The Rational Distance Problem for Equilateral Triangles

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Abstract. We provide a complete characterization of all equilateral triangles $T$ for which there exists a point in the plane of $T$, that is at rational distance from each vertex of $T$.

Keywords. Equilateral triangle; Rational distance problem; Bi-quadric number; Legendre’s symbol; Non-degenerated triangle; Primitive integral triangle

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1. Introduction

Let $(P)$ denote the problem of existence of a point in the plane of a given triangle $T$, that is at rational distance from all the vertices of $T$. Answer to $(P)$ is positive if $T$ has a rational side and the square of all sides are rational (see [1]). In [3], a complete solution to $(P)$ is given for all isosceles triangles with one rational side. In this article, we provide a complete solution to $(P)$ for all equilateral triangles.

In all what follows, $\theta$ denotes an arbitrary positive real number and $T = [\theta]$ denotes the equilateral triangle with side-length $\theta$. For convenience, we say that $\theta$ is “good” (or “suitable”) if answer to $(P)$ is positive for the triangle $T = [\theta]$. Clearly, the property “$\theta$ is good” is invariant by any rational re-scaling of $\theta$. 
It turns out that the good \( \theta \) must have algebraic degree 1, 2 or 4, and they form a subclass of the positive bi-quadric numbers, that is, the positive roots of equations of the form \( x^4 + ux^2 + v = 0 \), \( u, v \in \mathbb{Q} \). The general form of such numbers is
\[
\sqrt{\alpha \pm \sqrt{\beta}}, \quad \alpha, \beta \in \mathbb{Q}, \ \beta \geq 0, \ \alpha \pm \sqrt{\beta} \geq 0
\]
that includes positive numbers of the form
\[
\alpha, \sqrt{\alpha}, \alpha \pm \sqrt{\beta} \sqrt{\alpha} \pm \sqrt{\beta}, \quad \alpha, \beta \in \mathbb{Q}, \ \alpha, \beta \geq 0.
\]

**Notations and Conventions**

\((x, y)\) and \((x, y, z)\) denote the gcd \((\frac{x}{y})\) denotes Legendre’s symbol. A triangle with side-lengths \( a, b, c \) is denoted by \( T = [a, b, c] \). A triangle is non-degenerated if it has positive area. A radical is non-degenerated if it is irrational.

### 2. The results

**Theorem 2.1.** If \( \theta \) is good, then \( \theta \) is bi-quadric. More precisely, \( \theta^2 = \alpha \pm \sqrt{\beta} \) for some \( \alpha, \beta \in \mathbb{Q} \), \( \beta \geq 0 \) and \( \alpha \) positive.

**Theorem 2.2.** Suppose \( \theta \notin \mathbb{Q} \) and \( \theta^2 \in \mathbb{Q} \). Then, \( \theta \) is good \( \iff \theta \) has the form \( \theta = \lambda \sqrt{p_1 \ldots p_r} \), where \( \lambda \in \mathbb{Q}, \ \lambda > 0, \ r \geq 1, \ p_1, \ldots, p_r \) are distinct odd primes, \( p_i \) is either 3 or of the form \( 6k + 1 \).

**Theorem 2.3.** Suppose \( \theta^2 = \alpha \pm \sqrt{\beta}, \ \alpha, \beta \in \mathbb{Q}, \ \alpha, \beta > 0, \ \sqrt{\beta} \notin \mathbb{Q} \). Then, \( \theta \) is good \( \iff \) up to a rational re-scaling of \( \theta \), \( \theta \) is described as follows:
\[
2\theta^2 = (a^2 + b^2 + c^2) \pm 4\Delta \sqrt{3},
\]
where \([a, b, c]\) is a non-degenerated primitive integral triangle with area \( \Delta \) such that \( 4\Delta \sqrt{3} \notin \mathbb{Q} \).

**Remark.** \( \Delta \) is given by Hero’s formula, \( \Delta = \sqrt{s(s-a)(s-b)(s-c)}, \ s = \frac{1}{2}(a+b+c) \). Equivalently, \( 4\Delta \sqrt{3} = \sqrt{3}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \), and the condition \( 4\Delta \sqrt{3} \notin \mathbb{Q} \) means that this latter radical is non-degenerated.

### 3. Proofs of Theorems 2.1 and 2.2

**Proof of Theorem 2.1** Suppose \( \theta \) good. Let \( M \) be a point in the plane of triangle \( T = [\theta] \), whose distances from the vertices of \( T \) are all rational. The following fundamental relation is well-known (see [4]):
\[
3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2.
\]
Expanding \((\bullet)\) yields a relation as \( \theta^4 - u\theta^2 + v = 0 \), where \( u, v \in \mathbb{Q} \) and \( u = a^2 + b^2 + c^2 > 0 \). Solving for \( \theta^2 \) yields \( \theta^2 = \alpha \pm \sqrt{\beta} \), with \( \alpha, \beta \in \mathbb{Q} \) and \( \alpha = \frac{1}{2}u > 0 \).

**Lemma 3.1.** Let \( q > 1 \) be a square-free integer. Then, we have: The equation \( x^2 + 3y^2 = qz^2 \) has a solution in integers \( x, y, z \) with \( z \neq 0 \) if and only if any prime factor of \( q \) is either 3 or of the form \( 6k + 1 \).
Proof. Suppose first that $q$ has only prime factors as 3 or $6k + 1$. Since the quadratic form $x^2 + 3y^2$, $x, y \in \mathbb{Z}$, represents 3 and every prime $p = 6k + 1$, and since the set $\{x^2 + 3y^2, x, y \in \mathbb{Z}\}$ is closed by multiplication, we conclude that the equation $x^2 + 3y^2 = q \cdot z^2$ has a solution in integers $x, y, z$ with $z = 1$.

Conversely, suppose that $x^2 + 3y^2 = q \cdot z^2$ has a solution in integers $x, y, z$, $z \neq 0$. Pick such a solution with $|z|$ minimum. Clearly, $(x, y) = 1$. We claim that $q$ is odd and has no prime factor $6k - 1$. For the purpose of contradiction, we consider two cases:

Case 1: $q$ is even. Set $q = 2w$, $w$ odd. From $x^2 + 3y^2 = 2wz^2$, we see that $x \equiv y \pmod{2}$. As $(x, y) = 1$, $x$ and $y$ must be odd, so $x^2 + 3y^2 \equiv 4 \pmod{8}$. Now, $4/2wz^2$ yields $wz^2$ even. But $w$ is odd, hence $z$ is even, so $2wz^2 \equiv 0 \pmod{8}$. We get a contradiction.

Case 2: $q = p \cdot w$ for some prime $p = 6k - 1$. $x^2 + 3y^2 = pwz^2$ yields $x^2 + 3y^2 \equiv 0 \pmod{p}$. As $(x, y) = 1$, $p$ cannot divide $y$. Hence for some $t \in \mathbb{Z}$, $yt \equiv 1 \pmod{p}$. Therefore, $x^2 t^2 + 3y^2 t^2 \equiv x^2 z^2 + 3 \equiv 0 \pmod{p}$, so $-3 \equiv (xt)^2 \pmod{p}$. Hence $(-3/p) = +1$ contradicting $p = 6k - 1$.

Lemma 3.2. Let $\theta = \lambda \sqrt{q}$, $\lambda \in \mathbb{Q}$, $\lambda > 0$, $q > 1$ square-free integer. We have: $\theta$ is good $\iff$ There are $a, b, e, r, s \in \mathbb{Q}$, $e \neq 0$, such that

\begin{align}
& a^2 + 3b^2 = q, \\ & (a + e)^2 + 3(b + e)^2 = qr^2, \\ & (a - e)^2 + 3(b + e)^2 = qs^2.
\end{align}

Proof. By re-scaling, we take $\theta = 2\sqrt{q}$. Let $T = ABC = [\theta]$. Choose a $x - y$ axis to get the coordinates $A(0, \sqrt{3q}), B(-\sqrt{q}, 0), C(\sqrt{q}, 0)$.

Suppose first that $\theta$ is good:

There is a point $M = M(x, y)$ in the plane of $T$ such that $MA, MB, MC \in \mathbb{Q}$. Clearly, $M \neq A, B, C$.

Set $w = \frac{MA}{q}$, $r = \frac{MB}{wq}$, $s = \frac{MC}{wq}$. Then, $w, r, s \in \mathbb{Q} - \{0\}$.

The Pythagoras relations are:

\begin{align}
& MA^2 = x^2 + (y - \sqrt{3q})^2 = w^2 q^2, \\
& MB^2 = (x + \sqrt{q})^2 + y^2 = w^2 q^2 r^2, \\
& MC^2 = (x - \sqrt{q})^2 + y^2 = w^2 q^2 s^2.
\end{align}

Subtracting (3.2) and (3.3) yields $x = \frac{1}{q} w q (r^2 - s^2) \cdot \sqrt{q}$, that is,

\begin{equation}
\tag{3.4}
x = a \sqrt{q}, \quad a \in \mathbb{Q}.
\end{equation}

Then (3.2) gives $y^2 \in \mathbb{Q}$, and then (3.1) gives $2y \sqrt{3q} \in \mathbb{Q}$, hence, $y = \gamma \sqrt{3q}$, $\gamma \in \mathbb{Q}$.

For convenience, we put $\gamma = \beta + 1$, obtaining

\begin{equation}
\tag{3.5}
y = (\beta + 1) \sqrt{3q}, \quad \beta \in \mathbb{Q},
\end{equation}

Due to (3.4) and (3.5), equations (3.1), (3.2), (3.3) become after dividing by $q$:

\begin{equation}
\tag{3.6}
a^2 + 3\beta^2 = qw^2,
\end{equation}
We may write:

\[(a + 1)^2 + 3(b + 1)^2 = qw^2r^2,\]
\[(a - 1)^2 + 3(b + 1)^2 = qw^2s^2.\]

Set \(a = \frac{a}{w}, b = \frac{b}{w}, e = \frac{1}{w}\). Dividing by \(w^2\), we get precisely relations \((3.1), (3.2), (3.3)\).

• Conversely suppose that relations \((3.1), (3.2), (3.3)\) hold with some \(a, b, e, r, s \in \mathbb{Q}, e \neq 0\). Define point \(M = M(x, y)\) in the plane of \(T\) by

\[x = \frac{a}{e} \sqrt{q}, \quad y = \left(\frac{b}{e} + 1\right) \sqrt{3q}.\]

We may write:

\[MA^2 = x^2 + (y - \sqrt{3q})^2 = \frac{a^2}{e^2} + 3q \frac{b^2}{e^2} = \frac{a^2}{e^2} + 3q = \left(\frac{q}{e}\right)^2,\]
\[MB^2 = \left(\left(\frac{a + e}{e} \sqrt{q}\right)^2 + \left(\frac{b + e}{e} \sqrt{3q}\right)^2\right) - \frac{q}{e^2} (a^2 + 3b^2) = \left(\frac{q}{e}\right)^2,\]
\[MC^2 = \left(\left(\frac{a - e}{e} \sqrt{q}\right)^2 + \left(\frac{b + e}{e} \sqrt{3q}\right)^2\right) - \frac{q}{e^2} (a^2 + 3b^2) = \left(\frac{qs}{e}\right)^2.\]

Therefore, \(MA, MB, MC\) are all rational. \(\square\)

**Proof of Theorem 2.2** Let \(\theta\) such that \(\theta \notin \mathbb{Q}\) and \(\theta^2 \in \mathbb{Q}\): \(\theta\) can be written as \(\theta = \lambda \sqrt{q}, \lambda \in \mathbb{Q}, \lambda > 0, q > 1\) square-free integer.

• Suppose first that \(q\) is even or has a prime factor \(6k - 1\). By Lemma 3.1 \(a^2 + 3b^2 = q, a, b \in \mathbb{Q}\), is impossible.

Hence, relation \((3.1)\) in Lemma 3.2 fails, so \(\theta\) is not good.

• Suppose now that \(q\) has only prime factors as \(3\) or \(6k + 1\). We show that \(\theta\) is good using the characterization of Lemma 3.2

By Lemma 3.1 for some \(a, b \in \mathbb{Q}\), we have \(a^2 + 3b^2 = q\). Set \(e = -\frac{a}{4b} = -\frac{(a + 3b)}{4b}, r = \frac{a - b}{2b}, s = \frac{a + b}{2b}\). We have

\[(a + e)^2 + 3(b + e)^2 = (a^2 + 3b^2) + 4e^2 + 2e(a + 3b)\]
\[= q + \frac{q^2}{4b^2} - \frac{q}{2b}(a + 3b)\]
\[= \frac{q}{4b^2}(4b^2 + q - 2b(a + 3b))\]
\[= \frac{q}{4b^2}(4b^2 + a^2 + 3b^2 - 2ab - 6b^2)\]
\[= \frac{q}{4b^2}(a^2 + b^2 - 2ab)\]
\[= \frac{q}{4b^2}(a^2 + b^2 - 2ab)\]
\[= q \cdot r^2\]

and

\[(a - e)^2 + 3(b + e)^2 = (a^2 + 3b^2) + 4e^2 - 2e(a - 3b)\]
\[ q + \frac{q}{4b^2} + \frac{q}{2b}(a-3b) \]
\[ = \frac{q}{4b^2}(4b^2 + q + 2b(a-3b)) \]
\[ = \frac{q}{4b^2}(4b^2 + a^2 + 3b^2 + 2ab - 6b^2) \]
\[ = \frac{q}{4b^2}(a^2 + b^2 + 2ab) \]
\[ = q \cdot s^2. \]

\[ \square \]

### 4. Proof of Theorem 2.3

**Lemma 4.1.** Let \( x, y, z, t \) be positive real numbers such that
\[ 3(x^4 + y^4 + z^4 + t^4) = (x^2 + y^2 + z^2 + t^2)^2. \]
Then, any three of \( x, y, z, t \) satisfy the triangle inequality.

**Proof.** Since \( x, y, z, t \) play symmetric roles, it suffices to show that \( x, y, z \) satisfy the triangle inequality. Write (\( \odot \)) as
\[ t^4 - (x^2 + y^2 + z^2)t^2 + (x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2) = 0. \]
The discriminant \( \Delta \) of this trinomial in \( t^2 \) must be non-negative. But, \( \Delta = 6(x^2y^2 + y^2z^2 + z^2x^2) - 3(x^4 + y^4 + z^4) \) that factors as \( \Delta = 3(x+y+z)(-x-y+z)(x-y+z)(x+y-z). \)

Hence, \((x+y+z)(x-y+z)(x+y-z) \geq 0). The reader can easily check (using contraposition) that \( x, y, z \) must satisfy the triangle inequality. \( \square \)

**Lemma 4.2.** Let \( T = ABC = [\theta] \). Let \( a, b, c \) be positive real numbers satisfying
\[ 3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2. \]
Then, there is a point \( M \) in the plane of \( T \) such that \( MA = a, MB = b \) and \( MC = c \).

**Proof.** By Lemma 4.1, \( a, b \) and \( \theta \) satisfy the triangle inequality. In particular, \( a + b \geq \theta \).

It follows that the circle \( \mathcal{C}(A,a) \) intersects the circle \( \mathcal{C}(B,b) \) at two points \( M_1 \) and \( M_2 \) (\( M_1 = M_2 \) if \( a + b = \theta \)). Set \( c_1 = M_1C \) and \( c_2 = M_2C \).

By the fundamental relation (\( \star \)), we have \( 3(a^4 + b^4 + c_1^4 + \theta^4) = (a^2 + b^2 + c_1^2 + \theta^2)^2 \) and \( 3(a^4 + b^4 + c_2^4 + \theta^4) = (a^2 + b^2 + c_2^2 + \theta^2)^2 \). Therefore, \( c_1^2 \) and \( c_2^2 \) are the roots of the trinomial in \( T \)
\[ T^2 - (a^2 + b^2 + \theta^2)T + (a^4 + b^4 + \theta^4 - a^2b^2 - b^2\theta^2 - \theta^2a^2) = 0. \]

Since by hypothesis \( c^2 \) is also a root of this trinomial, we must have \( c^2 = c_1^2 \) or \( c^2 = c_2^2 \). Hence \( c = c_1 \) or \( c = c_2 \). Therefore, \( a, b \) and \( c \) are the distances from either point \( M_1 \) or \( M_2 \) to the vertices \( A, B \) and \( C \) of \( T \). \( \square \)

**Proof of Theorem 2.3.** Let \( \theta > 0 \) such that \( \theta^2 = a \pm \sqrt{\beta}, a, \beta \in Q, a, \beta > 0, \sqrt{\beta} \notin Q \).
• Suppose first that \( \theta \) is good:
Let \( P \) be a point in the plane of \( T = ABC = [\theta] \) such that \( PA = a, PB = b, PC = c \) are all rational. We have
\[
3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2. \tag{1}
\]
By Lemma 4.1, \( a, b \) and \( c \) satisfy the triangle inequality. Relation (*) yields
\[
\theta^4 - U\theta^2 + V = 0 \quad \text{with} \quad U = a^2 + b^2 + c^2 \quad \text{and} \quad V = a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \quad (U, V \in \mathbb{Q}).
\]
Solving for \( \theta^2 \), we get
\[
2\theta^2 = (a^2 + b^2 + c^2) \pm \sqrt{3(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}. \tag{\star}
\]
Since \( \theta^2 \) has algebraic degree 2, then, the radical in (\star) is non-degenerated. In particular, the triangle \([a, b, c]\) is non degenerated. Select a sufficiently large positive integer \( N \) such that \( Na, Nb, Nc \) are all integers and set \( D = (Na, Nb, Nc) \). If we multiply relation (\star) by \( \frac{N^2}{D^2} \), this results in replacing in (\star) \( \theta \) by \( \frac{N}{D} \cdot \theta \) and \( a, b, c \) by the integers \( \frac{Na}{D}, \frac{Nb}{D}, \frac{Nc}{D} \), respectively. As an outcome, we obtain essentially the same relation (\star) where \( \theta \) has been re-scaled by the rational \( \frac{N}{D} \), and where the new symbols \( a, b, c \) represent relatively prime positive integers, satisfying the triangle inequality.

• Conversely, suppose that for some positive rational \( \lambda, \theta_0 = \lambda \cdot \theta \) is described precisely as in Theorem 2.3. Eliminating the radical
\[
4\Delta\sqrt{3} = \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}
\]
in the relation \( 2\theta_0^2 = (a^2 + b^2 + c^2) \pm 4\Delta\sqrt{3} \) leads to
\[
3(a^4 + b^4 + c^4 + \theta_0^4) = (a^2 + b^2 + c^2 + \theta_0^2)^2.
\]
By Lemma 4.2 there is a point \( M \) in the plane of \( T = [\theta_0] \) that is at distances \( a, b, c \) from the vertices of \( T \). Since \( a, b, c \) are integers, then, \( \theta_0 \) is good. Therefore, \( \theta = \lambda^{-1}\theta_0 \) is also good. \( \Box \)

### 5. Exercises

1. Check which are “good” among the radicals: \( \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10} \).
2. Show that the positive real number \( \theta = \sqrt{25 + 12\sqrt{3}} \) is “good”.
3. Suppose that \( 2\theta^2 = a + \sqrt{b}, a, b \in \mathbb{Q}, a, b > 0, \sqrt{b} \notin \mathbb{Q}, \) and \( a^2 < b \). Show that \( \theta \) is not good.
4. Produce solution-points to problem (P) for the triangle \( T = [\sqrt{3}] \).
5. Let \( \theta = a + \beta\sqrt{q} > 0, a, \beta \in \mathbb{Q}, \beta \neq 0, q > 1 \) square-free integer. Show that \( \theta \) is not good.
6. Suppose that \( 2\theta^2 = a \pm \sqrt{b} > 0, a, b \in \mathbb{Q}, a, b > 0, \sqrt{b} \notin \mathbb{Q} \). Write the fraction \( a \) in lowest terms as \( a = \frac{m}{n} \) \((m, n \) positive integers\) and suppose that \( mn \) has the form \( mn = 4^l(8k + 7), \) \( k, l \) non-negative integers. Then, prove that \( \theta \) is not good.

### 6. Conclusion

We have a complete answer to problem (P) for equilateral triangles \( T = [\theta] \):
If $\theta$ is transcendental or has algebraic degree $\geq 5$, then, $\theta$ is not good. If $\theta$ has algebraic degree 3, or if both $\theta$ and $\theta^2$ have algebraic degree 4, then, $\theta$ is not good. If $\theta$ is irrational and $\theta^2 \in \mathbb{Q}$, so $\theta$ has the form $\lambda \sqrt{q}$, where $\lambda \in \mathbb{Q}$, $\lambda > 0$ and $q > 1$ is a square-free integer, then, $\theta$ is good if and only if $q$ is odd and has no prime factor $6k - 1$. Finally, if $\theta^2 = \alpha \pm \sqrt{\beta}$, $\alpha, \beta \in \mathbb{Q}$, $\beta > 0$, $\sqrt{\beta} \not\in \mathbb{Q}$, then, $\theta$ is not good if $\alpha \leq 0$ or if $\alpha^2 < \beta$, while if $\alpha > 0$ and $\alpha^2 > \beta$, $\theta$ is good if and only if $\theta$ satisfies the geometric property described in Theorem 2.3.

**Competing Interests**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

**References**


