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Research Article

An Essential Remark on Relation-Theoretic Metrical Fixed Point Results

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Abstract. In this short note, we notice that the relation-theoretic metrical fixed point results are equivalent with the fixed point results in α-complete metric spaces. We observe that any arbitrary binary relation on a non empty set X can be defined in terms of an arbitrary real valued function defined on $X \times X$. Consequently we show that the results of Alam and Imdad (J. *Fixed Point Theory Appl.* **17** (4) (2015)) and Ahmadullah *et al*. (to appear in *Fixed Point Theory*) do not contribute anything new in the literature.

Keywords. Complete metric space; Binary relation; Fixed point

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1. Introduction

The mathematician M. Turinici [8, 9] initiated the study of metric fixed point theory equipped with arbitrary binary relations. Later on, Ran and Reurings [6] established some fixed point results in partially ordered metric spaces during the investigation of solutions to some special matrix equations. Thereafter, many mathematicians have delivered their significant contributions on numerous types of metric fixed point problems endowed with different kind of binary relations such as partial order, preorder, strict order, pseudo order, tolerance, transitive etc.

In another direction, several mathematicians have carried out their research works on the concept of α-ϕ-contraction mapping and established several important fixed point results. Before going into our main results, at first we recall some basic definitions and important results related to our work.

Let Φ denotes the family of functions such that

$$\Phi = \left\{\phi : [0,\infty) \to [0,\infty) \text{ is nondecreasing function with } \sum_{n=1}^{\infty} \phi^n(t) < \infty \text{ for all } t > 0\right\},$$

where ϕ^n denotes the nth iteration of ϕ. In 2012, Samet *at al.* [7] defined the notion of α-admissible mapping given by:

Definition 1.1 ([7]). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. Then T is said to be an α-admissible mapping if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1.$$

Definition 1.2 ([7]). Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is α-ϕ contractive mapping if there exist two mappings $\alpha : X \to X$ and $\phi \in \Phi$ such that for all $x, y \in X$

$$\alpha(x,y)d(Tx,Ty) \leq \phi(d(x,y)).$$

In this direction, they presented the following result.

Theorem 1.3 ([7]). *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that T is a α-ϕ contractive mapping. Suppose the following conditions hold:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *T is continuous*

then there exists $z \in X$ such that $z = Tz$.

They also ensured the uniqueness of fixed point in addition of the following condition with Theorem 1.3.

(•) **For every pair of elements $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.**

After soon, a lot of researchers have done their work using this concept and introduced several useful concepts such as α-complete metric space, α-continuous mappings etc. which are given by:

Definition 1.4 ([3]). Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a function. The metric space (X,d) is said to be an α-complete metric space if and only if every Cauchy sequence with $\alpha(x_n, x_{n+1}) \geq 1$, $\forall\ n \in \mathbb{N} \cup \{0\}$, converges in (X,d).

Definition 1.5 ([3])**.** Let (X,d) be a metric space. Let $T: X \to X$ be a mapping and $\alpha: X \times X \to [0,\infty)$ be a function. Then T is said to be an α-continuous mapping on (X,d), if for given $x \in X$ and sequence (x_n) with

$$\begin{aligned} &x_n \to x, \quad \text{as } n \to \infty, \\ &\alpha(x_n, x_{n+1}) \geq 1, \quad \forall\, n \in \mathbb{N} \cup \{0\} \Rightarrow Tx_n \to Tx. \end{aligned}$$

Very recently, Alam and Imdad [2], ahmadullah et al. [1] extended several well known fixed point results in metric spaces endowed with an arbitrary binary relation. To make this article self-contained, we need to recall some definitions and results relevant to this literature.

Definition 1.6 ([5])**.** Let X be a non-empty set and $\mathcal{R}$ be a binary relation defined on $X \times X$. Then, x is $\mathcal{R}$-related to y if and only if $(x,y) \in \mathcal{R}$.

Definition 1.7 ([4])**.** A binary relation $\mathcal{R}$ defined on X is said to be complete if for all $x,y \in X$, $[x,y] \in \mathcal{R}$, where $[x,y] \in \mathcal{R}$ stands for either $(x,y) \in \mathcal{R}$ or $(y,x) \in \mathcal{R}$.

Definition 1.8 ([2])**.** Suppose $\mathcal{R}$ is a binary relation defined on a non-empty set X. Then a sequence (x_n) in X is said to be $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Definition 1.9 ([2])**.** A metric space (X,d) endowed with a binary relation $\mathcal{R}$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in X.

Definition 1.10 ([2])**.** Let X be a non-empty set and f be a self-map defined on X. Then a binary relation $\mathcal{R}$ on X is said to be f-closed if $(x,y) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R}$.

Definition 1.11 ([2])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$. Then, $\mathcal{R}$ is said to be d-self-closed if every $\mathcal{R}$-preserving sequence with $x_n \to x$ there is a subsequence (x_{n_k}) of (x_n) such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N} \cup \{0\}$.

Definition 1.12 ([2])**.** Let (X,d) be a metric space endowed with a binary relation $\mathcal{R}$ and T be a self mapping defined on X. Then, T is said be $\mathcal{R}$-continuous if for every $\mathcal{R}$-preserving sequence with $x_n \to x$, we have $Tx_n \to Tx$.

Before proceeding further, we record the following results.

Theorem 1.13 ([2])**.** *Let (X,d) be a complete metric space equipped with a binary relation $\mathcal{R}$. Suppose T is a self-mapping on X such that*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$,*

(ii) *$\mathcal{R}$ is T-closed,*

(iii) *either T is continuous,*

(iv) *there exists $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq k d(x,y) \quad \forall\, x,y \in X \text{ with } (x,y) \in \mathcal{R}.$$

Then $F(T)$ is non empty.

2. Main Results

In this section, we show that there is a one to one correspondence between the fixed point results in α-complete metric spaces and the relation-theoretic metrical fixed point results.

Theorem 2.1. *Theorem* 1.13 *is equivalent with Theorem* 1.3.

Proof. Let us consider that all the hypotheses of Theorem 1.13 hold. Then (X,d) be a metric space endowed with an arbitrary relation $\mathcal{R}$. Now, we define a function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \forall\ x,y \text{ with } x\mathcal{R}y; \\ 0, & \text{otherwise.} \end{cases}$$

Then by the hypothesis of Theorem 1.13:

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$ which implies that $\alpha(x_0, Tx_0) \geq 1$;

(ii) given that $\mathcal{R}$ is T-closed, that is, whenever $(x,y) \in \mathcal{R}$ then $(Tx,Ty) \in \mathcal{R}$. This implies that whenever $\alpha(x,y) \geq 1$ then $\alpha(Tx,Ty) \geq 1$, i.e., T is α-admissible;

(iii) T is continuous;

(iv) for all $x,y \in X$ with $(x,y) \in \mathcal{R}$, we have

$$\begin{aligned} & d(Tx,Ty) \leq kd(x,y) \\ \Rightarrow \quad & \alpha(x,y)d(Tx,Ty) \leq \phi(d(x,y)) \end{aligned}$$

for all $x,y \in X$ with $\alpha(x,y) \geq 1$, $\phi(d(x,y)) = kd(x,y)$ and $k \in [0,1)$, i.e., T is an α-ϕ-contraction.

This shows that all the hypotheses of Theorem 1.3 also satisfy. Hence, Theorem 1.13 implies Theorem 1.3.

Conversely, we consider that all the hypotheses of Theorem 1.3 satisfy. Then, (X,d) is a metric space and $\alpha : X \times X \to [0,\infty)$ is an arbitrary function. We define a binary relation $\mathcal{R}$ on $X \times X$ by

$$(x,y) \in \mathcal{R} \quad \text{whenever } \alpha(x,y) \geq 1.$$

Then by the hypotheses of Theorem 1.3, we obtain:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, which implies $(x_0, Tx_0) \in \mathcal{R}$;

(ii) given that T is α-admissible, i.e., whenever $\alpha(x,y) \geq 1$ then $\alpha(Tx,Ty) \geq 1$. Then it is clear that $\mathcal{R}$ is T-closed, that is, $(x,y) \in \mathcal{R} \Rightarrow (Tx,Ty) \in \mathcal{R}$;

(iii) T is continuous;

(iv) by setting, $\phi(t) = kt$, $k \in [0,1)$ and for all $x,y \in X$ with $\alpha(x,y) \geq 1$, we have

$$\begin{aligned} & \alpha(x,y)d(Tx,Ty) \leq \phi(d(x,y)) \\ \Rightarrow \quad & d(Tx,Ty) \leq \alpha(x,y)d(Tx,Ty) \leq kd(x,y) \\ \Rightarrow \quad & d(Tx,Ty) \leq kd(x,y) \end{aligned}$$

with $(x,y) \in \mathcal{R}$.

Therefore, all the hypotheses of Theorem 1.13 also satisfy. Hence, Theorem 1.3 is equivalent with Theorem 1.13. □

In both cases of Theorem 1.3 and Theorem 1.13, continuity of T can be replaced by the following conditions respectively:

(C1) x_n be a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x$ as $n\to\infty$, then there exists a subsequence (x_{n_k}) of (x_n) such that $\alpha(x_{n_k},x)\geq 1$ or $\alpha(x,x_{n_k})\geq 1$ for all k.

(C2) $\mathcal{R}$ is d-self-closed.

Proposition 2.2. *Condition* (C1) *and* (C2) *are equivalent.*

Proof. Suppose condition (C1) holds. By setting the binary relation $\mathcal{R}$ on $X\times X$ as before in Theorem 2.1, we get (x_n) as a $\mathcal{R}$-preserving sequence such that there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k},x)\in\mathcal{R}$ for all k, i.e., $\mathcal{R}$ is d-self-closed.

Conversely, let $\mathcal{R}$ is d-self-closed. Then, we get (x_n) as a $\mathcal{R}$-preserving sequence such that there exists a subsequence (x_{n_k}) of (x_n) such that $[x_{n_k},x]\in\mathcal{R}$ for all k. This implies that $\alpha(x_{n_k},x)\geq 1$ or $\alpha(x,x_{n_k})\geq 1$ for all k. Thus condition (C2) implies condition (C1). Hence, (C1) and (C2) are equivalent. □

Uniqueness of fixed point of Theorem 1.13 is ensured by the following assumption:

($*$) **For every pair of elements $x,y\in X$, there exists $z\in X$ such that $(x,z)\in\mathcal{R}$ and $(y,z)\in\mathcal{R}$.**

Remark 2.3. Here it is easy to show that condition ($\bullet$) and ($*$) are also equivalent under the following assumption:

$$\alpha(x,y)=1 \quad \text{iff} \quad (x,y)\in\mathcal{R}.$$

In a similar fashion, one can deduce that

(A) (X,d) is α-complete iff (X,d) is $\mathcal{R}$-complete.

(B) T is α-continuous iff T is $\mathcal{R}$-continuous.

After showing the equivalence of Theorem 1.3 and Theorem 1.13, it is very natural to arise a question in mind that is it possible to define all kind of binary relations on an arbitrary set in terms of an arbitrary function? In this article, we give a answer to this question in affirmative sense.

Let X be a non-empty set and $\alpha: X\times X\to[0,\infty)$ be an arbitrary function. Now, we define a binary relation $\mathcal{R}\subseteq X\times X$ as

$$x\mathcal{R}y \quad \text{whenever} \quad \alpha(x,y)\geq 1.$$

Therefore,

- $\mathcal{R}$ is reflexive if for all $x\in X$, $\alpha(x,x)\geq 1$.
- $\mathcal{R}$ is irreflexive if for all $x\in X$, $\alpha(x,x)\ngeq 1$.

- $\mathcal{R}$ is symmetric if for all $x,y \in X$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(y,x) \geq 1$.
- $\mathcal{R}$ is antisymmetric if for all distinct $x,y \in X$, $\alpha(x,y) \geq 1$ does not implies $\alpha(y,x) \geq 1$.
- $\mathcal{R}$ is transitive if for all $x,y,z \in X$, $\alpha(x,z) \geq 1, \alpha(z,y) \geq 1 \Rightarrow \alpha(x,y) \geq 1$.
- $\mathcal{R}$ is said to be complete or connected or dichotomous is for all $x,y \in X$, $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$.
- $\mathcal{R}$ is said to be weakly complete or weakly connected or trichotomous is for all $x,y \in X$, $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$ or $x = y$.

In a similar way, we can define different types of binary relations such as strict order, preorder, near order, partial order, simple order, total order, equivalence relation etc. in terms of an arbitrary function.

3. Consequence

Ahmadullah et al. [1] extended and modified the result of Alam and Imdad [2] and presented the following theorem:

Theorem 3.1 ([1])**.** *Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. Suppose T is a self-mapping on X with the following conditions:*

(i) *there exists $Y \subseteq X, TX \subseteq Y$ such that (Y,d) is $\mathcal{R}$-complete,*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$*

(iii) *$\mathcal{R}$ is T-closed,*

(iv) *either T is $\mathcal{R}$-continuous or $\mathcal{R}|_Y$ is d-self-closed,*

(v) *there exists $\phi \in \Phi$ such that*

$$d(Tx,Ty) \leq \phi(M_T(x,y)) \quad \forall\ x,y \in X \ \ \text{with} \ \ (x,y) \in \mathcal{R},$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Then $F(T)$ is non empty.

Equivalent version of the above theorem is given by:

Theorem 3.2. *Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be an arbitrary function. Suppose T is a self-mapping on X with the following conditions:*

(i) *there exists $Y \subseteq X, TX \subseteq Y$ such that (Y,d) is α-complete,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*

(iii) *T is α-admissible,*

(iv) *either T is α-continuous or if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence (x_{n_k}) of (x_n) such that $\alpha(x_{n_k}, x) \geq 1$ for all k,*

(v) *there exists* $\phi \in \Phi$ *such that*

$$\alpha(x,y)d(Tx,Ty) \le \phi(M_T(x,y)) \quad \forall\ x,y \in X,$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Then $F(T)$ *is non empty.*

Corollary 3.3. *Let* (X,d) *be a metric space equipped with a binary relation* $\mathcal{R}$. *Suppose* T *is a self-mapping on* X *with the following conditions:*

(i) *there exists* $Y \subseteq X$, $TX \subseteq Y$ *such that* (Y,d) *is* $\mathcal{R}$*-complete,*

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in \mathcal{R}$

(iii) $\mathcal{R}$ *is* T*-closed,*

(iv) *either* T *is* $\mathcal{R}$*-continuous or* $\mathcal{R}|_Y$ *is* d*-self-closed,*

(v) *there exists* $k \in \left[0, \frac{1}{2}\right)$ *such that*

$$d(Tx,Ty) \le k[d(Tx,y) + d(x,Ty)] \quad \forall\ x,y \in X \text{ with } (x,y) \in \mathcal{R}.$$

Then $F(T)$ *is non empty.*

Corollary 3.4. *Let* (X,d) *be a metric space and* $\alpha : X \times X \to [0,\infty)$ *be an arbitrary function. Suppose* T *is a self-mapping on* X *with the following conditions:*

(i) *there exists* $Y \subseteq X, TX \subseteq Y$ *such that* (Y,d) *is* α*-complete,*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$

(iii) T *is* α*-admissible,*

(iv) *either* T *is* α*-continuous or if* x_n *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \ge 1$ *for all* n *and* $x_n \to x$ *as* $n \to \infty$, *then there exists a subsequence* (x_{n_k}) *of* (x_n) *such that* $\alpha(x_{n_k}, x) \ge 1$ *or* $\alpha(x, x_{n_k}) \ge 1$ *for all* k,

(v) *there exists* $k \in \left[0, \frac{1}{2}\right)$ *such that*

$$\alpha(x,y)d(Tx,Ty) \le k[d(Tx,y) + d(x,Ty)] \quad \forall\ x,y \in X.$$

Then $F(T)$ *is non empty.*

Remark 3.5. Corollary 3.3 and Corollary 3.4 are equivalent.

Corollary 3.6. *Let* (X,d) *be a metric space equipped with a binary relation* $\mathcal{R}$. *Suppose* T *is a self-mapping on* X *with the following conditions:*

(i) *there exists* $Y \subseteq X, TX \subseteq Y$ *such that* (Y,d) *is* $\mathcal{R}$*-complete,*

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in \mathcal{R}$

(iii) $\mathcal{R}$ *is* T*-closed,*

(iv) *either* T *is* $\mathcal{R}$*-continuous or* $\mathcal{R}|_Y$ *is* d*-self-closed,*

(v) *there exists* $k \in [0, \frac{1}{2})$ *such that*

$$d(Tx, Ty) \leq k[d(Ty, y) + d(x, Tx)] \quad \forall\, x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then $F(T)$ *is non empty.*

Corollary 3.7. *Let* (X, d) *be a metric space and* $\alpha : X \times X \to [0, \infty)$ *be an arbitrary function. Suppose* T *is a self-mapping on* X *with the following conditions:*

(i) *there exists* $Y \subseteq X, TX \subseteq Y$ *such that* (Y, d) *is* α*-complete,*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$

(iii) T *is* α*-admissible,*

(iv) *either* T *is* α*-continuous or if* x_n *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *as* $n \to \infty$*, then there exists a subsequence* (x_{n_k}) *of* (x_n) *such that* $\alpha(x_{n_k}, x) \geq 1$ *or* $\alpha(x, x_{n_k}) \geq 1$ *for all* k*,*

(v) *there exists* $k \in [0, \frac{1}{2})$ *such that*

$$\alpha(x, y) d(Tx, Ty) \leq k[d(Ty, y) + d(x, Tx)] \quad \forall\, x, y \in X.$$

Then $F(T)$ *is non empty.*

Remark 3.8. Corollary 3.6 and Corollary 3.7 are equivalent.

4. Conclusion

After studying the literature concerning α-complete metric spaces and relation-theoretic metrical fixed point results simultaneously, we can conclude that the relation-theoretic metrical fixed point results do not contribute any thing new in the literature. Therefore, relation-theoretic metrical fixed point results, common fixed point results and the other results (published/unpublished) are equivalent to the corresponding existing results in α-complete metric spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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