Generalized Turan-Type Inequalities for the \((q,k)\)-Polygamma Functions

Kwara Nantomah
Department of Mathematics, University for Development Studies, Navrongo Campus, P.O. Box 24, Navrongo, UE/R, Ghana
*Corresponding author: knantomah@uds.edu.gh

Abstract. This study provides generalizations of the results of Merovci concerning some Turan-type inequalities involving the \((q,k)\)-Polygamma functions.

Keywords. \((q,k)\)-Polygamma function; Generalized Hölder’s inequality; Generalized Minkowski’s inequality; Weighted AM-GM inequality

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1. Introduction

In this paper, we provide generalizations of the results of Merovci concerning some Turan-type inequalities involving the \((q,k)\)-Polygamma functions. In the meantime, we present the following definitions and preliminary materials.

The \((q,k)\)-analogue of the Gamma function, \(\Gamma_{q,k}(x)\) is defined for \(x > 0\), \(q \in (0,1)\) and \(k > 0\) by any of the following equivalent definitions (see [3], [4] and the references therein).

\[
\Gamma_{q,k}(x) = \int_0^{\left(\frac{x}{1-qk}\right)^{\frac{1}{k}} \Gamma^{-1}\frac{k}{\frac{k}{qk}} q^{-\frac{k}{qk}} \left(x - \frac{k}{qk}\right) \Gamma_{q,k}(x) \, dq \, d \frac{k}{qk} = \frac{(1-q^k)^{\frac{x}{qk}}}{(1-q)^{\frac{x}{qk}}}.
\]
where $(1 + x)_{q,k}^t = \left(\frac{1}{1 - q}\right)_{q,k}^{\sum_{i=0}^{t} \frac{q^i x}{q^i}}$, $(1 + x)_{q,k}^{\infty} = \prod_{i=0}^{\infty} (1 + q^i k x)$ for $x, t \in \mathbb{R}$ and $E_{q,k}^{x} = \sum_{n=0}^{\infty} \frac{q^n (n-1)! x^n}{[n]_{q,k}!} = (1 + (1 - q^k) x)_{q,k}^{\infty}$ is the $(q,k)$-analogue of the classical exponential function. The function $\Gamma_{q,k}(x)$ satisfies the basic identities \cite{3}.

In order to achieve this, the following lemmas shall be employed.

Theorem 1.1. If $m$ is even, then $\psi_{q,k}(x) + \psi_{q,k}(y) \leq (\psi_{q,k}(x))^\frac{1}{2} + (\psi_{q,k}(y))^\frac{1}{2}$, \hspace{1cm} (1.2)

where $m+n$ is an integer, $a > 1, \frac{1}{a} + \frac{1}{b} = 1$.

Theorem 1.2. For $m, n = 1, 2, 3, \ldots$,

$\left(\psi_{q,k}(x) + \psi_{q,k}(y)\right)^\frac{1}{2} \leq (\psi_{q,k}(x))^\frac{1}{2} + (\psi_{q,k}(y))^\frac{1}{2}$, \hspace{1cm} (1.3)

where $m+n$ is an integer, $u \geq 1$.

Theorem 1.3. For every $x > 0$ and integers $m \geq 1$, we have:

If $m$ is odd, then $\left(\exp \psi_{q,k}(x)\right)^2 \geq \exp \psi_{q,k}(x) \cdot \exp \psi_{q,k}(x)$; \hspace{1cm} (1.4)

If $m$ is even, then $\left(\exp \psi_{q,k}(x)\right)^2 \leq \exp \psi_{q,k}(x) \cdot \exp \psi_{q,k}(x)$. \hspace{1cm} (1.5)

Also, in a recent paper \cite{9}, the author gave improvements of Theorems 1.1 and 1.2.

In the present work, the main objective is to provide generalizations of the above inequalities. In order to achieve this, the following lemmas shall be employed.
Lemma 1.4 (Generalized Hölder’s Inequality). Let \( i = 1, 2, 3, \ldots, N \) and \( n = 1, 2, 3, \ldots, T \) such that the sums exist. Then the inequality
\[
\sum_{n=1}^{T} \left| \prod_{i=1}^{N} Q_{i,n} \right| \leq \prod_{i=1}^{N} \left( \sum_{n=1}^{T} |Q_{i,n}|^{\alpha_i} \right)^{\frac{1}{\alpha_i}}
\] (1.6)
is valid for \( \alpha_i > 1 \) such that \( \sum_{i=1}^{N} \frac{1}{\alpha_i} = 1 \).

Lemma 1.5 (Generalized Minkowski’s Inequality). Let \( i = 1, 2, 3, \ldots, N \) and \( n = 1, 2, 3, \ldots, T \) such that the sums exist. Then the inequality
\[
\left( \sum_{n=1}^{T} \left| \prod_{i=1}^{N} Q_{i,n} \right|^{u} \right)^{\frac{1}{u}} \leq \prod_{i=1}^{N} \left( \sum_{n=1}^{T} |Q_{i,n}|^{u} \right)^{\frac{1}{u}}
\] (1.7)
is valid for \( u \geq 1 \).

For proofs of Lemmas 1.4 and 1.5 see [8] and the references therein.

Lemma 1.6 (Weighted AM-GM Inequality). For \( i = 1, 2, 3, \ldots, n \), let \( Q_{i} \geq 0 \) and \( \lambda_{i} \geq 0 \) such that \( \sum_{i=1}^{n} \lambda_{i} = 1 \). Then the inequality
\[
\sum_{i=1}^{n} \lambda_{i} Q_{i} \geq \prod_{i=1}^{n} Q_{i}^{\lambda_{i}}
\] (1.8)
is valid .

Lemma 1.6 is well-known in the literature. See for instance [5] and the related references.

2. Main Results

In this section, we present generalizations of the results of Merovci as shown in Theorem 1.1, 1.2 and 1.3.

Theorem 2.1. For \( i = 1, 2, \ldots, N \), let \( \alpha_i > 1 \), \( \sum_{i=1}^{N} \frac{1}{\alpha_i} = 1 \) and \( m_i \in \mathbb{N} \) such that \( \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \in \mathbb{N} \). Then the inequality
\[
\left| \psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i} \right)} \left( \sum_{i=1}^{N} x_i \right)^{\frac{1}{\alpha_i}} \right| \leq \prod_{i=1}^{N} \left| \psi_{q,k}^{\left(\frac{m_i}{\alpha_i}\right)}(x_i) \right|^{\frac{1}{\alpha_i}}
\] (2.1)
is valid for \( x_i > 0 \).

Proof. From (1.1) and (1.6) we obtain
\[
\left| \psi_{q,k}^{\left(\sum_{i=1}^{N} \frac{m_i}{\alpha_i} \right)} \left( \sum_{i=1}^{N} x_i \right)^{\frac{1}{\alpha_i}} \right| = \left| \ln q \right| \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \sum_{n=1}^{\infty} \frac{\left( n k \right)^{\sum_{i=1}^{N} \frac{x_i}{\alpha_i}} q^{n k \sum_{i=1}^{N} \frac{x_i}{\alpha_i}}}{1 - q^{n k}}
\]
\[
= \sum_{n=1}^{\infty} \left| \ln q \right| \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \left( n k \right)^{\sum_{i=1}^{N} \frac{x_i}{\alpha_i}} q^{n k \sum_{i=1}^{N} \frac{x_i}{\alpha_i}} \frac{1}{1 - q^{n k}}
\]
\[
= \sum_{n=1}^{\infty} \left| \ln q \right| \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \left( n k \right)^{\sum_{i=1}^{N} \frac{x_i}{\alpha_i}} q^{n k \sum_{i=1}^{N} \frac{x_i}{\alpha_i}} \frac{1}{1 - q^{n k}}
\]
which yields the result \( (2.1) \).

**Remark 2.2.** Let \( N = 2 \), \( a_1 = a \), \( a_2 = b \), \( m_1 = m \), \( m_2 = n \), \( x_1 = x \) and \( x_2 = y \) in Theorem 2.1. Then, we get

\[
\left| \psi_{q,k}^{(m_1)} \left( \frac{x}{a} + \frac{y}{b} \right) \right| \leq \left| \psi_{q,k}^{(m_1)} (x) \right|^{\frac{1}{u}} \left| \psi_{q,k}^{(m_1)} (y) \right|^{\frac{1}{u}}
\]

as obtained in [9]. Particularly, if \( m \) and \( n \) are odd, then we obtain the result \( (1.2) \).

**Remark 2.3.** By letting \( q \to 1 \) as \( k \to 1 \) in Theorem 2.1, we recover Theorem 2.1 of [1].

**Theorem 2.4.** For \( i = 1, 2, \ldots, N \), let \( m_i \in \mathbb{N} \). Then the inequality

\[
\left( \sum_{i=1}^{N} \left| \psi_{q,k}^{(m_i)} (x_i) \right| \right)^{\frac{1}{u}} \leq \sum_{i=1}^{N} \left| \psi_{q,k}^{(m_i)} (x_i) \right|^{\frac{1}{u}}
\]

(2.2)

is valid for \( x_i > 0 \) and \( u \geq 1 \).

**Proof.** From (1.1) we obtain

\[
\left( \sum_{i=1}^{N} \left| \psi_{q,k}^{(m_i)} (x_i) \right| \right)^{\frac{1}{u}} = \left( \sum_{i=1}^{N} \sum_{n=1}^{\infty} \frac{\left| \ln q \right|^{m_i+1} (nk)^{m_i} q^{nk x_i}}{1-q^{nk}} \right)^{\frac{1}{u}}
\]

\[
= \left( \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{\left| \ln q \right|^{m_i+1} \left( \frac{m_i}{n} q^{nk x_i} \right)}{1-q^{nk}} \right)^{\frac{1}{u}}
\]

\[
\leq \left( \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{\left| \ln q \right|^{m_i+1} \left( \frac{m_i}{n} q^{nk x_i} \right)}{1-q^{nk}} \right)^{\frac{1}{u}}
\]

\[
= \sum_{i=1}^{N} \sum_{n=1}^{\infty} \frac{\left| \ln q \right|^{m_i+1} (nk)^{m_i} q^{nk x_i}}{1-q^{nk}}
\]

\[
= \sum_{i=1}^{N} \left| \psi_{q,k}^{(m_i)} (x_i) \right|^{\frac{1}{u}}
\]
which yields the result \((2.2)\). (Note. The first inequality follows from the fact that \(\sum_{i=1}^{n} a_i^u \leq \left( \sum_{i=1}^{n} a_i \right)^u\), for \(a_i \geq 0, u \geq 1, n \in \mathbb{N}\) whiles the second inequality is as a result of the generalized Minkowski’s inequality \((1.7)\).

\[\text{Remark 2.5.}\]

By letting \(N = 2, m_1 = m, m_2 = n, x_1 = x\) and \(x_2 = y\) in Theorem 2.4 we get

\[\left( |\psi_{q,k}^{(m)}(x)| + |\psi_{q,k}^{(n)}(y)| \right)^{\frac{1}{u}} \leq |\psi_{q,k}^{(m)}(x)|^{\frac{1}{u}} + |\psi_{q,k}^{(n)}(y)|^{\frac{1}{u}}\]

as obtained in \([9]\). Particularly, if \(m\) and \(n\) are odd, then we obtain the result \((1.3)\).

\[\text{Theorem 2.6.}\]

For \(i = 1, 2, \ldots, N\), let \(\alpha_i > 1, \sum_{i=1}^{N} \frac{1}{\alpha_i} = 1\) and \(m_i \in \mathbb{N}\) such that \(\sum_{i=1}^{N} \frac{m_i}{\alpha_i} \in \mathbb{N}\). Then, the following inequalities are valid for \(x_i > 0\).

\[
\exp \left( \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \right) \left( \sum_{i=1}^{N} \frac{x_i}{\alpha_i} \right) \leq \prod_{i=1}^{N} \left( \exp \left( \psi_{q,k}^{(m_i)}(x_i) \right) \right)^{\frac{1}{\alpha_i}} \quad \text{if } m_i, \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \text{ are odd} \tag{2.3}
\]

and

\[
\exp \left( \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \right) \left( \sum_{i=1}^{N} \frac{x_i}{\alpha_i} \right) \geq \prod_{i=1}^{N} \left( \exp \left( \psi_{q,k}^{(m_i)}(x_i) \right) \right)^{\frac{1}{\alpha_i}} \quad \text{if } m_i, \sum_{i=1}^{N} \frac{m_i}{\alpha_i} \text{ are even} \tag{2.4}
\]

\[\text{Proof.}\]

Let \(m_i\) and \(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\) both be odd. Then,

\[
\psi_{q,k}^{(m_i)} \left( \sum_{i=1}^{N} \frac{x_i}{\alpha_i} \right) - \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i} = 1 - q^{nk} \sum_{i=1}^{N} \frac{x_i}{\alpha_i} - \sum_{i=1}^{N} \frac{1}{\alpha_i} \sum_{n=1}^{\infty} \frac{(\ln q)^{m_i+1}(nk)^{m_i}q^{nkx_i}}{1 - q^{nk}}
\]

which follows from the weighted AM-GM inequality \((1.8)\). Hence

\[
\psi_{q,k}^{(m_i)} \left( \sum_{i=1}^{N} \frac{x_i}{\alpha_i} \right) \leq \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i} \tag{2.5}
\]

Similarly, if \(m_i\) and \(\sum_{i=1}^{N} \frac{m_i}{\alpha_i}\) are both even, then we obtain

\[
\psi_{q,k}^{(m_i)} \left( \sum_{i=1}^{N} \frac{x_i}{\alpha_i} \right) \geq \sum_{i=1}^{N} \frac{\psi_{q,k}^{(m_i)}(x_i)}{\alpha_i} \tag{2.6}
\]

Finally, by exponentiating \((2.5)\) and \((2.6)\), we obtain \((2.3)\) and \((2.4)\), respectively. \qed
3. Concluding Remarks

By using the generalized Hölder’s inequality, the generalized Minkowski’s inequality and the weighted AM-GM inequality, we provide generalizations of the results of Merovci [6]. Interested readers can also refer to the work [7] for similar results involving the m-th derivative of the (q,k)-Gamma function.

Competing Interests
The author declares that he has no competing interests.

Authors’ Contributions
The author wrote, read and approved the final manuscript.

References


