Double Total Domination in Harary Graphs

Adel P. Kazemi* and Behnaz Pahlavsay

Department of Mathematics, University of Mohaghegh Ardabili, P.O. Box 5619911367, Ardabil, Iran
*Corresponding author: adelpkazemi@yahoo.com

Abstract. Let $G$ be a graph with minimum degree at least 2. A set $D \subseteq V$ is a double total dominating set of $G$ if each vertex is adjacent to at least two vertices in $D$. The double total domination number $\gamma_{\times 2,t}(G)$ of $G$ is the minimum cardinality of a double total dominating set of $G$. In this paper, we will find double total domination number of Harary graphs.

Keywords. Double total domination number; Harary graph

MSC. 05C69

Received: November 7, 2013 Accepted: December 31, 2013

Copyright © 2017 Adel P. Kazemi and Behnaz Pahlavsay. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let $G$ be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. The open neighborhood and the closed neighborhoods of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

The research of domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi and Slater [1, 2]. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. For example, the $k$-tuple total domination number is defined in [3] by Henning and Kazemi, which is an extension of the total domination number (for more information see [4, 5]).
Here, we find double domination number of Harary graphs $H_{m,n}$. The next known results are useful for our investigations.

**Definition 1.1** ([3]). Let $k \geq 1$ be an integer and let $G$ be a graph with $\delta(G) \geq k$. A subset $S \subseteq V(G)$ is called a $k$-tuple total dominating set, briefly $kTDS$, in $G$, if for each $x \in V(G)$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a $k$-tuple total dominating set in a graph $G$ is called the $k$-tuple total domination number of $G$ and denoted by $\gamma_{\times,k,t}(G)$.

When $k = 2$, $k$-tuple total dominating set and $k$-tuple total domination number are known as double total dominating set and double total domination number, respectively.

**Harary Graph** ([7]). Given $m < n$, place $n$ vertices $1, 2, \ldots, n$ around a circle, equally spaced. If $m$ is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m}{2}$ vertices in each direction around the circle. If $m$ is odd and $n$ is even, form $H_{m,n}$ by making each vertex adjacent to the nearest $\frac{m-1}{2}$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{m,n}$ is $m$-regular. When $m$ and $n$ are both odd, index the vertices by the integers modulo $n$. Construct $H_{m,n}$ from $H_{m-1,n}$ by adding the edges $i \leftrightarrow i + \frac{n-1}{2}$ for $0 \leq i \leq \frac{n-1}{2}$.

Here, we find double domination number of Harary graphs $H_{m,n}$. The next known results are useful for our investigations.

**Proposition 1.2** (Henning and Kazemi [3]). For any graph $G$ of order $n$ with $\delta(G) \geq k$,

(i) $\max(\gamma_{\times,k}(G),k + 1) \leq \gamma_{\times,k,t}(G) \leq n$,

(ii) if $G$ is a spanning subgraph of a graph $H$, then $\gamma_{\times,k,t}(H) \leq \gamma_{\times,k,t}(G)$,

(iii) if $v$ is a degree-$k$ vertex in $G$, then $N_G(v)$ is a subset of every $kTDS$ of $G$.

**Proposition 1.3** (Henning and Kazemi [4]). Let $G$ be a graph of order with minimum degree at least $k$. Then $\gamma_{\times,k,t}(G) \geq \left\lceil \frac{kn}{\Delta(G)} \right\rceil$.

2. Harary Graphs

As a first result of Proposition 1.3, we obtain the next result.

**Proposition 2.1.**

(i) $\gamma_{\times,k,t}(H_{2m,n}) \geq \left\lceil \frac{kn}{2m} \right\rceil$,

(ii) $\gamma_{\times,k,t}(H_{2m+1,2n}) \geq \left\lceil \frac{2kn}{2m + 1} \right\rceil$,

(iii) $\gamma_{\times,k,t}(H_{2m+1,2n+1}) \geq \left\lceil \frac{k(2n + 1)}{2m + 2} \right\rceil$.

In this section, we calculate double total domination number of Harary graphs.

**Theorem 2.2.** Let $G = H_{2m,n}$ be a Harary graph. Then $\gamma_{\times,2t}(G) = \left\lceil \frac{n}{m} \right\rceil$.
Proof. Let $V(G) = \{i \mid 1 \leq i \leq n\}$. Since $S = \{im + 1 \mid 0 \leq i \leq \left\lfloor \frac{n}{m} \right\rfloor - 1\}$ is a 2TDS of $G$, Proposition 2.1(i) implies $\gamma_{x_2,t}(G) = \left\lfloor \frac{n}{m} \right\rfloor$.

Theorem 2.3. Let $G = H_{2m+1,2n+1}$ be a Harary graph. Let $2n = (2m + 1)\ell + r$, where $0 \leq r < 2m$, $\ell \geq 1$. Also let $\ell + r = 2m\ell' + r'$, where $0 \leq r' < 2m$ and $\ell' \geq 0$. Then

$$\left\lfloor \frac{4n}{2m + 1} \right\rfloor \leq \gamma_{x_2,t}(G) \leq \left\lfloor \frac{4n}{2m + 1} \right\rfloor + 1,$$

if $1 \leq r \leq m$ and $(r', \ell') \not\in \{(1, 2, \ldots, m) \times \{0\}$, and $\gamma_{x_2,t}(G) = \left\lfloor \frac{4n}{2m + 1} \right\rfloor$ otherwise.

Proof. Let $V(G) = \{i \mid 1 \leq i \leq 2n\}$. Proposition 2.1(ii) implies that $\gamma_{x_2,t}(G) \geq \left\lfloor \frac{4n}{2m + 1} \right\rfloor = \begin{cases} 2\ell & \text{if } r = 0, \\ 2\ell + 1 & \text{if } 1 \leq r \leq m, \\ 2\ell + 2 & \text{if } m + 1 \leq r \leq 2m, \end{cases}$

where $r$ and $\ell$ are both even or both odd. If $(\ell, r, m) = (1, 1, 1)$, then $G = K_4$ and $\gamma_{x_2,t}(G) = 3 = \left\lfloor \frac{8}{3} \right\rfloor$.

Now let $(\ell, r, m) \neq (1, 1, 1)$. If $r = 0$, set $S = ((2m + 1)i + 1, (2m + 1)i + (m + 1)|0 \leq i \leq \ell - 1|,$

which is a 2TDS of cardinality $\left\lfloor \frac{4n}{2m + 1} \right\rfloor$. Now let $r \neq 0$. If $\ell' = 0$ and $1 \leq r' < m$, set $S = (2mi + 1, 2mi + (m + 1)|0 \leq i \leq \ell - 1| \cup 2n - m + 1),$

which is a 2TDS of $G$ of cardinality $2\ell + 1 = \left\lfloor \frac{4n}{2m + 1} \right\rfloor$. Otherwise, for odd $r$, set

$$S = \left\{(2m + 1)i + (n + m + 1), (2m + 1)i + (n + 2m + 2)|0 \leq i \leq \left\lfloor \frac{n}{2m + 1} \right\rfloor \right\}$$

$$\cup \left\{(2m + 1)i + 1, (2m + 1)i + (m + 1)|0 \leq i \leq \left\lfloor \frac{n}{2m + 1} \right\rfloor - 1\right\}$$

$$\cup \left\{n + 1, (2m + 1)\left\lfloor \frac{n}{m + 1} \right\rfloor - 1 \right\} + (n + m + 1),$$

and for even $r$, set

$$S = \left\{(2m + 1)i + (n + 1), (2m + 1)i + (n + m + 1)|0 \leq i \leq \frac{\ell}{2} - 1\right\}$$

$$\cup \left\{(2m + 1)i + 1, (2m + 1)i + (m + 1)|0 \leq i \leq \frac{\ell}{2} - 1\right\}$$

$$\cup \left\{n + 1 - \frac{r}{2}. \left\lfloor 2n + 1 - \frac{r}{2} \right\rfloor \right\}. 
$$

In each case, $S$ is a 2TDS of $G$ of cardinality $2\ell + 2$, and this completes our proof. \hfill \Box

Theorem 2.4. Let $G = H_{2m+1,2n+1}$ be a Harary graph. Let $2n + 1 = (2m + 1)\ell + r$, where $0 \leq r < 2m$ and $\ell \geq 1$. Also let $\ell + r = 2m\ell' + r'$, where $0 \leq r' < 2m$, and $\ell' \geq 0$. Then

$$\gamma_{x_2,t}(G) = \left\lfloor \frac{4n + 1}{2m + 1} \right\rfloor$$
Therefore one can verify that\[ S \subseteq V(G) \]
This is counted at least two times in the union of the neighborhoods of the vertices of \( S \), we have \( \sum_{j \in S} \deg(j) \geq 2(2n + 1) \). So \( |S|(2m + 1) + 1 \geq 2(2n + 1) \), and this implies \( |S| \geq \left\lceil \frac{4n + 1}{2m + 1} \right\rceil \). Hence \( \gamma_{x,2}(G) \geq \left\lceil \frac{4n + 1}{2m + 1} \right\rceil \).

Now let \( 2n + 1 = 2m \ell + (\ell + r) \), and let \( \ell + r = 2m \ell' + r' \), where \( 0 \leq r' < 2m \) and \( \ell' \geq 0 \). Notice that \( \ell \) is odd if and only if \( r \) is even. Recall that
\[
\left\lceil \frac{4n + 1}{2m + 1} \right\rceil = \begin{cases} 
2\ell & \text{if } r = 0, \\
2\ell + 1 & \text{if } 1 \leq r \leq m, \\
2\ell + 2 & \text{if } m + 1 \leq r \leq 2m.
\end{cases}
\]

Let \( \alpha_i = (2m + 1)i \) be an arbitrary vertex of \( G \). Continue the proof in the following cases.

**Case 1:** \( 0 \leq r \leq 1 \).
For \( r = 0 \), set
\[
S_0 = \{ \alpha_i + 1, \alpha_i + (m + 1), \alpha_i + (n + 1), \alpha_i + (n + m + 1) \mid 0 \leq i \leq \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \} \\
\quad \cup \{ n - m + 1, 2n - m + 1, 2n + 1 \},
\]
and for \( r = 1 \), set
\[
S_1 = (S_0 - \{ n - m + 1, 2n - m + 1, 2n + 1 \}) \cup \{ 2n + 1 \}.
\]
Then \( S_0 \) and \( S_1 \) are two double total dominating sets of \( G \) of cardinality \( 2\ell + 1 \).

**Case 2:** \( 2 \leq r \leq 2m \).
Let \( 1 \leq r' = \ell + r \leq m \). Then \( S = \{ 2mi + 1, 2mi + (m + 1) \mid 0 \leq i \leq \ell - 1 \} \cup \{ 2n + 2 - m \} \) is a 2TDS of \( G \) of cardinality \( 2\ell + 1 \). Otherwise, if \( r \) is odd, set
\[
S_0 = \{ \alpha_i + 1, \alpha_i + (m + 1), \alpha_i + (n + 1), \alpha_i + (n + m + 1) \mid 0 \leq i \leq \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \} \\
\quad \cup \{ n + 1 - \left( \frac{r - 1}{2} \right), 2n + 1 - \left( \frac{r - 1}{2} \right) \},
\]
and for \( r = 1 \), set
\[
S_1 = (S_0 - \{ n - m + 1, 2n - m + 1, 2n + 1 \}) \cup \{ 2n + 1 \}.
\]
and if $r$ is even, set
\[
S_e = \left\{ \alpha_i + 1, \alpha_i + (m+1), \alpha_i + (n+1), \alpha_i + (n+m+1) \mid 0 \leq i \leq \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \right\} \\
\cup \left\{ n - \left\lfloor \frac{r-2}{2} \right\rfloor, n - m - \left\lfloor \frac{r-2}{2} \right\rfloor, 2n + 1 - \left\lfloor \frac{r-2}{2} \right\rfloor, 2n + 1 - m - \left\lfloor \frac{r-2}{2} \right\rfloor \right\}.
\]
In each case, the given set is a 2TDS of $G$ of cardinality $\left\lceil \frac{4n+1}{2m+1} \right\rceil$ or $\left\lceil \frac{4n+1}{2m+1} \right\rceil + 1$, and this completes our proof.

3. Some Problems

We finish our paper with the following problems.

**Problem 3.1.** Characterize Harary graphs in which their double total domination number achieves the lower or upper bounds given in Theorem 2.3 or Theorem 2.4.

**Problem 3.2.** Find general formula for $\gamma_{k,t}(G)$, when $k \geq 2$ and $G$ is a graph from some of the known family of graphs, such as Supergeneralized Petersen graphs, which was proved the special case by A.P. Kazemi and B. Pahlavsay in [6].

4. Conclusion

The double total domination number in Harary graphs was proved as an algorithm because of the importance of this graphs. A natural way to extend this study is finding a general formula for $\gamma_{k,t}(H_{m,n})$.

Competing Interests

The authors declare that they have no competing interests.

Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References


