A New Study on Generalized Absolute Matrix Summability

Hikmet Seyhan Özarslan

Department of Mathematics, Erciyes University, Kayseri, Turkey
seyhan@erciyes.edu.tr; hseyhan38@gmail.com

Abstract. In this paper, a general theorem on $|A, p_n; \delta|_k$ summability factors, which generalizes a theorem of Bor [4] on $|N, p_n|_k$ summability factors, has been proved by using almost increasing sequences.

Keywords. Summability factors; Absolute matrix summability; Almost increasing sequence; Infinite series; Hölder inequality; Minkowski inequality

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1. Introduction

A positive sequence $(b_n)$ is said to be almost increasing if there exists a positive increasing sequence $(c_n)$ and two positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1). \quad (1.1)$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (1.2)$$
defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\tilde{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [5]).

The series \(\sum a_n\) is said to be summable \(|\tilde{N}, p_n|_k\), \(k \geq 1\), if (see [2])
\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{P_n}\right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty,
\]
where
\[
\Delta \sigma_{n-1} = -\frac{P_n}{P_n} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.
\]

Let \(A = (a_{nv})\) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \(A\) defines the sequence-to-sequence transformation, mapping the sequence \(s = (s_n)\) to \(As = (A_n(s))\), where
\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots .
\]

The series \(\sum a_n\) is said to be summable \(|A, p_n; \delta|_k\), \(k \geq 1\) and \(\delta \geq 0\), if (see [6])
\[
\sum_{n=1}^{\infty} \left(\frac{P_n}{P_n}\right)^{\delta k+k-1} |\tilde{\Delta} A_n(s)|^k < \infty,
\]
where
\[
\tilde{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).
\]

If we set \(\delta = 0\), then \(|A, p_n; \delta|_k\) summability reduces to \(|A, p_n|_k\) summability (see [5]). If we take \(a_{nv} = \frac{P_v}{P_n}\) and \(\delta = 0\), then \(|A, p_n; \delta|_k\) summability reduces to \(|\tilde{N}, p_n|_k\) summability. In the special case \(\delta = 0\) and \(p_n = 1\) for all \(n\), \(|A, p_n; \delta|_k\) summability is the same as \(|A|_k\) summability (see [9]). Also if we take \(a_{nv} = \frac{P_v}{P_n}\), then \(|A, p_n; \delta|_k\) summability is the same as \(|\tilde{N}, p_n; \delta|_k\) summability (see [3]).

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix \(A = (a_{nv})\), we associate two lower semimatrices \(\hat{A} = (\hat{a}_{nv})\) and \(\hat{\hat{A}} = (\hat{\hat{a}}_{nv})\) as follows:
\[
\hat{a}_{nv} = \sum_{i=0}^{n} a_{ni}, \quad n, v = 0, 1, \ldots .
\]
and
\[
\hat{\hat{a}}_{00} = \hat{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n-1,v}, \quad n = 1, 2, \ldots .
\]

It may be noted that \(\hat{A}\) and \(\hat{\hat{A}}\) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have
\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \hat{a}_{nv} a_v
\]
and
\[
\hat{\hat{A}} A_n(s) = \sum_{v=0}^{n} \hat{\hat{a}}_{nv} a_v .
\]
2. Known Result

In [4], the following theorem dealing with $|\vec{N}, p_n|_k$ summability factors of infinite series has already been proved.

**Theorem 2.1.** Let $(X_n)$ be an almost increasing sequence and let there be sequences $(\beta_n)$ and $(\lambda_n)$ such that

\[
|\Delta \lambda_n| \leq \beta_n, \quad (2.1)
\]
\[
\beta_n \to 0 \quad \text{as} \quad n \to \infty, \quad (2.2)
\]
\[
\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \quad (2.3)
\]
\[
|\lambda_n| X_n = O(1). \quad (2.4)
\]

If
\[
\sum_{n=1}^{m} \frac{|\lambda_n|}{n} = O(1) \quad \text{as} \quad m \to \infty, \quad (2.5)
\]
\[
\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty \quad (2.6)
\]

and $(p_n)$ is a sequence such that
\[
\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \quad (3.3)
\]

where $(t_n)$ is the $n$th $(C,1)$ mean of the sequence $(na_n)$, then the series $\sum a_n \lambda_n$ is summable $|\vec{N}, p_n|_k$, $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 to $|A, p_n; \delta|_k$ summability. Now, we shall prove the following theorem.

**Theorem 3.1.** Let $A = (a_{nv})$ be a positive normal matrix such that

\[
a_{\eta n 0} = 1, \quad n = 0,1,\ldots, \quad (3.1)
\]
\[
a_{n-1,v} \geq a_{nv}, \quad \text{for} \quad n \geq v + 1, \quad (3.2)
\]
\[
a_{nn} = O\left(\frac{p_n}{P_n}\right), \quad (3.3)
\]

and $(X_n)$ be an almost increasing sequence. If the conditions (2.1)-(2.5) of Theorem 2.1 and the conditions
\[
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta h-1} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \quad (3.4)
\]
\[
\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|t_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \quad (3.5)
\]
To complete the proof of Theorem 3.1, by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k} |\Delta_n \tilde{a}_{n+1}| = O \left( \left( \frac{P_n}{P_n} \right)^{\delta k - 1} \right) \quad \text{as} \quad m \to \infty,
\]

(3.6)

\[
\sum_{n=1}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k} |\tilde{a}_{n+1}| = O \left( \left( \frac{P_n}{P_n} \right)^{\delta k} \right) \quad \text{as} \quad m \to \infty
\]

(3.7)

are satisfied, then the series \( \sum a_n \lambda_n \) is summable \(|A, p_n; \delta|, k \geq 1 \) and \( 0 \leq \delta < 1/k \).

We need the following lemma for the proof of Theorem 3.1.

**Lemma 3.2** ([4]). Under the conditions on \((X_n), (\beta_n)\) and \((\lambda_n)\) as taken in the statement of Theorem 3.1, the following conditions hold:

\[
n \beta_n X_n = O(1) \quad \text{as} \quad n \to \infty,
\]

(3.8)

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]

(3.9)

### 4. Proof of Theorem 3.1

Let \((I_n)\) denotes A-transform of the series \( \sum a_n \lambda_n \). Then, by (1.9) and (1.10), we have

\[
\tilde{a}_n = \sum_{v=1}^{n} \tilde{a}_{n+1} a_v \lambda_v
\]

\[
= \sum_{v=1}^{n} \tilde{a}_{n+1} \frac{\lambda_v}{v} a_v.
\]

Using Abel’s transformation, we have that

\[
\tilde{a}_n = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\tilde{a}_{n+1} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\tilde{a}_{n+1} \lambda_n}{n} \sum_{r=1}^{n} r a_r
\]

\[
= \frac{n+1}{n} \tilde{a}_{n+1} \lambda_n t_n + \sum_{v=1}^{n-1} \Delta_v (\tilde{a}_{n+1} \lambda_v) t_v + \sum_{v=1}^{n-1} \frac{\lambda_v}{v} \tilde{a}_{n+1} \lambda_{v+1} t_v + \sum_{v=1}^{n-1} \frac{1}{v} \tilde{a}_{n+1} \lambda_{v+1} t_v
\]

\[
= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\]

To complete the proof of Theorem 3.1 by Minkowski’s inequality, it is sufficient to show that

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |I_{n,r}| < \infty, \quad \text{for} \quad r = 1, 2, 3, 4.
\]

First, we have that

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |I_{n,1}|^k = O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |\lambda_n| |t_n|^k \tilde{a}_{n+1}^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |\lambda_n| |\lambda_n| |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |\lambda_n| |t_n|^k
\]

\[
= O(1) \sum_{n=1}^{m} \Delta |\lambda_n| \sum_{r=1}^{n} \left( \frac{P_r}{P_r} \right)^{\delta k - 1} |t_r|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} |t_n|^k
\]
by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

Now, when $k > 1$, applying Hölder’s inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $I_{n,1}$, we have that

$$
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} |\Delta_v(nu)| \mid \lambda_v \mid |t_v|^k \right)^k
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Again, we have that

$$
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} |I_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} \mid \hat{\alpha}_{n,v+1} \mid |\Delta_v| \mid |t_v|^k \right)^k
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. Again, we have that

$$
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid \hat{\alpha}_{n,1} \mid |t_v|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} \mid \hat{\alpha}_{n,v+1} \mid \beta_v \mid |t_v|^k \right)^k
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Finally, we have that
\[
\sum_{n=2}^{m+1} \left( \frac{p_n}{p_m} \right)^{\delta k + k - 1} |I_{n,A}|^k \leq \sum_{n=2}^{m+1} \left( \frac{p_n}{p_m} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}| \left| \lambda_{v+1} \right| \frac{|t_v|}{v} \right)^k
\]
\[
\leq \sum_{n=2}^{m+1} \left( \frac{p_n}{p_m} \right)^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}| \left| \lambda_{v+1} \right| \frac{|t_v|}{v} \right)^k \left( \sum_{v=1}^{n-1} |\tilde{a}_{n,v+1}| \left| \lambda_{v+1} \right| \frac{|t_v|}{v} \right)^{-k-1}
\]
\[
= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{p_n}{p_m} \right)^{\delta k} |\tilde{a}_{n,v+1}|
\]
\[
= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k + O(1) |\lambda_{m+1}||t_m|^k
\]
by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.

This completes the proof of Theorem 3.1. If we take \(a_{nv} = \frac{p_n}{p_m}\) and \(\delta = 0\) in Theorem 3.1, then we get Theorem 2.1. Also, if we take \(\delta = 0\) in Theorem 3.1, then we obtain a known theorem on \(|A, p_n|_k\) summability method (see [7]).

## 5. Conclusions

In this study, we have generalized a known theorem dealing with absolute summability method to absolute matrix summability method by using almost increasing sequences. And so it has been brought a different perspective and studying field.

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## Competing Interests

The author declares that she has no competing interests.

## Authors’ Contributions

The author wrote, read and approved the final manuscript.

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