On the Solution of Stochastic Generalized Burgers Equation

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Abstract. We are interested in one dimensional nonlinear stochastic partial differential equation: the generalized Burgers equation with homogeneous Dirichlet boundary conditions, perturbed by additive space-time white noise. We establish a result of existence and uniqueness of the local solution to the viscous equation using fixed point argument, then if we impose a condition to the viscosity coefficient we can prove that this solution is global.

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1. Introduction

It is well known that the Burgers equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity.

However the situation is totally different when the force is a random one. Several authors have, indeed, suggested to use the stochastic Burgers equation as a simple model for turbulence ([1], [2], [3], [9]). The equation has also been proposed in ([10]) to study the dynamics of interfaces.
Here we consider the generalized Burgers equation with a random force which is a space-time white noise
\[ \frac{\partial u(t,x)}{\partial t} = \rho \frac{\partial^2 u(t,x)}{\partial x^2} - \partial_x f(u(t,x)) + \frac{\partial^2 \tilde{W}}{\partial t \partial x}, \tag{1} \]
where \( \rho \) is the viscosity coefficient and, \( \tilde{W}(t,x), t \geq 0, x \in \mathbb{R} \) is a zero mean Gaussian process whose covariance function is given by
\[ E[\tilde{W}(t,x)\tilde{W}(s,y)] = (t \wedge s)(x \wedge y), \quad t, s \geq 0, x, y \in \mathbb{R}. \]
Alternatively, we can consider a cylindrical Wiener process \( W \) by setting
\[ W(t) = \frac{\partial \tilde{W}}{\partial x} = \sum_{h=1}^{\infty} \beta_h e_h, \tag{2} \]
where \( \{e_h\} \) is an orthonormal basis of \( L^2(0,2\pi) \) and \( \{\beta_h\} \) is a sequence of mutually independent real Brownian motions in a fixed probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) adapted to a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). The series (2) defining \( W \) does not converge in \( L^2(0,2\pi) \) but it is convergent in any Hilbert space \( U \) such that the embedding
\[ L^2(0,2\pi) \subset U \]
is Hilbert-Schmidt ([5]).

In the following we shall write (1) as follows:
\[ du(t,x) = \left( \rho \frac{\partial^2 u(t,x)}{\partial x^2} - \partial_x f(u(t,x)) \right) dt + dW, \quad x \in [0,2\pi], t > 0, \tag{3} \]
where \( W \) is defined by (2). We assume that \( f: \mathbb{R} \rightarrow \mathbb{R} \) is a locally Lipschitz continuous function.

Equation (3) is supplemented with Dirichlet boundary conditions
\[ u(0,t) = u(2\pi,t) = 0, \tag{4} \]
and the initial condition
\[ u(x,0) = u_0(x), \quad x \in [0,2\pi]. \tag{5} \]
Our aim in this paper is to prove problem (3) with boundary and initial conditions (4), (5) has a unique global solution.

The next section, we set the notations, introduce the stochastic convolution and prove local existence in time.

## 2. Local Existence in Time

Define the unbounded self-adjoint operator \( A \) on \( L^2(0,2\pi) \) by
\[ Au = \rho \frac{\partial^2 u}{\partial x^2}, \]
for \( u \) on the domain
\[ D(A) = \{ u \in H^2(0,2\pi) : u(0) = u(2\pi) = 0 \}. \]
Denote \( e^{tA}, t \geq 0 \) the semigroup on \( L^2(0,2\pi) \) generated by \( A \). It is well known that \( e^{tA}, t \geq 0 \), has a natural extension, that we still denote by \( e^{tA}, t \geq 0 \), as a contraction semigroup on \( L^2(0,2\pi) \).
for any $p \geq 1$. Finally, we denote by $(e_k)$ the complete orthonormal system on $L^2(0,2\pi)$ which diagonalizes $A$ and $(\lambda_k)$ the corresponding eigenvalues. We have

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x, \quad k = 1, 2, \ldots$$

and

$$\lambda_k = -\pi^2 k^2, \quad k = 1, 2, \ldots.$$

Now, we rewrite (3), (4), (5) as the abstract differential stochastic equation

$$\begin{cases}
    du = (Au - \partial_x f(u)) dt + dW, \\
    u(0) = u_0.
\end{cases}$$

(6)

Recall that the solution to the linear problem

$$\begin{cases}
    du = Au dt + dW, \\
    u(0) = u_0
\end{cases}$$

(7)

is unique and given by the so-called stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s).$$

(8)

It can be shown that $W_A$ is a Gaussian process and it is mean square continuous with values in $L^2(0,2\pi)$. Moreover, $W_A$ has a version which is, a.s. for $\omega \in \Omega$, $\alpha$-Hölder continuous with respect to $(t,x)$ for any $\alpha \in [0,1/4]$. We set

$$v(t) = u(t) - W_A(t), \quad t \geq 0,$$

then $u$ satisfies (6) if and only if $v$ is a solution of

$$\begin{cases}
    \frac{dv}{dt} = Av - \partial_x f(v + W_A), \\
    v(0) = u_0.
\end{cases}$$

(9)

From now we will study equation (9) a.s. $\omega \in \Omega$ and consider for the moment that $W_A$ is an $\alpha$-Hölder continuous function with respect to $(t,x)$ for any $\alpha \in [0,1/4]$. We will return to the stochastic point of view (and to equation (6)) at the end of § 3.

Let us write (9) as

$$v(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A} \partial_x f(v + W_A) ds;$$

(10)

then if $v$ satisfies (10) we say that it is a mild solution of (9).

We are going to solve equation (10) by a fixed point argument in the space $C([0,T^*];L^p(0,2\pi))$ for $p > 1$ and for some $T^* > 0$. We set

$$\Sigma_p(m,T^*) = \{ v \in C([0,T^*];L^p(0,2\pi)) : |v(t)|_{L^p(0,2\pi)} \leq m, \text{ for all } t \in [0,T^*] \},$$

and consider an initial datum $u_0$ $\mathcal{F}_0$-measurable and belonging to $L^p(0,2\pi)$, $\omega \in \Omega$ a.s. We will see, in the proof of the Lemma 2.1 below that if $z(t)$ is, a bounded function from $[0,T]$ into $L^p(0,2\pi)$, then, for $t > 0$, the function $e^{tA} \frac{\partial}{\partial x} f(z)$ is also in $L^p(0,2\pi)$. Hence the integral in (10) is convergent in $L^p(0,2\pi)$ a.s. Thus (10) has a meaning as an equality in $L^p(0,2\pi)$.
Lemma 2.1. For any $p \geq 2$ and $m > |u_0|_{L^p(0,2\pi)}$, there exists a stopping time $T^* > 0$ such that (10) has a unique solution in $\Sigma_p(m, T^*)$.

Proof. Take any $v$ in $\Sigma_p(m, T^*)$ and define $z = Gv$ by

$$z(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}\partial_x f(v + W_A)ds,$$

where $G : C([0,T^*]; L^p(0,2\pi)) \to C([0,T^*]; L^p(0,2\pi))$ is a non-linear operator. Then

$$|z(t)|_{L^p(0,2\pi)} \leq |e^{tA}u_0|_{L^p(0,2\pi)} + \int_0^t |e^{(t-s)A}\partial_x f(v + W_A)|_{L^p(0,2\pi)}ds.$$

As we noticed before, $e^{tA}$, $t \geq 0$ is a contraction semigroup on $L^p(0,2\pi)$ which has a regularizing effect and, for any $s_1 \leq s_2$ in $\mathbb{R}$, and $r \geq 1$, $e^{tA}$ maps $W^{s_1,r}(0,2\pi)$ into $W^{s_2,r}(0,2\pi)$, for all $t > 0$. Moreover, the following estimate holds

$$|e^{tA}z|_{W^{s_2,r}(0,2\pi)} \leq C_1(t^{1-s_2} + 1)|z|_{W^{s_1,r}(0,2\pi)}$$

for all $z \in W^{s_1,r}(0,2\pi)$. The constant $C_1$ depends only on $s_1, s_2$ and $r$, see for instance (11).

Using the Sobolev embedding theorem we have

$$|e^{(t-s)A}\partial_x f(v + W_A)|_{L^p(0,2\pi)} \leq C_2|e^{(t-s)A}\partial_x f(v + W_A)|_{W^{1,p}(0,2\pi)}$$

and, thanks to (11) with $s_1 = -1, s_2 = 1/p, r = p/2$

$$|e^{(t-s)A}\partial_x f(v + W_A)|_{L^p(0,2\pi)} \leq C_1 C_2((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1)|\partial_x f(v + W_A)|_{W^{1,p}(0,2\pi)} \leq C_1 C_2((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1)|f(v + W_A)|_{L^p(0,2\pi)}.$$

Therefore,

$$|z(t)|_{L^p(0,2\pi)} \leq |u_0|_{L^p(0,2\pi)} + C_1 C_2 \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1)|f(v + W_A)|_{L^p(0,2\pi)} ds$$

$$\leq |u_0|_{L^p(0,2\pi)} + C_1 C_2 Lip_1 \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1)(1 + |v + W_A|_{L^p(0,2\pi)}) ds$$

$$\leq |u_0|_{L^p(0,2\pi)} + C_1 C_2 Lip_2 \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1)(1 + |v|_{L^p(0,2\pi)} + |W_A|_{L^p(0,2\pi)}) ds$$

$$\leq |u_0|_{L^p(0,2\pi)} + C_1 C_2 Lip_1 (1 + (2\pi)\frac{1}{p} m + \mu_1) \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{p}} + 1) ds$$

$$\leq |u_0|_{L^p(0,2\pi)} + C_1 C_2 Lip_1 (1 + (2\pi)\frac{1}{p} m + \mu_1) \left(\frac{2p}{p - 1}(T^*)^{-\frac{1}{2} - \frac{1}{p}} + T^*\right) \leq m.$$

where $Lip_1$ is the Lipschitz constant of $f$ which depend on $m + \mu_1$, and

$$\mu_1 = \sup_{t \in [0, T^*]} |W_A(t)|_{L^p(0,2\pi)}.$$

Hence $|z(t)|_{L^p(0,2\pi)} \leq m$ for all $t \in [0, T^*]$ provided

$$|u_0|_{L^p(0,2\pi)} + C_1 C_2 Lip_1 (1 + (2\pi)\frac{1}{p} m + \mu_1) \left(\frac{2p}{p - 1}(T^*)^{-\frac{1}{2} - \frac{1}{p}} + T^*\right) \leq m. \tag{12}$$

It is clear that for any $m > |u_0|_{L^p(0,2\pi)}$ there exists a $T^*$ satisfying (12). Now consider $v_1, v_2 \in \Sigma_p(m, T^*)$ and set $z_i = Gv_i, i = 1, 2$ and $z = z_1 - z_2$. 
Then
\[
z(t) = \int_0^t e^{(t-s)A} \frac{\partial}{\partial x} [f(u_1 + W_A) - f(u_2 + W_A)] ds,
\]
and we derive as above
\[
|z(t)|_{L^p(0, 2\pi)} \leq C_1 C_2 \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{2p}} + 1)|f(u_1 + W_A) - f(u_2 + W_A)|_{L^p(0, 2\pi)} ds.
\]
According to the hypothesis on \(f\), we have
\[
|f(u_1 + W_A) - f(u_2 + W_A)|_{L^p(0, 2\pi)} \leq Lip_2 |u_1 - u_2|_{L^p(0, 2\pi)} \leq Lip_2(2\pi)^{\frac{1}{p}} |u_1 - u_2|_{L^p(0, 2\pi)} = C_3 |u_1 - u_2|_{L^p(0, 2\pi)},
\]
where \(Lip_2\) is the Lipschitz constant of \(f\) which depend on \(m + \mu_p\), and \(C_3 = Lip_2(2\pi)^{\frac{1}{p}}\), hence
\[
|z(t)|_{L^p(0, 2\pi)} \leq C_1 C_2 C_3 \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{2p}} + 1)|u_1 - u_2|_{L^p(0, 2\pi)} ds
\]
\[
\leq C \max_{0 \leq s \leq t} |u_1(s) - u_2(s)|_{L^p(0, 2\pi)} \int_0^t ((t-s)^{-\frac{1}{2} - \frac{1}{2p}} + 1) ds
\]
\[
\leq C \left( \frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^* \right) |u_1 - u_2|_{C([0, T^*]; L^p(0, 2\pi))}
\]
for all \(t \in [0, T^*]\) provided
\[
|Gv_1 - Gv_2|_{C([0, T^*]; L^p(0, 2\pi))} \leq C \left( \frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^* \right) |u_1 - u_2|_{C([0, T^*]; L^p(0, 2\pi))}.
\]
We take \(T^*\) such that
\[
C \left( \frac{2p}{p-1} (T^*)^{\frac{1}{2} - \frac{1}{2p}} + T^* \right) < 1
\]
and (12) holds so that \(G\) is a strict contraction on \(\Sigma_p(m, T^*)\). \(\square\)

**Remark 2.1 ([4]).** As mentioned before all the previous results are valid a.s. for \(\omega \in \Omega\); in particular \(\mu_p\) and \(T^*\) depend on \(\omega\). In the next section we will show that \(T^* = T\) a.s. for \(\omega \in \Omega\) and hence remove the dependence on \(\omega\) for the time interval on which the solution exists.

### 3. Global Existence

We are still considering equation (10) as a deterministic one, working a.s. for \(\omega \in \Omega\).

**Theorem 3.1** (Global existence). Let \(u_0\) be given which is \(\mathcal{F}_0\)-measurable and such that for some \(p \geq 2\), \(u_0 \in L^p(0, 2\pi)\) a.s. If \(\rho \geq \frac{Lip_1 c}{2}\) then there exists a unique mild solution of equation (6), which belongs a.s. to \(C([0, T]; L^p(0, 2\pi))\).

In the following lemma, we derive a priori estimate which yields global existence.

**Lemma 3.1.** If \(v \in C([0, T]; L^p(0, 2\pi))\) satisfies (10) and \(\rho \geq \frac{Lip_1 c}{2}\), then
\[
|v(t)|_{L^p(0, 2\pi)} \leq e^{T Lip_1 e^{\frac{\rho - 1}{2} \mu_0^2}} |u_0|_{L^p(0, 2\pi)},
\]
where \(c = (2\pi)^{\frac{1}{p} - \frac{1}{2p}}\) and \(\mu_\infty = \sup_{t \in [0, T]} |W_A(t)|_{L^\infty(0, 2\pi)}\).
We integrate by parts the last integral
\[ u^n_0 \rightarrow u_0, \text{ in } L^p(0,2\pi), \]
and let \( \{W^n\} \) be a sequence of regular processes such that
\[ W^n_A(t) = \int_0^t e^{(t-s)A} dW^n(s) \rightarrow W_A(t) \]
in \( C([0,T] \times [0,2\pi]) \) a.s. for \( \omega \in \Omega \).

Let \( v^n \) be the solution of
\[ v^n(t) = e^{tA}u^n_0 - \int_0^t e^{(t-s)A} \partial_x f(v^n + W^n_A) \, ds \]
given by Lemma 2.1. It is easy to see that \( v^n \) does exist on an interval of time \([0,T_n]\) such that \( T_n \rightarrow T^* \) a.s. and that \( v^n \) converges to \( v \) in \( C([0,T^*];L^p(0,2\pi)) \) a.s. Moreover, \( v^n \) is regular a.s. and satisfies
\[ \frac{\partial v^n}{\partial t} - \rho \frac{\partial^2 v^n}{\partial x^2} + \partial_x f(v^n + W^n_A) = 0. \]  
(13)

Multiplying (13) by \(|v^n|^{p-2} v^n\) and integrating over \([0,2\pi]\), we find
\[ \frac{1}{p} \frac{\partial}{\partial t} |v^n|_{L^p(0,2\pi)}^p + \rho(p-1) \int_0^{2\pi} |v^n|^{p-2} \left| \frac{\partial v^n}{\partial x} \right|^2 \, dx + \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W^n_A) |v^n|^{p-2} v^n \, dx = 0. \]  
(14)

We integrate by parts the last integral
\[ \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W^n_A) |v^n|^{p-2} v^n \, dx = - \int_0^{2\pi} f(v^n + W^n_A) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx, \]
then
\[ \left| \int_0^{2\pi} \frac{\partial}{\partial x} f(v^n + W^n_A) |v^n|^{p-2} v^n \, dx \right| = (p-1) \left| \int_0^{2\pi} f(v^n + W^n_A) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx \right| \]
\[ \leq (p-1) \int_0^{2\pi} \left| f(v^n + W^n_A) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \right| \, dx \]
\[ \leq (p-1) \int_0^{2\pi} \text{Lip}_1(1 + |v^n + W^n_A|) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx \]
\[ \leq (p-1) \int_0^{2\pi} \text{Lip}_1 |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx \]
\[ + (p-1) \int_0^{2\pi} \text{Lip}_1 |W^n_A| |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx \].

The first term is zero, indeed
\[ \int_0^{2\pi} |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx = - \int_0^{2\pi} (p-2) |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx. \]
Hence
\[ (p-1) \int_0^{2\pi} |v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx = 0. \]
In the same way, we can prove that the second term is also zero.
According to the Hölder’s and Cauchy’s inequalities we bound the third term as follows

\[ Lip_1(p-1) \int_0^{2\pi} |W_A^n|v^n|^{p-2} \frac{\partial}{\partial x} v^n \, dx \]

\[ \leq Lip_1(p-1)|W_A^n|L^{\infty}(0,2\pi)|v^n|^{p-2}L^{p-2}(0,2\pi) \left( \int_0^{2\pi} |v^n|^{p-2} \left( \frac{\partial}{\partial x} v^n \right)^2 \, dx \right)^{\frac{1}{2}} \]

\[ \leq Lip_1 c (p-1) \mu_{n,\infty} |v^n|^{p-2}L^{p}(0,2\pi) + Lip_1 c (p-1) \frac{1}{2} \int_0^{2\pi} |v^n|^{p-2} \left( \frac{\partial}{\partial x} v^n \right)^2 \, dx, \]

where \( c = (2\pi)^{\frac{2}{p-2}} \) and \( \mu_{n,\infty} = \sup_{t \in [0,T]} |W_A^n(t)|L^{\infty}(0,2\pi) \) for a.s. \( \omega \in \Omega \).

Going back to (14) we obtain

\[ \frac{1}{p} \frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} + \rho (p-1) \int_0^{2\pi} |v^n|^{p-2} \left( \frac{\partial v^n}{\partial x} \right)^2 \, dx \]

\[ \leq Lip_1 c (p-1) \mu_{n,\infty} |v^n|^{p-2}L^{p}(0,2\pi) + Lip_1 c (p-1) \frac{1}{2} \int_0^{2\pi} |v^n|^{p-2} \left( \frac{\partial v^n}{\partial x} \right)^2 \, dx. \]

It follows

\[ \frac{1}{p} \frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} + (p-1) \left( \rho - \frac{Lip_1 c}{2} \right) \int_0^{2\pi} |v^n|^{p-2} \left( \frac{\partial v^n}{\partial x} \right)^2 \, dx \leq Lip_1 c (p-1) \frac{1}{2} \mu_{n,\infty} |v^n|^{p-2}L^{p}(0,2\pi). \]

If we take \( \rho \) and \( Lip_1 \) such that

\[ \rho \geq \frac{Lip_1 c}{2}. \]

We obtain

\[ \frac{\partial}{\partial t} |v^n|^p_{L^p(0,2\pi)} \leq Lip_1 c \frac{p(p-1)}{2} \mu_{n,\infty} |v^n|^p_{L^p(0,2\pi)} \]

and, according to Gronwall’s lemma

\[ |v^n|^p_{L^p(0,2\pi)} \leq e^{t Lip_1 c \frac{p(p-1)}{2} \mu_{n,\infty}} |u_0|^p_{L^p(0,2\pi)}. \]

Taking the limit as \( n \to \infty \), we see that a.s.

\[ |v|^p_{L^p(0,2\pi)} \leq e^{t Lip_1 c \frac{p(p-1)}{2} \mu_{\infty}} |u_0|^p_{L^p(0,2\pi)}. \]

It follows

\[ |v|^p_{L^p(0,2\pi)} \leq e^{t Lip_1 c \frac{p(p-1)}{2} \mu_{\infty}} |u_0|^p_{L^p(0,2\pi)}. \]

and the assertion of the lemma follows.

**Proof of Theorem 3.1** It is easily deduced from Lemma 2.1 and Lemma 3.1.

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All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

**References**


