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Research Article

Klein-Gordon-Maxwell System with Partially Sublinear Nonlinearity

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Abstract. In this paper we shown that a class of sublinear Klein-Gordon-Maxwell system has infinitely many solutions by using a critical point theorem established by Liu and Wang and Moser iteration method.

Keywords. Klein-Gordon-Maxwell system; Variational methods; Critical point theorem; Sublinear

MSC. 35J61; 35C06; 35J20

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1. Introduction

In this paper, we are study the following Klein-Gordon-Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{$\mathcal{KGM}$}$$

where $\omega > 0$ is a constant, u, $\phi : \mathbb{R}^3 \to \mathbb{R}$, $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function.

The following system

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi + u^2\phi = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

was first introduced in [4] as a model describing the nonlinear Klein–Gordon field interacting with the electromagnetic field. Later, many authors studied this system and interested in the case $f(x,u)=|u|^{p-2}u$ for $2<p\le 6$ (see [1,2,4–8,10–15,18,21–24]). The other case such as semiclassical states [22], nonhomogeneous case [5,10] and critical exponent case [6–8,24] are also studied. Very recently, the authors [9,11,16,17] investigated the existence of solutions of the problem $(\mathcal{KGM})$. Especially, Li and Tang [19] use the genus properties to obtain the following theorem.

Theorem 1.1 ([19]). *Assume that V and f satisfy the following conditions:*

(V) $V(x)\in C(\mathbb{R}^3,\mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x)=V_0>0$ *and there exists* $v_0>0$ *such that*

$$\lim_{|y|\to\infty} meas\{x\in\mathbb{R}^3: |x-y|\le v_0, V(x)\le M\}=0 \ \text{ for all } M>0;$$

(B_1) *There exist* p, σ, $\gamma\in(1,2)$ *and* $\nu\in(2,6)$ *such that*

$$b(x)|t|^p\le f(x,t)t \text{ and } |f(x,t)|\le m(x)|t|^{\sigma-1}+h(x)|t|^{\gamma-1}+C|t|^{\nu-1}$$

for all $(x,t)\in\mathbb{R}^3\times R$, *where* $b,m,h:\mathbb{R}^3\to\mathbb{R}$ *are positive continuous functions satisfying* $b\in L^{\frac{2}{2-p}}(\mathbb{R}^3)$, $m\in L^{\frac{2}{2-\sigma}}(\mathbb{R}^3)$, $h\in L^{\frac{2}{2-\gamma}}(\mathbb{R}^3)$;

(B_2) $f(x,-z)=-f(x,z)$, $(x,z)\in\mathbb{R}^3\times\mathbb{R}$.

Then $(\mathcal{KGM})$ *has infinitely many solutions.*

The main aim of this paper is to complement Theorem 1.1. We want to study the following problem

$$\begin{cases} -\Delta u+V(x)u-(2\omega+\phi)\phi u=K(x)f(x,u), & x\in\mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2, & x\in\mathbb{R}^3. \end{cases} \tag{2}$$

Our result is as follows.

Theorem 1.2. *Assume that V satisfies (V) and f satisfies (B_2) and the following conditions:*

(f_1) *There exist* $\delta>0$, $1\le\gamma<2$, $C>0$ *such that* $f\in C(\mathbb{R}^3\times[-\delta,\delta],\mathbb{R})$ *and* $|f(x,z)|\le C|z|^{\gamma-1}$;

(f_2) $\lim_{z\to 0} F(x,z)/|z|^2=+\infty$ *uniformly in some ball* $B_r(x_0)\subset\mathbb{R}^3$, *where* $F(x,z)=\int_0^z f(x,s)ds$;

(f_3) $K:\mathbb{R}^3\to\mathbb{R}^+$ *is a continuous function such that* $K>0$ *for all* $x\in\mathbb{R}^3$ *and* $K\in L^{\frac{2}{2-\gamma}}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$.

Then (2) *has infinitely many solutions* $\{u_k\}$ *such that* $\|u_k\|_{L^\infty}\to 0$ *as* $k\to\infty$.

Remark 1.3. It should be noted that the authors [19] had obtained infinitely many solutions when nonlinearity is sublinear at zero globally. In this paper, if we have a control at infinity, the nonlinearity can be generalized to partially sublinear and get more information about the solutions (such as the solutions are convergent to zero in $L^\infty(\mathbb{R}^3)$).

Throughout the paper, we denote by $C>0$ various positive constants which may vary from line to line.

2. Preliminaries

In this section, we shall give some notations and propositions that will be used throughout this paper.

For any $1 \le s < \infty$, $L^s(\mathbb{R}^3)$ denotes the usual Lebesgue space equipped with the norm $\|u\|_s := \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{1/s}$. $H^1(\mathbb{R}^3)$ is the usual Sobolev space with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx\right)^{1/2},$$

and the function space

$$\mathcal{D}^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$$

equipped with the norm

$$\|u\|_{\mathcal{D}^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}.$$

In our problem, the function space E is defined by

$$E := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx < \infty\right\}.$$

Thus, E is a Hilbert space with the inner product $(u,v)_E := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx$, and its norm is $\|u\|_E = (u,u)_E^{1/2}$. Since $V(x) \ge V_0 > 0$, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for any $s \in [2,6]$.

Next, we need the following compactness result proved in [3].

Proposition 2.1. *Under the assumption* (V)*, the embedding* $E \hookrightarrow L^q(\mathbb{R}^3)$*,* $2 \le q < 2^* = 6$ *is compact.*

Now, we need the following technical results established in [4] (see also [13]).

Proposition 2.2. *For any fixed* $u \in H^1(\mathbb{R}^3)$*, there exists a unique* $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *which solves equation*

$$-\Delta\phi + u^2\phi = -\omega u^2. \tag{3}$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \Phi[u] := \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *is continuously differentiable, and*

(i) $-\omega \le \phi_u \le 0$ *on the set* $\{x | u(x) \ne 0\}$*;*

(ii) $\|\phi_u\|_{\mathcal{D}^{1,2}} \le C\|u\|_{L^2(\mathbb{R}^3)}$.

To get infinitively many solutions, we show the following theorem established by Liu and Wang [20] which is an extension of Clark's theorem.

Theorem 2.3 ([20])**.** *Let* X *be a Banach space,* $\Psi \in C^1(X,\mathbb{R})$*. Assume* Ψ *is even and satisfies the (PS) condition, bounded from below, and* $\Psi(0) = 0$*. If for any* $k \in \mathbb{N}$*, there exists a* k*-dimensional subspace* X^k *of* X *and* $\rho_k > 0$ *such that* $\sup_{X^k \cap S_{\rho_k}} \Psi < 0$*, where* $S_\rho = \{u \in X | \|u\| = \rho\}$*, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points* $\{u_k\}$ *satisfying* $\Psi(u_k) < 0$ *for all* k *and* $\|u_k\| \to 0$ *as* $k \to \infty$.

(ii) *There exists* $r > 0$ *such that for any* $0 < a < r$ *there exists a critical point* u *such that* $\|u\| = a$ *and* $\Psi(u) = 0$.

3. Proof of the Main Result

Proof of Theorem 1.2. Firstly, choose $\hat{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ such that $\hat{f}$ is odd in $u \in \mathbb{R}$ and

$$\hat{f}(x,u) = \begin{cases} f(x,u), & \text{if } |u| < \frac{\delta}{2}, \\ 0, & \text{if } |u| > \delta. \end{cases}$$

In order to obtain solutions of (2), we now consider the following problem

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = K(x)\hat{f}(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{4}$$

As is known, (4) is variational and its solutions are the critical points of the functional defined in E by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right) dx - \int_{\mathbb{R}^3} K(x)\hat{F}(x,u)dx,$$

where $\hat{F}(x,u) = \int_0^u \hat{f}(x,s)ds$ denotes a primitive of $\hat{f}$. From (f_1) and (f_3), it is not hard to check that $\mathcal{J}$ is well defined on E and $\mathcal{J} \in C^1(E,\mathbb{R})$ (see [19] for more details). Moreover,

$$\langle \mathcal{J}'(u), v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv - K(x)\hat{f}(x,u)v\right) dx, \ v \in E.$$

Note that $\mathcal{J}$ is even, and $\mathcal{J}(0) = 0$. For $u \in E$,

$$\int_{\mathbb{R}^3} K(x)|\hat{F}(x,u)|dx \le C\int_{\mathbb{R}^3} K(x)|u|^\gamma dx \le C\|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3)} \|u\|^\gamma_{L^2(\mathbb{R}^3)} \le C\|u\|^\gamma.$$

Hence, by Proposition 2.2

$$\mathcal{J}(u) \ge \frac{1}{2}\|u\|^2 - C\|u\|^\gamma, \ u \in E. \tag{5}$$

Now, we show that $\mathcal{J}$ satisfies the Palais-Smale condition. Let $\{u_n\}$ be a Palais-Smale sequence in E, that is $\mathcal{J}(u_n)$ is bounded and $\mathcal{J}'(u_n) \to 0$. We will prove that $\{u_n\}$ has a strongly convergent subsequence in E. Due to (5), we get $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in E. By Proposition 2.1, $u_n \to u$ in $L^q(\mathbb{R}^3)$ for any $2 \le q < 6$. Observe that

$$\begin{aligned} \|u_n - u\|^2 &= \langle \mathcal{J}'(u_n) - \mathcal{J}'(u), u_n - u\rangle - \int_{\mathbb{R}^3}[(2\omega + \phi_{u_n})\phi_{u_n}u_n + (2\omega + \phi_u)\phi_u u](u_n - u)dx \\ &\quad + \int_{\mathbb{R}^3} K(x)(\hat{f}(x,u_n) - \hat{f}(x,u))(u_n - u)dx \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

where ϕ_v is the solution of $\Delta\phi = (\omega + \phi)v^2$ established in Proposition 2.2.

It is clear that

$$I_1 \to 0, \quad \text{as } n \to \infty. \tag{6}$$

Next, we estimate I_2. By the Hölder inequality, the Sobolev inequality, and Proposition 2.2, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)dx\right| &\le \|\phi_{u_n}-\phi_u\|_{L^6(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\|u_n\|_{L^2(\mathbb{R}^3)}\\ &\le C\|\nabla(\phi_{u_n}-\phi_u)\|_{L^2(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\|u_n\|_{L^2(\mathbb{R}^3)}\\ &\le C\left(\|u_n\|_{L^2(\mathbb{R}^3)}+\|u\|_{L^2(\mathbb{R}^3)}\right)\|u_n\|_{L^2(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\\ &\le C\|u_n-u\|_{L^3(\mathbb{R}^3)}.\end{aligned}$$

So, we get

$$\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)dx\to 0,\ \text{as } n\to\infty$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\phi_u(u_n-u)(u_n-u)dx\right| &\le \|\phi_u\|_{L^6(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\|u_n-u\|_{L^2(\mathbb{R}^3)}\\ &\le C\|\nabla\phi_u\|_{L^2(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\|u_n-u\|_{L^2(\mathbb{R}^3)}\\ &\le C\|u\|_{L^2(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\|u_n-u\|_{L^2(\mathbb{R}^3)}\\ &\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\int_{\mathbb{R}^3}(\phi_{u_n}u_n+\phi_u u)(u_n-u)dx &= \int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)dx+\int_{\mathbb{R}^3}\phi_u(u-u_n)(u_n-u)dx\\ &\to 0,\quad \text{as } n\to\infty.\end{aligned}\tag{7}$$

By the Hölder inequality and Proposition 2.2 again, we obtain

$$\begin{aligned}\|\phi_{u_n}^2u_n\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le \|\phi_{u_n}\|^2_{L^6(\mathbb{R}^3)}\|u_n\|_{L^3(\mathbb{R}^3)}\\ &\le \|\nabla\phi_{u_n}\|^2_{L^2(\mathbb{R}^3)}\|u_n\|_{L^3(\mathbb{R}^3)}\\ &\le \|u_n\|^2_{L^2(\mathbb{R}^3)}\|u_n\|_{L^3(\mathbb{R}^3)}\\ &\le C.\end{aligned}$$

This shows that the sequence $\{\phi_{u_n}^2u_n\}$ is bounded in $L^{\frac{3}{2}}(\mathbb{R}^3)$. Then we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\phi_{u_n}^2u_n+\phi_u^2u)(u_n-u)dx\right| &\le \|\phi_{u_n}^2u_n+\phi_u^2u\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\|u_n-u\|_{L^3(\mathbb{R}^3)}\\ &\le \left(\|\phi_{u_n}^2u_n\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}+\|\phi_u^2u\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\right)\|u_n-u\|_{L^3(\mathbb{R}^3)}\\ &\le C\|u_n-u\|_{L^3(\mathbb{R}^3)}\\ &\to 0,\quad \text{as}\ \ n\to\infty.\end{aligned}\tag{8}$$

Therefore, from (7)–(8) we show that

$$I_2\to 0,\quad \text{as}\ \ n\to\infty.\tag{9}$$

Last, we estimate I_3. By (f_1), for any $R>0$, there holds

$$\int_{\mathbb{R}^3}K(x)|\widehat{f}(x,u_n)-\widehat{f}(x,u)||u_n-u|dx$$

$$\begin{aligned}
&\le C\int_{\mathbb{R}^3\setminus B_R(0)} K(x)(|u_n|^\gamma+|u|^\gamma)dx + C\int_{B_R(0)}(|u_n|^{\gamma-1}+|u|^{\gamma-1})|u_n-u|dx\\
&\le C\left(\|u_n\|^\gamma_{L^2(\mathbb{R}^3\setminus B_R(0))}+\|u\|^\gamma_{L^2(\mathbb{R}^3\setminus B_R(0))}\right)\|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3\setminus B_R(0))}\\
&\quad + C\left(\|u_n\|^{\gamma-1}_{L^\gamma(B_R(0))}+\|u\|^{\gamma-1}_{L^\gamma(B_R(0))}\right)\|u_n-u\|_{L^\gamma(B_R(0))}\\
&\le C\|K\|_{L^{\frac{2}{2-\gamma}}(\mathbb{R}^3\setminus B_R(0))}+C\|u_n-u\|_{L^\gamma(B_R(0))}.
\end{aligned}$$

this implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)|\hat{f}(x,u_n)-\hat{f}(x,u)||u_n-u|dx=0, \text{ as } n\to\infty.$$

Hence

$$I_3\to 0, \quad \text{as} \quad n\to\infty. \tag{10}$$

Combining (6), (9), and (10) together, we get that $\{u_n\}$ converges strongly in E and thus the Palais-Smale condition holds for $\mathcal{I}$.

By (f_2) and (f_3), for any $L>0$, there exists $\delta=\delta(L)>0$ such that if $u\in C_0^\infty(B_r(x_0))$ and $|u|_\infty<\delta$, then $K(x)\hat{F}(x,u(x))\ge L|u(x)|^2$, and it follows from Proposition 2.2 that

$$\mathcal{I}(u)\le \frac{1}{2}\|u\|^2+\frac{1}{4}\|u\|^4-L\|u\|^2_{L^2(\mathbb{R}^3)}.$$

This shows that for any $k\in\mathbb{N}$, if X^k is a k-dimensional subspace of $C_0^\infty(B_r(x_0))$ and ρ_k is sufficiently small then $\sup_{X^k\cap S_{\rho_k}}\mathcal{I}(u)<0$, where $S_\rho=\{u\in\mathbb{R}^3|\|u\|=\rho\}$. Now, we can apply Theorem 2.3 to obtain infinitely many solutions $\{u_k\}$ for (4) such that

$$\|u_k\|\to 0, \quad k\to\infty. \tag{11}$$

Finally, we get $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$. Let u be a solution of (4). Let $M>0$ and define

$$u^M(x):=\begin{cases}-M, & \text{if } u(x)<-M,\\ u(x), & \text{if } |u(x)|\le M,\\ M, & \text{if } u(x)>M.\end{cases}$$

For $\alpha>0$, it is easy to see that $|u^M|^\alpha u^M\in E$. Then multiplying $(4)_1$ by $|u^M|^\alpha u^M$ and integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[(\alpha+1)|u^M|^\alpha\nabla u\cdot\nabla u^M+V(x)u|u^M|^\alpha u^M\right]dx\\
&\quad=\int_{\mathbb{R}^3}(2\omega+\phi)\phi u|u^M|^\alpha u^M dx+\int_{\mathbb{R}^3}K(x)\hat{f}(x,u)|u^M|^\alpha u^M dx.
\end{aligned} \tag{12}$$

Due to the definition of u^M and Proposition 2.2 (i), it follows that

$$\int_{\mathbb{R}^3}(2\omega+\phi)\phi u|u^M|^\alpha u^M dx=\int_{\{u\neq 0\}}(2\omega+\phi)\phi u|u^M|^\alpha u^M dx\le 0. \tag{13}$$

Substituting (13) into (12), this shows

$$\int_{\mathbb{R}^3}\left[(\alpha+1)|u^M|^\alpha\nabla u\cdot\nabla u^M+V(x)u|u^M|^\alpha u^M\right]dx\le\int_{\mathbb{R}^3}K(x)\hat{f}(x,u)|u^M|^\alpha u^M dx,$$

and we get

$$\frac{4(\alpha+1)}{(\alpha+2)^2}\int_{\mathbb{R}^3}|\nabla|u^M|^{\frac{\alpha}{2}+1}|^2dx \le C\int_{\mathbb{R}^3}|u^M|^{\alpha+1}dx.$$

Together with Gagliardo-Nirenberg-Sobolev inequality, it follows that

$$\|u^M\|_{L^{3\alpha+6}(\mathbb{R}^3)} \le (C_1(\alpha+2))^{\frac{2}{\alpha+2}}\|u^M\|_{L^{\alpha+1}(\mathbb{R}^3)}^{\frac{\alpha+1}{\alpha+2}} \tag{14}$$

for some $C_1 \ge 1$ independent of u and α. Set $\alpha_0 = 5$ and $\alpha_k = 3(\alpha_{k-1}+2)-1$. Doing iteration by (14), it follows that

$$\|u^M\|_{L^{\alpha_{k+1}+1}(\mathbb{R}^3)} \le \exp\left(\sum_{i=0}^{k}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)\|u^M\|_{L^6(\mathbb{R}^3)}^{\mu_k}, \tag{15}$$

where $\mu_k = \prod_{i=0}^{k}\frac{\alpha_i+1}{\alpha_i+2}$. Letting M to infinity and then k to infinity, we obtain from (15) that

$$\|u\|_{L^\infty(\mathbb{R}^3)} \le \exp\left(\sum_{i=0}^{\infty}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)\|u\|_{L^6(\mathbb{R}^3)}^{\mu},$$

where $\mu = \prod_{i=0}^{\infty}\frac{\alpha_i+1}{\alpha_i+2}$ is a number in $(0,1)$ and $\exp\left(\sum_{i=0}^{\infty}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)$ is a positive number. Therefore, we obtain that $\|u_k\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $k\to\infty$, and u_k are the solutions of (2) as k sufficiently large. This complete the proof. □

4. Conclusion

In this paper, we consider a class of Klein-Gordon-Maxwell system with partial sublinear nonlinearity, which improved the previous work.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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