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Research Article

Domination in Cayley Digraphs of Right and Left Groups

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Abstract. Let $Cay(S,A)$ denote a Cayley digraph of a semigroup S with a connection set A. A semigroup S is said to be a right group if it is isomorphic to the direct product of a group and a right zero semigroup and S is called a left group if it is isomorphic to the direct product of a group and a left zero semigroup. In this paper, we attempt to find the value or bounds for the domination number of Cayley digraphs of right groups and left groups. Some examples which give sharpness of those bounds are also shown. Moreover, we consider the total domination number and give the necessary and sufficient conditions for the existence of total dominating sets in Cayley digraphs of right groups and left groups.

Keywords. Cayley digraph; Right group; Left group; Domination number; Total domination number

MSC. 05C20; 05C25; 05C69; 05C99; 20M99

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1. Introduction

Let S be a semigroup and A a subset of S. Recall that the Cayley digraph $Cay(S,A)$ of S with the connection set A is defined as the digraph with a vertex set S and an arc set $E(Cay(S,A)) = \{(x,xa) \mid x \in S, a \in A\}$ (see [7]). Clearly, if A is an empty set, then $Cay(S,A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. Cayley graphs of groups have been extensively studied and many interesting results have been obtained (see for examples, [1], [12], [13], and [14]). Also, the Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have recently appeared in various journals (see for examples, [4], [8], [9], [10], [12], [15], [16], [17], [18], and [19]). Furthermore, some properties of the Cayley digraphs of left groups and right groups are obtained by some authors (see for examples, [11], [16], [17], [18], and [21]).

The concept of domination for Cayley graphs has been studied by various authors (see for examples, [3], [5], [22], [24], and [25]). The total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi (see [3]) and is now well studied in graph theory. Tamizh Chelvam and Rani (see for examples, [24], [25], and [26]) have obtained bounds for various domination parameters for a class of Circulant graphs. Here we shall study some domination parameters of Cayley digraphs of right groups and left groups. All graphs considered in this paper are finite directed graphs. The terminologies and notations which related to our paper will be defined in the next section.

2. Preliminaries and Notations

In this section, we give some preliminaries needed in what follows on digraphs, semigroups, Cayley graphs of semigroups, domination number, and total domination number. For more information on digraphs, we refer to [2], and for semigroups see [6]. All sets in this paper are assumed to be finite.

A semigroup S is said to be a *right* (*left*) *zero semigroup* if $xy = y(xy = x)$ for all $x, y \in S$. A semigroup S is called a *right (left) group* if it is isomorphic to the direct product $G \times R_m(G \times L_m)$ of a group G and an m-element right (left) zero semigroup $R_m(L_m)$. If $m = 1$, then we can consider a Cayley digraph of a right (left) group $G \times R_m(G \times L_m)$ as a Cayley digraph of a group G. So in this paper, we will consider in the case where $m \geq 2$.

Let $D = (V, E)$ be a digraph. A set $X \subseteq V$ of vertices in a digraph D is called a *dominating set* if every vertex $v \in V \setminus X$, there exists $x \in X$ such that $(x, v) \in E$ and we call that x dominates v or v is dominated by x. The *domination number* $\gamma(D)$ of a digraph D is the minimum cardinality of a dominating set in D and the corresponding dominating set is called a γ*-set*. A set $X \subseteq V$ is called a *total dominating set* if every vertex $v \in V$, there exists $x \in X$ such that $(x, v) \in E$. The *total domination number* $\gamma_t(D)$ equals the minimum cardinality among all total dominating sets in D and the corresponding total dominating set is called a γ_t*-set*.

For any family of nonempty sets $\{X_i | i \in I\}$, we write $\dot{\bigcup}_{i\in I} X_i := \bigcup_{i\in I} X_i$ if $X_i \cap X_j = \emptyset$ for all $i \neq j$. Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (\dot{\bigcup}_{i=1}^{n} V_i, \dot{\bigcup}_{i=1}^{n} E_i)$. It is easy to verify that $\gamma(\dot{\bigcup}_{i=1}^{n}(V_i, E_i)) = \sum\limits_{i=1}^{n} \gamma(V_i, E_i)$, $\gamma_t(\dot{\bigcup}_{i=1}^{n}(V_i, E_i)) = \sum\limits_{i=1}^{n} \gamma_t(V_i, E_i)$, and if for each $i, j \in I$ with $(V_i, E_i) \cong (V_j, E_j)$, then $\gamma(V_i, E_i) = \gamma(V_j, E_j)$ and $\gamma_t(V_i, E_i) = \gamma_t(V_j, E_j)$. From now on, $|A|$ denotes the cardinality of A where A is any finite set and p_i denotes the projection map on the i^{th} coordinate of a triple where $i \in \{1, 2, 3\}$. A subdigraph F of a digraph G is called a strong subdigraph of G if and only if whenever u and v are vertices of F and (u,v) is an arc in G, then (u,v) is an arc in F as well. Moreover, we denote by G a finite group and G_k a group of order k, and let $R_m(L_m)$ denote a right (left) zero semigroup with m elements. Now, we recall some results which are needed in the sequel as below for further references.

Lemma 2.1 ([20]). *Let $S = G \times R_m$ be a right group, A a nonempty subset of S such that $p_2(A) = R_m, G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1,2,\dots,k\}$. Then*

(1) *$S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times R_m | i \in I\}$ and $S = \dot{\bigcup\limits_{i\in I}}(g_i\langle p_1(A)\rangle \times R_m)$,*

(2) *$Cay(S,A) = \dot{\bigcup}_{i\in I}((g_i\langle p_1(A)\rangle \times R_m), E_i)$ where $((g_i\langle p_1(A)\rangle \times R_m), E_i)$ is a strong subdigraph of $Cay(S,A)$ with $((g_i\langle p_1(A)\rangle \times R_m), E_i) \cong Cay(\langle A\rangle, A)$ for all $i \in I$.*

Lemma 2.2 ([20]). *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. Then $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A)$ is a right group contained in S.*

Lemma 2.3 ([16]). *Let $S = G \times L_m$ be a left group and A a nonempty subset of S. Then the following conditions hold :*

(1) *for each $l \in L_m$, $Cay(G \times \{l\}, p_1(A) \times \{l\}) \cong Cay(G, p_1(A))$,*

(2) *$Cay(S,A) = \dot{\bigcup\limits_{l\in L_m}} Cay(G \times \{l\}, p_1(A) \times \{l\})$.*

Lemma 2.4 ([20]). *Let $S = G \times L_m$ be a left group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1,2,\dots,k\}$. Then*

(1) *$S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times \{l\} | i \in I, l \in L_m\}$ and $S = \dot{\bigcup}_{i\in I, l\in L_m}(g_i\langle p_1(A)\rangle \times \{l\})$,*

(2) *$Cay(S,A) = \dot{\bigcup}_{i\in I, l\in L_m}((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ where $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ is a strong subdigraph of $Cay(S,A)$ with $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) \cong Cay(\langle p_1(A)\rangle, p_1(A))$ for all $i \in I$, $l \in L_m$.*

The following lemmas give the results for the domination number and the total domination number of Cayley graphs of the group $\mathbb{Z}_n$ with respect to the specific connection sets.

Lemma 2.5 ([23]). *Let $n \geq 3$ be an odd integer, $m = \frac{n-1}{2}$ and $A = \{m, n-m, m-1, n-(m-1), \dots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$ where $1 \leq k \leq m$. Then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k} \rceil$.*

Lemma 2.6 ([23]). *Let $n \geq 3$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and $A = \{\frac{n}{2}, m, n-m, m-1, n-(m-1), \dots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$ where $1 \leq k \leq m$. Then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k+1} \rceil$.*

3. Main Results

In this section, we give some results for the domination parameters of Cayley digraphs of right groups and left groups related to the according connection sets such as the domination number and total domination number. We divide this section into two parts. The first part gives some results for the domination in Cayley digraphs of right groups and the second part for left groups. Hereafter, we will denote by D the Cayley digraph $Cay(S,A)$ of a semigroup S with a connection set A.

3.1 The domination parameters of Cayley digraphs of right groups

In this part, we study the domination parameters of Cayley digraphs of right groups related to the appropriate connection sets. We start with the theorem which describes the domination number in Cayley digraphs of right groups with arbitrary connection sets A where $|p_2(A)| \neq |R_m|$.

Theorem 3.1. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. If $|p_2(A)| \neq |R_m|$, then $\gamma(D) = (|R_m| - |p_2(A)|) \times |G|$.*

Proof. Suppose that $|p_2(A)| \neq |R_m|$, we have $|p_2(A)| < |R_m|$. Let $Y = \{(x,a) \in S | a \notin p_2(A)\}$. We will show that Y is a dominating set of D. Let $(b,c) \in S \setminus Y$. Then $b \in G$ and $c \in p_2(A)$, that is, there exists $d \in p_1(A) \subseteq G$ such that $(d,c) \in A$. Since G is a group, there exists $y \in G$ such that $b = yd$. From $|R_m| > |p_2(A)|$, we get that there exists $r \in R_m \setminus p_2(A)$ which leads to $(y,r) \in Y$. Thus $(b,c) = (yd,c) = (y,r)(d,c)$. Therefore, Y is a dominating set of D. Hence $\gamma(D) \leq |Y| = (|R_m| - |p_2(A)|) \times |G|$. Now, we assume in the contrary that $\gamma(D) < (|R_m| - |p_2(A)|) \times |G|$. Let $X \subseteq S$ be a dominating set of D such that X is a γ-set, that is, $|X| = \gamma(D) < (|R_m| - |p_2(A)|) \times |G|$. We have

$$\begin{aligned}
|S \setminus X| &= |S| - |X| \\
&> nm - [(|R_m| - |p_2(A)|) \times |G|] \\
&= nm - [(m - |p_2(A)|) \times n] \\
&= nm - nm + n(|p_2(A)|) \\
&= n(|p_2(A)|) \\
&= |G \times p_2(A)|.
\end{aligned}$$

Thus there exists at least one element $(a,b) \in (S \setminus X) \setminus (G \times p_2(A))$, that is, $(a,b) \in S \setminus X$ and $(a,b) \notin G \times p_2(A)$. Since $a \in G$, we obtain that $b \notin p_2(A)$. From $(a,b) \in S \setminus X$ and X is the dominating set of D, there exists $(x,y) \in X$ such that $((x,y),(a,b)) \in E(D)$. Thus $(a,b) = (x,y)(c,d) = (xc,yd) = (xc,d)$ for some $(c,d) \in A$. We conclude that $b = d \in p_2(A)$ which is a contradiction. Therefore, $\gamma(D) \not< (|R_m| - |p_2(A)|) \times |G|$, that is, $\gamma(D) = (|R_m| - |p_2(A)|) \times |G|$, as required. □

The next theorem gives the bounds of the domination number in Cayley digraphs of right groups with arbitrary connection sets A where $|p_2(A)| = |R_m|$.

Theorem 3.2. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. If $|p_2(A)| = |R_m|$, then $\frac{|S|}{|A|+1} \leq \gamma(D) \leq |G|$.*

Proof. Assume that $|p_2(A)| = |R_m|$. We first prove the right inequality, that is, $\gamma(D) \leq |G|$. For each $r \in R_m$, let $Y = \{(x,r) | x \in G\} = G \times \{r\}$. We will show that Y is a dominating set of D. Let $(a,b) \in S \setminus Y$. Then $a \in G$ and $b \in R_m$ such that $b \neq r$. Since $|p_2(A)| = |R_m|$ and $p_2(A) \subseteq R_m$, we get that $R_m = p_2(A)$ and then $b \in p_2(A)$. Thus there exists $c \in p_1(A) \subseteq G$ such that $(c,b) \in A$.

Since G is a group, there exists $g \in G$ such that $a = gc$. We obtain that $(a,b) = (gc,b) = (g,r)(c,b)$ where $(g,r) \in Y$. Hence Y is the dominating set of D. Therefore, $\gamma(D) \le |Y| = |G \times \{r\}| = |G|$.

Now, we will prove the left inequality. Let X be the dominating set of D such that X is a γ-set, that is, $|X| = \gamma(D)$. Then for each $(a,b) \in S \setminus X$, we get that $(a,b) = (x,y)(s,t)$ for some $(x,y) \in X$ and $(s,t) \in A$ which implies that $S \setminus X \subseteq XA$. Hence $|S \setminus X| \le |XA|$. Since every element of X has the same out-degree $|A|$, we obtain that

$$\gamma(D)|A| = |X||A| \ge |XA| \ge |S \setminus X| = |S| - |X| = |S| - \gamma(D).$$

Then $\gamma(D)|A| \ge |S| - \gamma(D)$ which leads to $|S| \le \gamma(D)|A| + \gamma(D) = \gamma(D)(|A|+1)$. Hence $\gamma(D) \ge \frac{|S|}{|A|+1}$. □

In the following example, we present the sharpness of the bounds given in Theorem 3.2.

Example 3.3. Let $\mathbb{Z}_3 \times R_2$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R_2 = \{r_1, r_2\}$ is a right zero semigroup.

(1) Consider the Cayley digraph $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$.

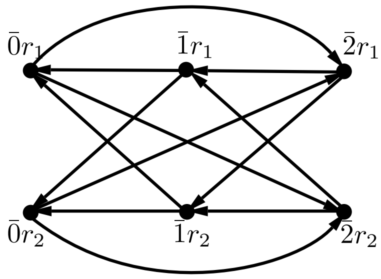

Figure 1. $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$.

We have $X = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1)\}$ is a γ-set of $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$ and $\gamma(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})) = |X| = 3 = |\mathbb{Z}_3|$. Similarly, $\gamma(Cay(\mathbb{Z}_n \times R_2, \{(\bar{2}, r_1), (\bar{2}, r_2)\})) = |\mathbb{Z}_n|$ where $n \in \mathbb{N}$.

(2) Consider the Cayley digraph $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$.

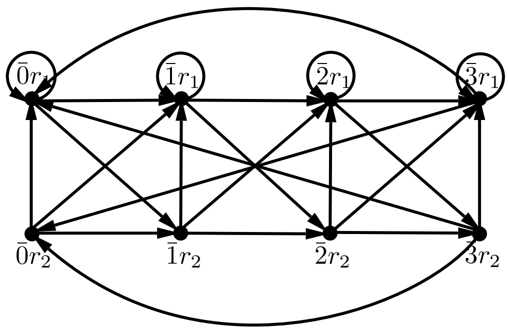

Figure 2. $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$.

We have $Y = \{(\bar{0}, r_2), (\bar{2}, r_2)\}$ is a γ-set of $Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$ and $\gamma(Cay(\mathbb{Z}_4 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})) = |Y| = 2 = \frac{|\mathbb{Z}_4 \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\}|+1}$.

Similarly, we also obtain $\gamma(Cay(\mathbb{Z}_{2k} \times R_2, \{(\bar{0},r_1),(\bar{1},r_1),(\bar{1},r_2)\})) = k = \frac{4k}{4} = \frac{|\mathbb{Z}_{2k} \times R_2|}{|\{(\bar{0},r_1),(\bar{1},r_1),(\bar{1},r_2)\}|+1}$ with a γ-set $\{(\bar{0},r_2),(\bar{2},r_2),(\bar{4},r_2),\dots,(\overline{2k-2},r_2)\}$ where $k \in \mathbb{N}$.

The following theorems show the values for the domination number in Cayley digraphs of right groups according to the specific connection sets. We begin with two lemmas which are referred in the proofs of theorems.

Lemma 3.4. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $p_1(A) = G$, $p_2(A) = R_m$, and $|A| = |R_m|$. For each $(x_1,r_1),(x_2,r_2) \in S$, if $(x_1,r_1)(y_1,s_1) = (x_2,r_2)(y_2,s_2)$ for some $(y_1,s_1),(y_2,s_2) \in A$, then $x_1 = x_2$.*

Proof. Let $(x_1,r_1),(x_2,r_2) \in S$ be such that $(x_1,r_1)(y_1,s_1) = (x_2,r_2)(y_2,s_2)$ for some (y_1,s_1), $(y_2,s_2) \in A$. Thus $(x_1y_1,r_1s_1) = (x_2y_2,r_2s_2)$, that is, $(x_1y_1,s_1) = (x_2y_2,s_2)$. Then $x_1y_1 = x_2y_2$ and $s_1 = s_2$. Since we know that $p_1(A) = G$, $p_2(A) = R_m$, and $|A| = |R_m|$, these imply $y_1 = y_2$. From $x_1y_1 = x_2y_2$ where x_1,x_2,y_1,y_2 are elements of a group G and $y_1 = y_2$, we can conclude that $x_1 = x_2$ by the cancellation law. □

Lemma 3.5. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Let Y be a dominating set of D. If there exists $x \in G$ such that $x \notin p_1(Y)$, then $(xa,r) \in Y$ for all $r \in R_m$.*

Proof. Let Y be a dominating set of D. Suppose that there exists $x \in G$ such that $x \notin p_1(Y)$ and assume in the contrary that there exists $r \in R_m$ such that $(xa,r) \notin Y$. Since Y is a dominating set of D, there exists $(y,r') \in Y$ such that $((y,r'),(xa,r)) \in E(D)$, that is, $(xa,r) = (y,r')(a,r)$ where $(a,r) \in A$. Hence $xa = ya$ which implies that $x = y \in p_1(Y)$ which contradicts to our supposition. Therefore, $(xa,r) \in Y$ for all $r \in R_m$. □

Theorem 3.6. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $p_1(A) = G$, $p_2(A) = R_m$, and $|A| = |R_m|$. Then $\gamma(D) = |G|$.*

Proof. Assume that the conditions hold. Since $|p_2(A)| = |R_m|$, we obtain that $\gamma(D) \leq |G|$ by Theorem 3.2. Now, suppose that there exists a dominating set Y such that $|Y| < |G|$. Then there exists $g \in G$ such that $g \notin p_1(Y)$. We first prove that for each $r \in R_m$, $(g,r)A \subseteq Y$. Let $r \in R_m$ and $(x,y) \in (g,r)A$. Then $(x,y) = (g,r)(g_1,r_1)$ for some $(g_1,r_1) \in A$. If $(x,y) \notin Y$, then there exists $(g',r') \in Y$ such that $(x,y) = (g',r')(g_2,r_2)$ for some $(g_2,r_2) \in A$ since Y is a dominating set of D. Thus $(g,r)(g_1,r_1) = (x,y) = (g',r')(g_2,r_2)$ where $(g_1,r_1),(g_2,r_2) \in A$. By Lemma 3.4, we can conclude that $g = g' \in p_1(Y)$ which is a contradiction. Hence $(x,y) \in Y$ which leads to $(g,r)A \subseteq Y$. Since $p_1(A) = G$, we obtain that the identity element e of G lies in $p_1(A)$. Then there exists $s \in p_2(A)$ such that $(e,s) \in A$ and $(g,s) = (g,r)(e,s) \in (g,r)A \subseteq Y$. Whence $g \in p_1(Y)$ which contradicts to the above supposition. Therefore, we can conclude that $\gamma(D) = |G|$. □

Theorem 3.7. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Then $\gamma(D) = |G|$.*

Proof. Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $A = \{a\} \times R_m$ where $a \in G$. Then $|p_2(A)| = |R_m|$. By Theorem 3.2, we obtain that $\gamma(D) \leq |G|$. Assume that there exists a dominating set Y of D such that $|Y| < |G| = n$. Then there exists $x \in G$ such that $x \notin p_1(Y)$. Let $U = \{u \in G | u \notin p_1(Y)\}$. Assume that $|U| = k$ where $1 \leq k \leq n-1$. For each $u \in U$, we obtain by Lemma 3.5 that $(ua, r) \in Y$ for all $r \in R_m$. Hence there exists at least one element $q \in p_1(Y)$ such that $(q, r) \in Y$ for all $r \in R_m$. Let $V = \{v \in p_1(Y) | (v, r) \in Y$ for all $r \in R_m\}$. Assume that $|V| = l$ where $1 \leq l \leq n-k$. By Lemma 3.5 again, we get that $|Y| \geq ml + [(n-k)-l] + (k-l) = ml + n - 2l = n + (m-2)l$. Since $m \geq 2$, we obtain that $|Y| \geq n + (m-2)l \geq n$, a contradiction. Therefore, $\gamma(D) = |G|$, as required. □

Theorem 3.8. *Let $S = G \times R_m$ be a right group and $A = K \times R_m$ a nonempty subset of S where K is any subgroup of G. Then $\gamma(D) = \frac{|G|}{|K|}$.*

Proof. Let $S = G \times R_m$ be a right group and $A = K \times R_m$ a nonempty subset of S where K is a subgroup of a group G. Consider the set of all left cosets of K in G, $G/K = \{g_1K, g_2K, \ldots, g_tK\}$, we obtain that the index of K in G equals t, that is, $[G : K] = t$. Let $I = \{1, 2, \ldots, t\}$ be an index set. By Lemma 2.1, we have $S/\langle A \rangle = \{g_iK \times R_m | i \in I\}$ such that $S = \dot{\bigcup}_{i \in I}(g_iK \times R_m)$ and $Cay(S, A) = \dot{\bigcup}_{i \in I}((g_iK \times R_m), E_i)$ where $((g_iK \times R_m), E_i)$ is a strong subdigraph of $Cay(S, A)$ with $((g_iK \times R_m), E_i) \cong Cay(\langle A \rangle, A)$ for all $i \in I$. Thus

$$\begin{aligned}
\gamma(D) &= \gamma(Cay(S, A)) \\
&= \gamma\Big(\dot{\bigcup_{i \in I}}((g_iK \times R_m), E_i)\Big) \\
&= \sum_{i \in I} \gamma((g_iK \times R_m), E_i) \\
&= |I|[\gamma(Cay(\langle A \rangle, A))] \\
&= t[\gamma(Cay(\langle A \rangle, A))].
\end{aligned}$$

By Lemma 2.2, we can conclude that $\langle A \rangle = \langle p_1(A) \rangle \times p_2(A) = \langle K \rangle \times R_m = K \times R_m = A$. Furthermore, we can prove that $\gamma(Cay(\langle A \rangle, A)) = 1$ which implies that $\gamma(D) = t = [G : K] = \frac{|G|}{|K|}$. □

The next theorem gives the necessary and sufficient conditions for the existence of the total dominating set in Cayley digraphs of right groups with connection sets.

Theorem 3.9. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S. Then the total dominating set of D exists if and only if $p_2(A) = R_m$.*

Proof. We first prove the necessary condition by assuming that the total dominating set of D exists, say T. We will show that $p_2(A) = R_m$. By the definition of the connection set A, we know that $p_2(A) \subseteq R_m$. Let $r \in R_m$. Then for each $a \in G$, we get that (a, r) is dominated by a vertex (x, y) for some $(x, y) \in T$ since T is the total dominating set of D. Thus there exists $(a', r') \in A$ such that $(a, r) = (x, y)(a', r') = (xa', yr') = (xa', r')$ which implies that $r = r'$, that is, $r \in p_2(A)$. Therefore, $p_2(A) = R_m$.

Conversely, we prove the sufficient condition by supposing that $p_2(A) = R_m$. We will prove that every vertex has an in-degree in D. Let $(g,r) \in G \times R_m$. Then $r \in R_m = p_2(A)$. Thus there exists $a \in p_1(A)$ such that $(a,r) \in A$. We obtain that $((ga^{-1},r'),(g,r)) = ((ga^{-1},r'),(ga^{-1},r')(a,r)) \in E(D)$, that is, (g,r) is dominated by (ga^{-1},r'). So we can conclude that every vertex of D always has an in-degree in D. If we take $T = V(D) = S$, then we can see that T is a total dominating set of D since for each $(x,y) \in S$, (x,y) is dominated by some vertices in T. Hence the total dominating set of D always exists if $p_2(A) = R_m$. □

Theorem 3.10. *Let $S = G \times R_m$ be a right group and A a nonempty subset of S such that $p_2(A) = R_m$. Then $\frac{|S|}{|A|} \le \gamma_t(D) \le |G|$.*

Proof. Let A be a connection set of D such that $p_2(A) = R_m$. We know that the total dominating set of D exists by Theorem 3.9. For each $r \in R_m$, let $T = \{(g,r) | g \in G\} = G \times \{r\}$. We will show that T is a total dominating set of D. Let $(x,y) \in S = G \times R_m$. Since $p_2(A) = R_m$, we get that $y \in p_2(A)$ which implies that there exists $z \in p_1(A)$ such that $(z,y) \in A$. Since G is a group and $x,z \in G$, we obtain that $x = hz$ for some $h \in G$.

Thus there exists $(h,r) \in T$ such that $(x,y) = (hz,y) = (h,r)(z,y)$. Hence (x,y) is dominated by the vertex (h,r) in T. We can conclude that T is the total dominating set of D which leads to $\gamma_t(D) \le |T| = |G \times \{r\}| = |G|$.

Next, we will show that $\gamma_t(D) \ge \frac{|S|}{|A|}$. Assume in the contrary that there exists a total dominating set T' such that $|T'| < \frac{|S|}{|A|}$. Thus $|T'A| \le |T'||A| < |S|$ which implies that there exists at least one element $(p,q) \in S$ but $(p,q) \notin T'A$. Hence there is no an element in T' which dominates (p,q), this contradicts to the property of the total dominating set T'. Consequently, $\gamma_t(D) \ge \frac{|S|}{|A|}$, as required. □

In the following example, we present the sharpness of the bounds given in Theorem 3.10.

Example 3.11. Let $\mathbb{Z}_3 \times R_2$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R_2 = \{r_1, r_2\}$ is a right zero semigroup.

(1) Consider the Cayley digraph $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2},r_1),(\bar{0},r_2),(\bar{2},r_2)\})$.

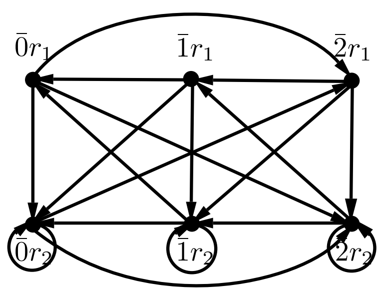

Figure 3. $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2},r_1),(\bar{0},r_2),(\bar{2},r_2)\})$.

We have $X = \{(\bar{0},r_1),(\bar{1},r_1),(\bar{2},r_1)\}$ is a γ_t-set of $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2},r_1),(\bar{0},r_2),(\bar{2},r_2)\})$ and $\gamma_t(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{2},r_1),(\bar{0},r_2),(\bar{2},r_2)\})) = |X| = 3 = |\mathbb{Z}_3|$.

Similarly, we can get that $\gamma_t(Cay(\mathbb{Z}_n \times R_2, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})) = |\mathbb{Z}_n|$ with a γ_t-set $\mathbb{Z}_n \times \{r_1\}$ where $n \in \mathbb{N}$.

(2) Consider $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$.

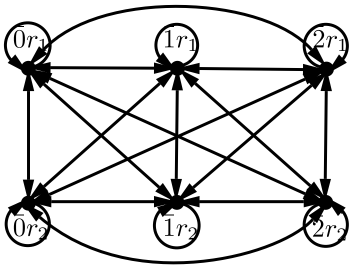

Figure 4. $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$.

We obtain that $Y = \{(\bar{0}, r_1)\}$ is a γ_t-set of $Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})$ and $\gamma_t(Cay(\mathbb{Z}_3 \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})) = |Y| = 1 = \dfrac{|\mathbb{Z}_3 \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\}|}$.

Similarly, $\gamma_t(Cay(\mathbb{Z}_{3k} \times R_2, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\})) = \dfrac{|\mathbb{Z}_{3k} \times R_2|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1), (\bar{0}, r_2), (\bar{1}, r_2), (\bar{2}, r_2)\}|}$ with the γ_t-set $\{(\bar{0}, r_1), (\bar{3}, r_1), (\bar{6}, r_1), \dots, (\overline{3k-3}, r_1)\}$ where $k \in \mathbb{N}$.

3.2 The domination parameters of Cayley digraphs of left groups

The following result gives us the domination number of a Cayley digraph of a left group $G \times L_m$ with a connection set A in the term of a domination number of a Cayley digraph of the subgroup $\langle p_1(A) \rangle$ of G.

Theorem 3.12. *Let $S = G \times L_m$ be a left group, A a nonempty subset of S, and $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_k\langle p_1(A)\rangle\}$. Then $\gamma(D) = m \cdot k \cdot \gamma(Cay(\langle p_1(A)\rangle, p_1(A)))$.*

Proof. Let $I = \{1, 2, \dots, k\}$. By Lemma 2.4, we have $D = \dot{\bigcup}_{i \in I, l \in L_m} ((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ such that a digraph $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ is the strong subdigraph of D with $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) \cong Cay(\langle p_1(A)\rangle, p_1(A))$ for all $i \in I, l \in L_m$. Therefore, $\gamma(D) = \sum_{i=1}^{k} \sum_{l=1}^{m} \gamma((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ and we can conclude that $\gamma(D) = m \cdot k \cdot \gamma(Cay(\langle p_1(A)\rangle, p_1(A)))$. □

Sometimes, it is not easy to find $\gamma(Cay(\langle p_1(A)\rangle, p_1(A)))$, so we can not find $\gamma(D)$ actually. However, we can know the bound of $\gamma(D)$ which is not depend on $\gamma(Cay(\langle p_1(A)\rangle, p_1(A)))$.

The next theorem gives the bounds of the domination number in Cayley digraphs of left groups with the according connection sets.

Theorem 3.13. *Let $S = G \times L_m$ be a left group and A a nonempty subset of S such that the identity of G lies in $p_1(A)$. If H is a subgroup of G with a maximum cardinality and contained in $p_1(A)$, then $\frac{|G|}{|p_1(A)|} \le \frac{\gamma(D)}{|L_m|} \le [G:H]$ where $[G:H]$ is the index of H in G.*

Proof. Suppose that H is the subgroup of G with a maximum cardinality such that $H \subseteq p_1(A)$. We will show that $\gamma(D) \le [G:H]|L_m|$. Let $[G:H] = k$ for some $k \in \mathbb{N}$. Consider the set of all left cosets of H in G, $\{g_1H, g_2H, \dots, g_kH\}$. Choose only one element from each left coset $g_1H, g_2H, \dots, g_kH$, say $g_1h_1, g_2h_2, \dots, g_kh_k$, respectively. Let $D_i = Cay(G \times \{l_i\}, p_1(A) \times \{l_i\})$ and $Y_i = \{g_1h_1, g_2h_2, \dots, g_kh_k\} \times \{l_i\} \subseteq G \times \{l_i\}$. We prove that Y_i is a dominating set of D_i. Let $(g, l_i) \in (G \times \{l_i\}) \setminus Y_i$. Since $g \in G = \bigcup_{t=1}^{k} g_tH$, we get that $g \in g_jH$ for some $1 \le j \le k$. Then $g = g_jh$ for some $h \in H$. Thus $(g_jh_j, l_i) \in Y_i$ and $h_j^{-1}h \in H \subseteq p_1(A)$. So there exists $l_q \in p_2(A)$ such that $(h_j^{-1}h, l_q) \in A$ and we have $(g, l_i) = (g_jh, l_i) = ((g_jh_j)(h_j^{-1}h), l_i) = (g_jh_j, l_i)(h_j^{-1}h, l_q) \in Y_iA$. Hence Y_i is the dominating set of D_i and then $\gamma(D_i) \le |Y_i| = k = [G:H]$. By Lemma 2.3, we can conclude that $\gamma(D) = \sum_{i=1}^{m} \gamma(D_i) = \gamma(D_i)|L_m| \le [G:H]|L_m|$. Now, we will prove that $\gamma(D) \ge \frac{|G|}{|p_1(A)|}|L_m|$. By Lemma 2.3(1), we have

$$Cay(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong Cay(G \times \{l_j\}, p_1(A) \times \{l_j\}) \text{ for all } l_i, l_j \in L_m.$$

For each $1 \le i \le m$, we will consider the domination number of D_i and let X_i be the dominating set of D_i such that X_i is a γ-set. Since the identity of G lies in $p_1(A)$ and X_i is the dominating set of D_i, we get that $(X_i)(p_1(A) \times \{l_i\}) = G \times \{l_i\}$. Hence $|G| = |G \times \{l_i\}| = |(X_i)(p_1(A) \times \{l_i\})| \le |X_i||p_1(A) \times \{l_i\}| = |X_i||p_1(A)|$. Thus $\gamma(D_i) = |X_i| \ge \frac{|G|}{|p_1(A)|}$. By Lemma 2.3(2), we obtain that $D = \dot{\bigcup}_{1 \le i \le m} D_i$. Then we conclude that $\gamma(D) = \gamma(\dot{\bigcup}_{1 \le i \le m} D_i) = \sum_{i=1}^{m} \gamma(D_i) = \gamma(D_i)|L_m| \ge \frac{|G|}{|p_1(A)|}|L_m|$. □

Corollary 3.14. *Let $S = G \times L_m$ be a left group and $A = K \times L_m$ a nonempty subset of S where K is any subgroup of G. Then $\gamma(D) = [G:K]|L_m|$.*

Proof. Since $A = K \times L_m$ where K is any subgroup of G, we obtain that the identity e of G lies in $K = p_1(A)$. Moreover, we get that K is the subgroup of G with a maximum cardinality that contained in $p_1(A)$. By Theorem 3.13, we obtain that $\frac{|G|}{|K|}|L_m| \le \gamma(D) \le [G:K]|L_m|$. Therefore, $\gamma(D) = [G:K]|L_m|$ since $[G:K] = \frac{|G|}{|K|}$. □

The following example gives the sharpness of bounds given in Theorem 3.13.

Example 3.15. Let $\mathbb{Z}_6 \times L_2$ be a left group where $\mathbb{Z}_6$ is a group of integers modulo 6 under the addition and $L_2 = \{l_1, l_2\}$ is a left zero semigroup.

(1) Consider the Cayley digraph $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.

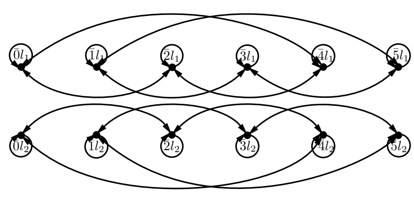

Figure 5. $Cay(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.

We have $X = \{(\bar{0},l_1),(\bar{0},l_2),(\bar{1},l_1),(\bar{1},l_2)\}$ is a γ-set of $Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),$ $(\bar{2},l_1),(\bar{4},l_1)\})$. Thus $\gamma(Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),(\bar{2},l_1),(\bar{4},l_1)\})) = |X| = 4 = 2(2) = [\mathbb{Z}_6 : H]|L_2|$ where $H = \{\bar{0},\bar{2},\bar{4}\}$ is the subgroup with a maximum cardinality of $\mathbb{Z}_6$ that contained in $p_1(\{(\bar{0},l_1),(\bar{2},l_1),(\bar{4},l_1)\})$.

Similarly, if $A = \{(\bar{0},l_1),(\bar{2},l_1),(\bar{4},l_1),\dots,(\overline{2k-2},l_1)\}$ is a nonempty subset of $\mathbb{Z}_{2k} \times L_2$ where $k \in \mathbb{N}$, then $\{(\bar{0},l_1),(\bar{0},l_2),(\bar{1},l_1),(\bar{1},l_2)\}$ is a γ-set of $Cay(\mathbb{Z}_{2k} \times L_2, A)$. Hence $\gamma(Cay(\mathbb{Z}_{2k} \times L_2,A)) = 4 = [\mathbb{Z}_{2k} : H]|L_2|$ where $H = \{\bar{0},\bar{2},\bar{4},\dots,\overline{2k-2}\}$ is the subgroup with a maximum cardinality of $\mathbb{Z}_{2k}$ that contained in $p_1(A)$.

(2) Consider the Cayley digraph $Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),(\bar{3},l_2)\})$.

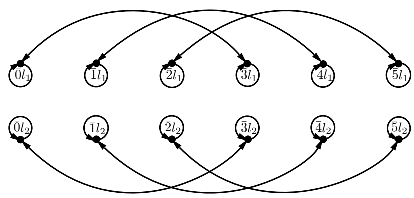

Figure 6. $Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),(\bar{3},l_2)\})$.

We have $Y = \{(\bar{0},l_1),(\bar{1},l_1),(\bar{2},l_1),(\bar{0},l_2),(\bar{1},l_2),(\bar{2},l_2)\}$ is a γ-set of $Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),(\bar{3},l_2)\})$ and $\gamma(Cay(\mathbb{Z}_6 \times L_2,\{(\bar{0},l_1),(\bar{3},l_2)\})) = |Y| = 6 = \frac{6}{2} \times 2 = \frac{|\mathbb{Z}_6|}{|p_1(\{(\bar{0},l_1),(\bar{3},l_2)\})|} \times |L_2|$.

Similarly, if $A = \{(\bar{0},l_1),(\bar{k},l_2)\}$ is a nonempty subset of $\mathbb{Z}_{2k} \times L_2$ where $k \in \mathbb{N}$, then $\{\bar{0},\bar{1},\bar{2},\dots,\overline{k-1}\} \times \{l_1,l_2\}$ is a γ-set of $Cay(\mathbb{Z}_{2k} \times L_2, A)$. Hence $\gamma(Cay(\mathbb{Z}_{2k} \times L_2,A)) = 2k = \frac{2k}{2} \times 2 = \frac{|\mathbb{Z}_{2k}|}{|p_1(A)|} \times |L_2|$.

Now, we show other results of the domination number of Cayley digraphs of $\mathbb{Z}_n$, the group of integers modulo n, with a connection set $\{\bar{1},\bar{t}\} \subseteq \mathbb{Z}_n$ in order to apply to the domination number of Cayley digraphs of left groups $\mathbb{Z}_n \times L$ where L is a left zero semigroup. Furthermore, let (V,E) be a digraph and for each $x \in V$, let $N(x) = \{y \in V | (x,y) \in E\}$ be the set of all neighbours of x and let $N[x] = N(x) \cup \{x\}$.

In general, it is easy to verify that $\lceil \frac{n}{3} \rceil \le \gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{t}\})) \le \lceil \frac{n}{2} \rceil$.

Proposition 3.16. *Let $n \ge 2$ be a positive integer.*
Then $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{2}\})) = \lceil \frac{n}{3} \rceil$.

Proof. We will consider the case $n \equiv 1 \pmod 3$.
It is easy to see that $\{\bar{1},\bar{4},\bar{7},\dots,\overline{n-3},\bar{n}\}$ is a dominating set of $Cay(\mathbb{Z}_n,\{\bar{1},\bar{2}\})$.
Hence $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{2}\})) \le |\{\bar{1},\bar{4},\bar{7},\dots,\overline{n-3},\bar{n}\}| = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil$.
Suppose that there exists a dominating set X such that $|X| < \frac{n+2}{3}$, that is, $|X| \le \frac{n-1}{3}$. Since $|N[x]| \le 3$ for all $x \in X$, we obtain that $|\bigcup_{x \in X} N[x]| \le 3|X| \le n-1 < n$ which is a contradiction.
Therefore, $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{2}\})) = \lceil \frac{n}{3} \rceil$ and we can prove other cases similarly. □

Lemma 3.17. *Let $n \ge 3$ be a positive integer and X a dominating set of $Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\})$. For each $x \in X, |N[x] \cap N[v]| \ge 1$ for some $v \in X \setminus \{x\}$.*

Proof. Let X be a dominating set of $Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\})$ and $x\in X$.
Then $N[x]=\{x,x+1,x+3\}$. Since $x+2\notin N[x]$ and $x+2$ has to be dominated, we can conclude that $x+2\in X$ or $x+2\in N[y]$ for some $y\in X$.
If $x+2\in X$, then $N[x+2]=\{x+2,x+3,x+5\}$, that is, $x+3\in N[x]\cap N[x+2]$ which implies that $|N[x]\cap N[x+2]|\geq 1$.
If $x+2\in N[y]$, then $y=x+1$ or $y=x-1$.
If $y=x+1$, then $x+1\in X$. Thus $x+1\in N[x]\cap N[x+1]$ which leads to $|N[x]\cap N[x+1]|\geq 1$.
If $y=x-1$, then $x-1\in X$. Thus $x\in N[x]\cap N[x-1]$ which implies that $|N[x]\cap N[x-1]|\geq 1$. □

Proposition 3.18. *Let $n\geq 3$ be a positive integer.*
Then $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\}))=\begin{cases}2\lceil\frac{n}{5}\rceil-1 & \text{if } n\equiv 1,2\ (\text{mod}\,5),\\ 2\lceil\frac{n}{5}\rceil & \text{if } n\equiv 0,3,4\ (\text{mod}\,5).\end{cases}$

Proof. We will consider the case $n\equiv 1\ (\text{mod}\,5)$. In this case, we can conclude that $T=\{\bar{1},\bar{2},\bar{6},\bar{7},\overline{11},\overline{12},\dots,\overline{n-5},\overline{n-4},\bar{n}\}$ is a dominating set which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\}))\leq|T|=\frac{2n+3}{5}=2\lceil\frac{n}{5}\rceil-1$. Next, suppose that there exists a dominating set X such that $|X|\leq 2\lceil\frac{n}{5}\rceil-2=\frac{2(n-1)}{5}$. For each $x\in X$, we have by Lemma 3.17 that $N[x]\cap N[y]\geq 1$ for some $y\in X\setminus\{x\}$. Since $|N[x]|\leq 3$, we have $|\bigcup_{x\in X}N[x]|\leq 3|X|-\lceil\frac{|X|}{2}\rceil\leq\frac{5|X|}{2}\leq n-1<n$, that is, $\bigcup_{x\in X}N[x]\subsetneq\mathbb{Z}_n$. Hence X does not dominate $\mathbb{Z}_n$ which is a contradiction. Therefore, $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\}))=|T|=2\lceil\frac{n}{5}\rceil-1$.

Similarly, we can prove the case $n\equiv 2\ (\text{mod}\,5)$.

Now, we will consider the case $n\equiv 3\ (\text{mod}\,5)$. We can obtain that $T=\{\bar{1},\bar{2},\bar{6},\bar{7},\overline{11},\overline{12},\dots,\overline{n-2},\overline{n-1}\}$ is a dominating set. Then $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\}))\leq|T|=\frac{2n+4}{5}=2\lceil\frac{n}{5}\rceil$. Assume in the contrary that there exists a dominating set X such that $|X|\leq 2\lceil\frac{n}{5}\rceil-1=\frac{2n-1}{5}$. Again by Lemma 3.17, we have $|\bigcup_{x\in X}N[x]|\leq\frac{5|X|}{2}\leq\frac{2n-1}{2}<\frac{2n}{2}=n$. Whence X does not dominate $\mathbb{Z}_n$ which contradicts to the property of the dominating set X. So we can conclude that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{3}\}))=|T|=2\lceil\frac{n}{5}\rceil$. For the cases $n\equiv 0,4\ (\text{mod}\,5)$, we can prove them similarly. □

Proposition 3.19. *Let $n\geq 4$ be a positive integer.*

Then $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\}))\leq\begin{cases}3\lceil\frac{n}{7}\rceil & \text{if } n\equiv 0,6\ (\text{mod}\,7),\\ 3\lceil\frac{n}{7}\rceil-1 & \text{if } n\equiv 4,5\ (\text{mod}\,7),\\ 3\lceil\frac{n}{7}\rceil-2 & \text{if } n\equiv 1,2,3\ (\text{mod}\,7).\end{cases}$

Proof. Let $n\geq 4$ be a positive integer.

For $n\equiv 0\ (\text{mod}\,7)$, we obtain that X_0 is a dominating set where
$X_0=\{\bar{1},\bar{8},\overline{15},\overline{22},\dots,\overline{n-6}\}\cup\{\bar{3},\overline{10},\overline{17},\overline{24},\dots,\overline{n-4}\}\cup\{\bar{6},\overline{13},\overline{20},\overline{27},\dots,\overline{n-1}\}$
which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\}))\leq|X_0|=3\lceil\frac{n}{7}\rceil$.

For $n\equiv 1\ (\text{mod}\,7)$, we obtain that X_1 is a dominating set where
$X_1=\{\bar{1},\bar{8},\overline{15},\overline{22},\dots,\bar{n}\}\cup\{\bar{3},\overline{10},\overline{17},\overline{24},\dots,\overline{n-5}\}\cup\{\bar{6},\overline{13},\overline{20},\overline{27},\dots,\overline{n-2}\}$
which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\}))\leq|X_1|=3\lceil\frac{n}{7}\rceil-2$.

For $n\equiv 2\ (\text{mod}\,7)$, we obtain that X_2 is a dominating set where
$X_2=\{\bar{1},\bar{8},\overline{15},\overline{22},\dots,\overline{n-1}\}\cup\{\bar{3},\overline{10},\overline{17},\overline{24},\dots,\overline{n-6}\}\cup\{\bar{6},\overline{13},\overline{20},\overline{27},\dots,\overline{n-3}\}$

which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\})) \le |X_2| = 3\lceil \frac{n}{7}\rceil - 2$.

For $n \equiv 3 \pmod 7$, we obtain that X_3 is a dominating set where

$$X_3 = \{\overline{1},\overline{8},\overline{15},\overline{22},\ldots,\overline{n-2}\} \cup \{\overline{3},\overline{10},\overline{17},\overline{24},\ldots,\overline{n-7}\} \cup \{\overline{6},\overline{13},\overline{20},\overline{27},\ldots,\overline{n-4}\}$$

which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\})) \le |X_3| = 3\lceil \frac{n}{7}\rceil - 2$.

For $n \equiv 4 \pmod 7$, we obtain that X_4 is a dominating set where

$$X_4 = \{\overline{1},\overline{8},\overline{15},\overline{22},\ldots,\overline{n-3}\} \cup \{\overline{3},\overline{10},\overline{17},\overline{24},\ldots,\overline{n-1}\} \cup \{\overline{6},\overline{13},\overline{20},\overline{27},\ldots,\overline{n-5}\}$$

which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\})) \le |X_4| = 3\lceil \frac{n}{7}\rceil - 1$.

For $n \equiv 5 \pmod 7$, we obtain that X_5 is a dominating set where

$$X_5 = \{\overline{1},\overline{8},\overline{15},\overline{22},\ldots,\overline{n-4}\} \cup \{\overline{3},\overline{10},\overline{17},\overline{24},\ldots,\overline{n-2}\} \cup \{\overline{6},\overline{13},\overline{20},\overline{27},\ldots,\overline{n-6}\}$$

which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\})) \le |X_5| = 3\lceil \frac{n}{7}\rceil - 1$.

For $n \equiv 6 \pmod 7$, we obtain that X_6 is a dominating set where

$$X_6 = \{\overline{1},\overline{8},\overline{15},\overline{22},\ldots,\overline{n-5}\} \cup \{\overline{3},\overline{10},\overline{17},\overline{24},\ldots,\overline{n-3}\} \cup \{\overline{6},\overline{13},\overline{20},\overline{27},\ldots,\overline{n}\}$$

which implies that $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{4}\})) \le |X_6| = 3\lceil \frac{n}{7}\rceil$. □

Proposition 3.20. *Let $n \ge 5$ be a positive integer. Then* $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{5}\})) \le \lceil \frac{n}{3}\rceil + 1$.

Proof. Let $n \ge 5$ be a positive integer.

For $n \equiv 0 \pmod 3$, we obtain that $X_0 = \{\overline{1},\overline{2},\overline{4},\overline{7},\overline{10},\overline{13},\ldots,\overline{n-2}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{5}\})) \le |X_0| = \lceil \frac{n}{3}\rceil + 1$.

For $n \equiv 1 \pmod 3$, we obtain that $X_1 = \{\overline{1},\overline{2},\overline{4},\overline{7},\overline{10},\overline{13},\ldots,\overline{n}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{5}\})) \le |X_1| = \lceil \frac{n}{3}\rceil + 1$.

For $n \equiv 2 \pmod 3$, we obtain that $X_2 = \{\overline{1},\overline{2},\overline{4},\overline{7},\overline{10},\overline{13},\ldots,\overline{n-1}\}$ is a dominating set which leads to $\gamma(Cay(\mathbb{Z}_n,\{\bar{1},\bar{5}\})) \le |X_2| = \lceil \frac{n}{3}\rceil + 1$. □

Since a Cayley digraph of a left group can be considered as the disjoint union of Cayley digraphs of groups as shown in Lemma 2.3, we can directly obtain some results for the domination number of Cayley digraphs of left groups as follows.

Theorem 3.21. *Let $n \ge 2$ be a positive integer. If $p_1(A) = \{\bar{1},\bar{2}\}$, then* $\gamma(Cay(\mathbb{Z}_n \times L, A)) = |L|\lceil \frac{n}{3}\rceil$.

Theorem 3.22. *Let $n \ge 3$ be a positive integer.*

If $p_1(A) = \{\bar{1},\bar{3}\}$, then $\gamma(Cay(\mathbb{Z}_n \times L, A)) = \begin{cases} |L|(2\lceil \frac{n}{5}\rceil - 1) & \text{if } n \equiv 1,2 \pmod 5, \\ 2|L|\lceil \frac{n}{5}\rceil & \text{if } n \equiv 0,3,4 \pmod 5. \end{cases}$

Theorem 3.23. *Let $n \ge 4$ be a positive integer.*

If $p_1(A) = \{\bar{1},\bar{4}\}$, then $\gamma(Cay(\mathbb{Z}_n \times L, A)) \le \begin{cases} 3|L|\lceil \frac{n}{7}\rceil & \text{if } n \equiv 0,6 \pmod 7, \\ |L|(3\lceil \frac{n}{7}\rceil - 1) & \text{if } n \equiv 4,5 \pmod 7, \\ |L|(3\lceil \frac{n}{7}\rceil - 2) & \text{if } n \equiv 1,2,3 \pmod 7. \end{cases}$

Theorem 3.24. *Let $n \ge 5$ be a positive integer. If $p_1(A) = \{\bar{1},\bar{5}\}$, then* $\gamma(Cay(\mathbb{Z}_n \times L, A)) \le |L|(\lceil \frac{n}{3}\rceil + 1)$.

Next, we give some results of the total domination number in Cayley digraphs of left groups with connection sets. We start with the lemma which gives the condition for the existence of the total dominating set in Cayley digraphs of left groups.

Lemma 3.25. *Let $S = G \times L_m$ be a left group and A a nonempty subset of S. Then the total dominating set of D exists if and only if $A \neq \emptyset$.*

Proof. Suppose that the total dominating set of D exists, say T. By the definition of T, we obtain that for each $(g,l) \in S$, (g,l) is dominated by (g_1,l_1) for some $(g_1,l_1) \in T$, that is, $((g_1,l_1),(g,l)) \in E(D)$. Then $(g,l) = (g_1,l_1)(a_1,l_2)$ where $(a_1,l_2) \in A$ which implies that $A \neq \emptyset$.

Conversely, assume that the connection set $A \neq \emptyset$, that is, there exists $(a,l) \in A$. Hence for each $(g_1,l_1) \in S$, we have $(g_1,l_1) = (g_1a^{-1},l_1)(a,l)$ where $(g_1a^{-1},l_1) \in S$. Thus (g_1,l_1) is dominated by (g_1a^{-1},l_1) in S. If we take $T = S$, then we can conclude that T is a total dominating set of D, that is, the total dominating set of D always exists when $A \neq \emptyset$. □

The following result gives us the total domination number of a Cayley digraph of a left group $G \times L_m$ with a connection set A in the term of a total domination number of a Cayley digraph of the subgroup $\langle p_1(A)\rangle$ of G.

Theorem 3.26. *Let $S = G \times L_m$ be a left group, A a nonempty subset of S, and $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_k\langle p_1(A)\rangle\}$. Then $\gamma_t(D) = m \cdot k \cdot \gamma_t(Cay(\langle p_1(A)\rangle, p_1(A)))$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.12. □

Proposition 3.27. *Let $n \geq 3$ be an odd integer and $c = \frac{n-1}{2}$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S such that $p_1(A) = \{c, n-c, c-1, n-(c-1), \dots, c-(k-1), n-(c-(k-1))\}$ where $1 \leq k \leq c$. Then $\gamma_t(D) = m\lceil \frac{n}{2k} \rceil$.*

Proof. The result of this proposition follows from Lemma 2.3 and Lemma 2.5, directly. □

Proposition 3.28. *Let $n \geq 3$ be an even integer and $c = \lfloor \frac{n-1}{2} \rfloor$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S such that $p_1(A) = \{\frac{n}{2}, c, n-c, c-1, n-(c-1), \dots, c-(k-1), n-(c-(k-1))\}$ where $1 \leq k \leq c$. Then $\gamma_t(D) = m\lceil \frac{n}{2k+1} \rceil$.*

Proof. This proposition follows from Lemma 2.3 and Lemma 2.6, directly. □

Before we give the next lemmas, we will define some notations which are used in the proof. Let $I = [a,b]$ be an interval of consecutive integers x such that $a \leq x \leq b$. Recall that $N(u)$ is the set of all neighbours of a vertex u and $N(A) = \bigcup_{a \in A} N(a)$ where A is a nonempty subset of a vertex set of any digraph.

Lemma 3.29. *Let $n \geq 3$ be an odd integer. Let $m = \frac{n-1}{2}$ and k be a fixed number such that $1 \leq k \leq m$. If $A = \{m, m-1, m-2, \dots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{k} \rceil$.*

Proof. Assume that $A = \{m, m-1, m-2, \dots, m-(k-1)\}$ and let $l = \lceil \frac{n}{k} \rceil$. Since every vertex in D has an out-degree k, from the definition of the total domination number, it follows that $\gamma_t(Cay(\mathbb{Z}_n, A)) \geq l$. Let $x = m+k+1$ and $X_t = \{x, x+k, x+2k, \dots, x+(l-1)k\}$. Note that $|X_t| = l$. Since $l = \lceil \frac{n}{k} \rceil$, we get that $n = (l-1)k + j$ for some $j \in \mathbb{N}$ with $1 \leq j \leq k$. Thus $V(Cay(\mathbb{Z}_n, A))$ can be partitioned into l intervals as follows:

$I_1 = [1,k]$, $I_2 = [k+1, 2k]$, $I_3 = [2k+1, 3k], \dots, I_{l-1} = [(l-2)k+1, (l-1)k]$, and $I_l = [(l-1)k+1, n]$.

Note that $|I_i| = k$ for all i with $1 \leq i \leq l-1$ and $1 \leq |I_l| \leq k$. For any $0 \leq i \leq l-2$, we have $x + ik \in X_t$ and $I_{i+1} = [ik+1, (i+1)k]$. Since $(x+ik) + (m-(k-1)) \equiv ik+1 \pmod{n}$ and A is a set of k consecutive integers with the least element $m-(k-1)$, we have $N(x+ik) = I_{i+1}$. Therefore,

$$(x+(l-1)k) + m - (k-1) \equiv (l-1)k + 1 \pmod{n} \text{ and so } I_l \subseteq N(x+(l-1)k).$$

Consequently,

$$\begin{aligned} V(Cay(\mathbb{Z}_n, A)) &= I_1 \cup I_2 \cup \ldots \cup I_{l-1} \cup I_l \\ &\subseteq N(x) \cup N(x+k) \cup \ldots \cup N(x+(l-2)k) \cup N(x+(l-1)k) \\ &= \bigcup_{y \in X_t} N(y) \\ &= N(X_t). \end{aligned}$$

Thus X_t is a total dominating set of $Cay(\mathbb{Z}_n, A)$. Hence $\gamma_t(Cay(\mathbb{Z}_n, A)) \leq |X_t| = l$ and then $\gamma_t(Cay(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k} \rceil$. □

Lemma 3.30. *Let $n \geq 3$ be an even integer. Let $m = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed number such that $1 \leq k \leq m$. If $A = \{\frac{n}{2}, m, m-1, \dots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(Cay(\mathbb{Z}_n, A)) = \lceil \frac{n}{k+1} \rceil$.*

Proof. Suppose that $A = \{\frac{n}{2}, m, m-1, \dots, m-(k-1)\}$. Then $|A| = k+1$ and let $l = \lceil \frac{n}{k+1} \rceil$. Since every vertex of $Cay(\mathbb{Z}_n, A)$ has an out-degree $k+1$, we also have $\gamma_t(Cay(\mathbb{Z}_n, A)) \geq l$. Let $x = m+k+2$ and $X_t = \{x, x+(k+1), x+2(k+1), \dots, x+(l-1)(k+1)\}$. By partitioning the set of all vertices in $Cay(\mathbb{Z}_n, A)$ into l intervals as follows:

$$I_1 = [1, k+1], I_2 = [(k+1)+1, 2(k+1)], \dots, I_{l-1} = [(l-2)(k+1)+1, (l-1)(k+1)], \text{ and}$$
$$I_l = [(l-1)(k+1)+1, n],$$

we can prove the remaining part of this lemma by applying the proof of the previous lemma, similarly. We also have $\gamma_t(Cay(\mathbb{Z}_n, A)) \leq |X_t| = l$. Thus $\gamma_t(Cay(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k+1} \rceil$. □

Now, we apply the above two lemmas to obtain the results for the total domination number of Cayley digraphs of left groups $\mathbb{Z}_n \times L_m$ with respect to according connection sets.

Theorem 3.31. *Let $n \geq 3$ be an odd integer. Let $c = \frac{n-1}{2}$ and k be a fixed number such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S. If $p_1(A) = \{c, c-1, c-2, \dots, c-(k-1)\}$, then $\gamma_t(D) = m\lceil \frac{n}{k} \rceil$.*

Proof. This theorem is a direct result from Lemma 2.3 and Lemma 3.29. □

Theorem 3.32. *Let $n \geq 3$ be an even integer. Let $c = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed number such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L_m$ be a left group and A a nonempty subset of S. If $p_1(A) = \{\frac{n}{2}, c, c-1, \dots, c-(k-1)\}$, then $\gamma_t(D) = m\lceil \frac{n}{k+1} \rceil$.*

Proof. This theorem follows from Lemma 2.3 and Lemma 3.30. □

4. Conclusion

In this paper, we give the backgrounds of the research and some preliminaries together with the notations in Section 1 and Section 2, respectively. In the third section, some results of the domination number and total domination number of Cayley digraphs of right groups and left groups with appropriate connection sets are obtained. In addition, we have the conditions for the existence of the total dominating sets of Cayley digraphs of right groups and left groups. Moreover, the sharpness of bounds for domination parameters in Cayley digraphs of right groups and left groups are also shown.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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