Research Article

Gaussian Pell-Lucas Polynomials

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Abstract. In this paper, we first define the Gaussian Pell-Lucas polynomial sequence. We then obtain Binet formula, generating function and determinantal representation of this sequence. Also, some properties of the Gaussian Pell-Lucas polynomials are investigated.

Keywords. Pell-Lucas numbers; Gaussian Pell-Lucas numbers; Gaussian Pell-Lucas polynomials

MSC. 11B37; 11B39; 11B75; 11B83

Received: February 4, 2018 Accepted: March 25, 2019

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1. Introduction

In 1963, the complex Fibonacci numbers are introduced by Horadam [6]. After this seminal paper, Gaussian Fibonacci, Lucas, Pell and Pell-Lucas numbers are studied by many authors [2,3,5,8]. The Gaussian Fibonacci and Lucas numbers are defined recursively by the relations $GF_{n+1} = GF_n + GF_{n-1}$, where $GF_0 = i$, $GF_1 = 1$, and $GL_{n+1} = GL_n + GL_{n-1}$ with initial conditions $GL_0 = 2 - i$, $GL_1 = 1 + 2i$, respectively. Also, the Gaussian Pell numbers are defined recursively by $GP_{n+1} = 2GP_n + GP_{n-1}$ with initial conditions $GP_0 = i$, $GP_1 = 1$, and the Gaussian Pell-Lucas numbers are defined as $GQ_{n+1} = 2GQ_n + GQ_{n-1}$, where $GQ_0 = 2 - 2i$, $GQ_1 = 2 + 2i$.

On the other hand, the Pell polynomial sequence is defined by the recurrence relation $P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x)$, where $P_0(x) = 0$, $P_1(x) = 1$. Similarly, the Pell-Lucas polynomial sequence is defined as $Q_0(x) = 2$, $Q_1(x) = 2x$, and $Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x)$. Moreover, some properties related with these sequences are studied by Horadam and Mahon [7].
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In [4], Halici and Oz introduced the Gaussian Pell polynomials satisfied the recurrence relation 
\[ GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x), \]
where \( GP_0(x) = i \) and \( GP_1(x) = 1 \). In a similar way, the Gaussian Jacobsthal and Jacobsthal-Lucas polynomials are studied in [1] by Asci and Gurel.

The main objective of this paper is to define the Gaussian Pell-Lucas polynomials, and to investigate some properties of these polynomials.

In Section 2, we define the Gaussian Pell-Lucas polynomial sequence that generalize the Gaussian Pell-Lucas number sequence given in [3]. Moreover, we give the generating function and Binet formula for the Gaussian Pell-Lucas polynomial sequence. We also obtain summation formula and determinantal representation of this sequence. In the rest of Section 2, by using Binet formula, we give well-known identities such as Catalan’s and d’Ocagne’s identities involving the Gaussian Pell-Lucas polynomial sequence.

### 2. Main Results

In this section, we first define the Gaussian Pell-Lucas polynomial sequence. Then we give generating function, Binet formula, determinantal representation and some properties of this sequence.

**Definition 2.1.** The Gaussian Pell-Lucas polynomial sequence \( \{GQ_n(x)\}_{n=0}^{\infty} \) is defined, for \( n \geq 1 \), recursively by

\[ GQ_{n+1}(x) = 2xGQ_n(x) + GQ_{n-1}(x) \]

with initial conditions \( GQ_0(x) = 2 - 2xi \) and \( GQ_1(x) = 2x + 2i \).

Clearly, if we take \( x = 1 \), we obtain the Gaussian Pell-Lucas numbers. Also, it is easy to see that

\[ GQ_n(x) = Q_n(x) + iQ_{n-1}(x), \]

where \( Q_n(x) \) is the \( n \)th Pell-Lucas polynomial.

The first few terms of the Gaussian Pell-Lucas polynomials are: \( 2 - 2xi, 2x + 2i, 4x^2 + 2 + 2xi, 8x^3 + 6x + (4x^2 + 2)i, 16x^4 + 16x^2 + 2 + (8x^3 + 6x)i. \)

We now give the generating function for the Gaussian Pell-Lucas polynomials by the following:

**Theorem 2.2.** The generating function of the Gaussian Pell-Lucas polynomial sequence \( \{GQ_n(x)\}_{n=0}^{\infty} \) denoted by \( g(t,x) \) is

\[ g(t,x) = \frac{2 - 2xt + (4x^2t + 2t - 2x)i}{1 - 2xt - t^2}. \]

**Proof.** The generating function for the sequence \( \{GQ_n(x)\}_{n=0}^{\infty} \) can be written in power series. Then, we have

\[ g(t,x) = \sum_{n=0}^{\infty} GQ_n(x)t^n = GQ_0(x) + GQ_1(x)t + GQ_2(x)t^2 + GQ_3(x)t^3 + GQ_4(x)t^4 \ldots, \]
The nth term of the Gaussian Pell-Lucas polynomial sequence is given by

\[ GQ_n(x) = a^n(x) + b^n(x) - [\beta(x)a^n(x) + \alpha(x)b^n(x)]i, \]

where \( a(x) = x + \sqrt{1 + x^2} \) and \( \beta(x) = x - \sqrt{1 + x^2} \) are the roots of the equation \( r^2 - 2xr - 1 = 0. \)

**Proof.** It is known that the general solution for the recurrence relation is given by \( GQ_n(x) = c_1a^n(x) + c_2b^n(x), \) where \( c_1 \) and \( c_2 \) are any constants. Plugging the general solution in the initial conditions gives the system

\[ c_1 + c_2 = 2 - 2xi, \quad c_1(x + \sqrt{1 + x^2}) + c_2(x - \sqrt{1 + x^2}) = 2x + 2i. \]

Then we obtain \( c_1 = 1 - \beta(x)i \) and \( c_2 = 1 - \alpha(x)i. \) Therefore, we get

\[ GQ_n(x) = a^n(x) + b^n(x) - \beta(x)a^n(x)i - \alpha(x)b^n(x)i \]

which completes the proof.

**Theorem 2.4.** For \( n \geq 1, \) the sum of the Gaussian Pell-Lucas polynomials is

\[ \sum_{k=1}^{n} GQ_k(x) = \frac{1}{2x}[GQ_{n+1}(x) + GQ_n(x) - 2x - 2(2x-2)i]. \]

**Proof.** From the recurrence relation of the Gaussian Pell-Lucas polynomial sequence, we have

\[ GQ_n(x) = \frac{1}{2x}(GQ_{n+1}(x) - GQ_{n-1}(x)). \]

Then, we get

\[ GQ_1(x) = \frac{1}{2x}(GQ_2(x) - GQ_0(x)) \]
\[ GQ_2(x) = \frac{1}{2x}(GQ_3(x) - GQ_1(x)) \]
\[ GQ_3(x) = \frac{1}{2x}(GQ_4(x) - GQ_2(x)) \]
\[ \vdots \]
\[ GQ_{n-1}(x) = \frac{1}{2x}(GQ_n(x) - GQ_{n-2}(x)) \]
\[ GQ_n(x) = \frac{1}{2x} (GQ_{n+1}(x) - GQ_{n-1}(x)) \]

Thus, we obtain
\[
\sum_{k=1}^{n} GQ_k(x) = \frac{1}{2x} [GQ_{n+1}(x) + GQ_n(x) - GQ_{n+1}(x) - GQ_0(x)] \\
= \frac{1}{2x} [GQ_{n+1}(x) + GQ_n(x) - 2x - 2 + (2x - 2)i].
\]

This completes the proof. \[\square\]

The following corollary follows from the above theorem.

**Theorem 2.5.** For \( n \geq 1 \), we have

(i) \[ \sum_{k=1}^{n} \sqrt{2} GQ_{2k}(x) = \frac{1}{2x} (GQ_{2n+1}(x) - 2x - 2i), \]

(ii) \[ \sum_{k=1}^{n} \sqrt{2} GQ_{2k-1}(x) = \frac{1}{2x} (GQ_{2n}(x) - 2 + 2i). \]

**Theorem 2.6.** For \( n \geq 1 \), let \( L_n(x) \) be an \( n \times n \) tridiagonal matrix defined by

\[ L_n(x) = \begin{pmatrix}
2x + 2i & 1 & 0 & 0 & \cdots & 0 \\
-2 + 2xi & 2x & 1 & 0 & \cdots & 0 \\
0 & -1 & 2x & 1 & \ddots & 0 \\
0 & 0 & -1 & 2x & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & 2x
\end{pmatrix} \]

and let \( L_0(x) = 2 - 2xi \). Then
\[ \det(L_n(x)) = GQ_n(x). \]

**Proof.** For the proof we use the mathematical induction on \( n \). For \( n = 1 \) and \( n = 2 \), we get
\[ \det(L_1(x)) = 2x + 2i = GQ_1(x) \quad \text{and} \quad \det(L_2(x)) = 4x^2 + 2 + 2xi = GQ_2(x). \]

Let us assume that the equality holds for \( n - 1 \) and \( n - 2 \), that is,
\[ \det(L_{n-1}(x)) = GQ_{n-1}(x) \quad \text{and} \quad \det(L_{n-2}(x)) = GQ_{n-2}(x). \]

Finally, for \( n \), we get
\[ \det(L_n(x)) = 2x \det(L_{n-1}(x)) + \det(L_{n-2}(x)) = 2xGQ_{n-1}(x) + GQ_{n-2}(x) \]

which completes the proof. \[\square\]

Now, we define the matrices \( Q \) and \( P \) as followings:
\[ Q = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 4x^2 + 2 + 2xi & 2x + 2i \\ 2x + 2i & 2 - 2xi \end{pmatrix}. \]

**Theorem 2.7.** For \( n \geq 1 \), we have
\[ Q^n P = \begin{pmatrix} GQ_{n+2}(x) & GQ_{n+1}(x) \\ GQ_{n+1}(x) & GQ_n(x) \end{pmatrix}. \]
Proof. The proof can be done easily by using the mathematical induction on \( n \).

The consequence of Theorem 2.7 which gives the Cassini’s identity for the Gaussian Pell-Lucas polynomial sequence is the following:

**Theorem 2.8** (Cassini’s Identity). For positive integer \( n \), we have

\[
GQ_{n-1}(x)GQ_{n+1}(x) - GQ_n^2(x) = 8(-1)^{n-1}(1 + x^2)(1 - xi).
\]

Proof. It is obvious that \( \det Q^{n-1} = (-1)^{n-1} \) and \( \det P = 8(1 + x^2)(1 - xi) \). By taking determinant of the matrix

\[
Q^{n-1}P = \begin{pmatrix} GQ_{n+1}(x) & GQ_n(x) \\ GQ_n(x) & GQ_{n-1}(x) \end{pmatrix},
\]

we get

\[
GQ_{n-1}(x)GQ_{n+1}(x) - GQ_n^2(x) = 8(-1)^{n-1}(1 + x^2)(1 - xi).
\]

Now, Catalan’s and d’Ocagne’s identities for the Gaussian Pell-Lucas polynomial sequence are given in the following theorems, respectively.

**Theorem 2.9** (Catalan’s Identity). For positive integers \( n \) and \( r \), we have

\[
GQ_{n-r}(x)GQ_{n+r}(x) - GQ_n^2(x) = 2(-1)^{n-r}(1 - xi)(a^r(x) - \beta^r(x))^2.
\]

Proof. From the Binet formula of the sequence \( \{GQ_n(x)\}_{n=0}^\infty \), we get

\[
GQ_{n-r}(x)GQ_{n+r}(x) - GQ_n^2(x) = \{a^{n-r}(x) + \beta^{n-r}(x) - [\beta(x)a^{n-r}(x) + \alpha(x)\beta^{n-r}(x)]i\}
\]

\[
\times \{a^{n+r}(x) + \beta^{n+r}(x) - [\beta(x)a^{n+r}(x) + \alpha(x)\beta^{n+r}(x)]i\}
\]

\[- \{a^n(x) + \beta^n(x) - [\beta(x)a^n(x) + \alpha(x)\beta^n(x)]i\}^2
\]

\[= (a(x)\beta(x))^{n-r}([a^{2r}(x) + \beta^{2r}(x) - 2a^r(x)\beta^r(x)](1 - (a(x)\beta(x)))
\]

\[- i(a(x)\beta(x))^{n-r}(a(x) + \beta(x))[a^{2r}(x) + \beta^{2r}(x) - 2a^r(x)\beta^r(x)]
\]

\[= (a(x)\beta(x))^{n-r}(a^r(x) - \beta^r(x))^2[1 - (a(x)\beta(x)) - i(a(x) + \beta(x))].
\]

Since \( a(x)\beta(x) = -1 \) and \( a(x) + \beta(x) = 2x \), we obtain

\[GQ_{n-r}(x)GQ_{n+r}(x) - GQ_n^2(x) = (-1)^{n-r}(a^r(x) - \beta^r(x))^2(2 - 2xi)
\]

which completes the proof.

Note that if we set \( r = 1 \) in Theorem 2.9, Cassini’s identity of the Gaussian Pell-Lucas polynomial sequence, which is given in Theorem 2.8, can be obtained again.

**Theorem 2.10** (d’Ocagne’s Identity). Let \( m \) and \( n \) be any positive integers. Then,

\[
GQ_m(x)GQ_{m+1}(x) - GQ_n(x)GQ_{n+1}(x) = 4(-1)^{n+1}\sqrt{1 + x^2}(1 - xi)(a^{m-n}(x) - \beta^{m-n}(x)).
\]

Proof. By using the Binet formula of the sequence \( \{GQ_n(x)\}_{n=0}^\infty \), we get

\[
GQ_m(x)GQ_{m+1}(x) - GQ_n(x)GQ_{n+1}(x)
\]

\[= \{a^m(x) + \beta^m(x) - [\beta(x)a^m(x) + \alpha(x)\beta^m(x)]i\} \{a^{n+1}(x) + \beta^{n+1}(x) - [\beta(x)a^{n+1}(x) + \alpha(x)\beta^{n+1}(x)]i\}
\]

\[= \{a^{m-n}(x) + \beta^{m-n}(x) - [\beta(x)a^{m-n}(x) + \alpha(x)\beta^{m-n}(x)]i\} \{a^{m-n}(x) + \beta^{m-n}(x) - [\beta(x)a^{m-n}(x) + \alpha(x)\beta^{m-n}(x)]i\}
\]

\[= 4(-1)^{n+1}\sqrt{1 + x^2}(1 - xi)(a^{m-n}(x) - \beta^{m-n}(x)).
\]
\[-\{\alpha^n(x) + \beta^n(x) - [\beta(x)\alpha^n(x) + \alpha(x)\beta^n(x)]i\}\{\alpha^{m+1}(x) + \beta^{m+1}(x) - [\beta(x)\alpha^{m+1}(x) + \alpha(x)\beta^{m+1}(x)]i\}\]
\[
= (\alpha(x) - \beta(x))[\alpha^n(x)\beta^m(x) - \alpha^m(x)\beta^n(x)] + i(\alpha^2(x) - \beta^2(x))[\alpha^m(x)\beta^n(x) - \alpha^n(x)\beta^m(x)]
\]
\[
= -2(\alpha(x) - \beta(x))[\alpha^n(x)\beta^m(x) - \alpha^m(x)\beta^n(x)] + i(\alpha^2(x) - \beta^2(x))[\alpha^m(x)\beta^n(x) - \alpha^n(x)\beta^m(x)]
\]
\[
= (\alpha(x) - \beta(x))(\alpha(x)\beta(x))\alpha^{m-n}(x) - \beta^{m-n}(x)][-2 + i(\alpha(x) + \beta(x))]
\]
\[
= 4(-1)^{n+1}\sqrt{1 + x^2}(1 - xi)(\alpha^{m-n}(x) - \beta^{m-n}(x)).
\]
This completes the proof. \(\square\)

### 3. Conclusion

In this study, we introduce the concept of the Gaussian Pell-Lucas polynomials. We also give some results including Binet formula, generating function, summation formula and determinantal representation for these polynomials. Moreover, we obtain some well-known identities, such as Catalan’s, Cassini’s and d'Ocagne’s identities, involving the Gaussian Pell-Lucas polynomials. In future, we plan to investigate some others identities and properties for these polynomials.

#### Acknowledgement

The author would like to thank the anonymous referee for his/her valuable comments and suggestions.

#### Competing Interests

The author declares that he has no competing interests.

#### Authors’ Contributions

The author wrote, read and approved the final manuscript.

#### References
