.
(i) If $f$ is fgb-$I$-continuous and surjective, then $g$ is fgb-$I$-closed.

(ii) If $g$ is fgb-$I$-irresolute and injective, then $f$ is fgb$^*$-$I$-closed.

Proof. (i) Let $F$ be a fuzzy $I$-closed set of $Y$. Then $f^{-1}(F)$ is fgb-$I$-closed in $X$ as $f$ is fgb-$I$-continuous. Since $g \circ f$ is fgb$^*$-$I$-closed, $(g \circ f)(f^{-1}(F)) = g(F)$ is a fgb-$I$-closed set in $Z$.

Hence $g : (Y, \tau, I) \rightarrow (Z, \gamma, I)$ is fgb-$I$-closed.

(ii) Let $F$ be a fgb-$I$-closed set in $X$. Then $(g \circ f)(F)$ is a fgb-$I$-closed set in $Z$. Since $g$ is fgb-$I$-irresolute and injective, $g^{-1}(g \circ f)(F) = f(F)$ is fgb-$I$-closed in $Y$. Hence $f$ is fgb$^*$-$I$-closed. 

Competing Interests
The authors declare that they have no competing interests.

Authors’ Contributions
All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References


