Strong and $\Delta$-Convergence Theorems for Asymptotically $k$-Strictly Pseudo-Contractive Mappings in CAT(0) Spaces

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Abstract. In this paper, we study and prove fixed point and convergence theorems of modified Mann iteration for asymptotically $k$-strictly pseudo-contractive mappings in CAT(0) spaces. Our result extend and improve many results in the literature.

Keywords. Fixed point; Asymptotically $k$-strictly pseudo-contractive mappings; Convergence theorems; CAT(0) spaces

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1. Introduction

Let $C$ be a nonempty subset of a real Hilbert space $X$. Let $T : C \rightarrow C$ be a self-mapping. Recall that a mapping $T$ is said to be:

(1) contractive if there exists a constant $k < 1$ such that $\|Tx - Ty\| \leq k\|x - y\|$ for all $x, y \in C$;

(2) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
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In 1953, Mann [15] introduced the following iteration for approximating a fixed point of a mapping \( T : C \to C \) is said to be asymptotically \( k \)-strictly pseudo-contractive if there exist a constant \( k \in (0, 1) \) and sequence \( \{k_n\} \in [1, \infty) \) with \( \lim_{n \to \infty} k_n = 1 \) such that

\[
d(T^n x, T^n y)^2 \leq k_n d(x, y)^2 + k d(x, T^n x) + d(y, T^n y)^2
\]

for all \( x, y \in C \). A point \( x \in C \) is called a fixed point of \( T \) if \( x = T(x) \). We denote with \( F(T) \) the set of fixed points of \( T \). A sequence \( \{x_n\} \) is called approximate fixed point sequence for \( T \) if

\[
\lim_{n \to \infty} (x_n, Tx_n) = 0.
\]

Kirk [10] first studied the theory of fixed point in CAT(0) space[1]. Later on, the fixed point theory for some mappings in CAT(0) spaces has been rapidly developed by many authors (see, e.g., [5, 7, 9, 13, 23, 24]).

In 1953, Mann [15] introduced the following iteration for approximating a fixed point of nonexpansive and pseudo-contractive mappings, sequence \( \{x_n\} \) is defined by

\[
x_{n+1} = x_n + (1 - \alpha_n)T x_n
\]

for all \( n \geq 1 \), where \( \{\alpha_n\} \) is an appropriate sequence \( (0, 1) \).

Motivate and inspired, we modify Mann’s iteration (1.2) to asymptotically nonexpansive and asymptotically \( k \)-strictly pseudo-contractive mappings in CAT(0) spaces is as below

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n
\]

for all \( n \geq 1 \), where \( \{\alpha_n\} \) is an appropriate sequence \( (0, 1) \).

Qihou [19] proved some convergence results for class of an asymptotically \( k \)-strictly pseudo-contractive mapping in Hilbert spaces.

The purpose of this paper, we prove strong and \( \Delta \)-convergence results by using modified Mann iteration process for an asymptotically \( k \)-strictly pseudo-contractive mapping in CAT(0) spaces. In section [2] and [3], we present preliminaries and main results, respectively.

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2. Preliminaries

Let \((X,d)\) be a metric space. A geodesic path joining \(x \in X\) to \(y \in X\) (or, more briefly, a geodesic from \(x\) to \(y\)) is a map \(c\) from closed interval \([0,r] \subset \mathbb{R}\) to \(X\) such that

\[
    c(0) = x, \quad c(r) = y \quad \text{and} \quad d(c(t), c(s)) = |t - s|
\]

for all \(s, t \in [0,r]\).

In particular, \(c\) is an isometry and \(d(x, y) = r\). The image of \(c\) is called a geodesic (or metric) segment joining \(x\) and \(y\). When it is unique, this geodesic is denoted by \([x, y]\). We denote the point \(w \in [x, y]\) if and only if there exists \(\alpha \in [0,1]\) such that

\[
    d(x, w) = \alpha d(x, y) \quad \text{by} \quad w = (1 - \alpha)x \oplus \alpha y.
\]

The space \((X, d)\) is called a geodesic space if any two points of \(X\) are joined by a geodesic and \(X\) is said to be uniquely geodesic if there is exactly one geodesic joining \(x\) and \(y\) for each \(x, y \in X\). A subset \(D \subseteq X\) is called convex if \(D\) includes geodesic segment joining every two points of itself. A geodesic triangle \(\triangle(x_1, x_2, x_3)\) in a geodesic metric space \((X, d)\) consist of three points (the vertices of \(\triangle\)) and a geodesic segment between each pair of vertices (the edges of \(\triangle\)). A comparison triangle for geodesic triangle (or \(\triangle(x_1, x_2, x_3)\)) in \((X, d)\) is a triangle \(\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)\) in the Euclidean plane \(\mathbb{R}^2\) such that

\[
    d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j) \quad \text{for} \quad i, j \in \{1, 2, 3\}.
\]

A geodesic metric space is said to be a CAT(0) space \([1]\) if all geodesic triangle satisfy the following comparison axiom. Let \(\Delta\) be a geodesic triangle in \(X\) and \(\overline{\Delta}\) be a comparison triangle for \(\Delta\). Then, \(\Delta\) is said to satisfy the CAT(0) inequality if for all \(x, y \in \Delta\) and all comparison points

\[
    d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).
\]

If \(x, y_1, y_2\) are points of a CAT(0) space and \(y_0\) is the midpoint of the segment \([y_1, y_2]\), which we will denote by \(\left(\frac{y_1 \oplus y_2}{2}\right)\), then the CAT(0) inequality implies

\[
    d \left( x, \frac{y_1 \oplus y_2}{2} \right)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2. \quad (2.1)
\]

The inequality (2.1) is called the (CN) inequality (see more details Bruhat and Titz \([3]\)).

A geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. A subset \(C\) of a CAT(0) space \(X\) is convex if for any \(x, y \in C\), then \([x, y] \subset C\).

Lemma 2.1 \([7]\). Let \(X\) be a CAT(0) space.

(i) For any \(x, y, z \in X\) and \(t \in [0,1]\), has

\[
    d\left( (1-t)x \oplus ty, z \right) \leq (1-t)d(x, z) + td(y, z). \quad (2.2)
\]

(ii) For any \(x, y, z \in X\) and \(t \in [0,1]\), has

\[
    d\left( (1-t)x \oplus ty, z \right) \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2. \quad (2.3)
\]

Next, we refer some elementary properties about CAT(0) spaces as follows: Let \(\{x_n\}\) be a bounded sequence in a CAT(0) space \((X, d)\). For all \(x \in X\), we set

\[
    r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]
The asymptotic radius \( r(x_n) \) of \( \{x_n\} \) is given by
\[
r(x_n) = \inf \{ r(x, x_n) : x \in X \},
\]
and the asymptotic center \( A(x_n) \) of \( \{x_n\} \) is the set
\[
A(x_n) = \{ x \in X : r(x, x_n) = r(x_n) \}.
\]
We know that in a complete CAT(0) space, (see [6]) \( A(x_n) \) consists of exactly one point.

In 1976, a concept of convergence in a general metric space introduced by Lim [14] setting which is called \( \Delta \)-convergent. In 2008, Kirk and Panyanak [12] used the concept of \( \Delta \)-convergent to prove on the CAT(0) space.

**Definition 2.2** ([4, 6, 14, 17, 21, 22]). A sequence \( \{x_n\} \) in \( X \) is said to be \( \Delta \)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{u_n\} \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). In this case we write \( \Delta - \lim_{n \to \infty} x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{x_n\} \).

**Lemma 2.3** ([7]). If \( \{x_n\} \) is a bounded sequence in a complete CAT(0) space with \( A(x_n) = \{x\} \), \( \{u_n\} \) is a subsequence of \( \{x_n\} \) with \( A(u_n) = \{u\} \) and the sequence of \( \{d(x_n, u)\} \) is convergent then \( x = u \).

Let \( C \) be a closed convex subset of a CAT(0) space \( X \) and \( \{x_n\} \) be a bounded sequence in \( C \). We given the notation as follows:
\[
x_n \rightharpoonup \omega \iff \Phi(\omega) = \inf_{x \in C} \Phi(x).
\]

**Proposition 2.4** ([17]). Let \( C \) be a closed convex subset of a CAT(0) space \( X \) and \( \{x_n\} \) be a bounded sequence in \( C \). Then \( \Delta - \lim_{n \to \infty} x_n = p \) implies that \( \{x_n\} \) \( \rightharpoonup p \).

**Lemma 2.5.**
(i) (see [11]) Every bounded sequences in a complete CAT(0) space always has an \( \Delta \)-convergent subsequence.
(ii) (see [5]) Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space and let \( \{x_n\} \) be a bounded sequence in \( C \). Then the asymptotic center of \( \{x_n\} \) is in \( C \).

**Definition 2.6** ([22]). Let \( (X, d) \) be a metric space and \( C \) be its nonempty subset. Then \( T : C \to C \) is said to be semi-compact if for a sequence \( x_n \) in \( C \) with \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightharpoonup p \in C \).

### 3. Main Results

Next, we state our results of my work.

**Theorem 3.1.** Let \( C \) be a nonempty bounded closed convex subset of a complete CAT(0) space \( X \) and let \( T : C \to C \) be an asymptotically \( k \)-strictly pseudo-contractive mapping. Then \( T \) has a fixed point. Moreover fixed point set \( F(T) \) is closed and convex.

**Proof.** \( F(T) \) closed is evident. Since \( T \) is continuous. We will show that the fixed point set of \( T \) is convex. Let \( p, q \in F(T) \) and \( t \in (0, 1) \). Setting \( z = (1 - t)p \oplus tq \) we get,
\[
d(z, p)^2 \leq t^2 d(p, q)^2 \quad \text{and} \quad d(z, q)^2 \leq (1 - t)^2 d(p, q)^2.
\]
Since $T$ is an asymptotically $k$-strictly pseudo-contractive mapping, from Lemma 2.1, we obtain

\[ d(z, T^nz)^2 = d((1-t)p \oplus tq, T^nz)^2 \]
\[ \leq (1-t)d(p, T^nz)^2 + td(q, T^nz)^2 - t(1-t)d(p, q)^2 \]
\[ \leq (1-t)(k_n d(z, p)^2 + k(d(z, T^nz) + d(p, p))^2) \]
\[ + t(k_n d(z, q)^2 + k(d(z, T^nz) + d(q, q))^2) - t(1-t)d(p, q)^2 \]
\[ = (1-t)(k_n t^2 d(p, q)^2 + kd(z, T^nz)^2) \]
\[ + t[k_n (1-t)^2 d(p, q)^2 + kd(z, T^nz)^2] - t(1-t)d(p, q)^2 \]
\[ = t(1-t)(tk_n + (1-t)k_n - 1)d(p, q)^2 + (1-t + t)kd(z, T^nz)^2 \]
\[ = t(1-t)(k_n - 1)d(p, q)^2 + kd(z, T^nz)^2, \]

where $k \in [0,1)$ and a sequence $(k_n)$ in $[1,\infty)$ such that $\lim_{n \to \infty} k_n = 1$ for any $n \geq 1$. It follow that

\[ d(z, T^nz)^2 \leq \frac{(1-t)d(p, q)^2}{1-k}(k_n - 1). \]

Hence, taking limit as $n \to \infty$ on the both side the above inequality and by using the fact of $k_n \to 1$ as $n \to \infty$, we have $\lim_{n \to \infty} d(z, T^nz) = 0$. From the continuity of $T$, we obtain

\[ Tz = T(\lim_{n \to \infty} T^nz) = \lim_{n \to \infty} T^{n+1}z = z. \]

Therefore $z \in F(T)$, that is, $F(T)$ is convex. This complete proof. 

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be an asymptotically $k$-strictly pseudo-contractive mappings such that $k \in [0, \frac{1}{4}]$ there exist sequence $(k_n) \subset [1,\infty)$ $\lim_{n \to \infty} k_n = 1$ and $F(T) \neq \emptyset$. Let $(x_n)$ be a bounded sequence in $C$ such that $\Delta - \lim_{n \to \infty} x_n = \omega$ and $\limsup_{m \to \infty} \sup_{n \to \infty} d(x_n, T^mx_n) = 0$. Then $T\omega = \omega$.

**Proof.** By the hypothesis, $\Delta - \lim_{n \to \infty} x_n = \omega$. By Proposition 2.4 we have $x_n \to \omega$.

Then, we have $A(x_n) = \omega$ by Lemma 2.5 (ii). Since $\limsup_{m \to \infty} \sup_{n \to \infty} d(x_n, T^mx_n) = 0$. Then, we have

\[ \Phi(x) = \limsup_{n \to \infty} \sup_{m \to \infty} d(T^mx_n, x) = \limsup_{n \to \infty} d(Tx_n, x) \quad (3.1) \]

for all $x \in C$. From (3.1) taking $x = T^m\omega$, we obtain

\[ \Phi(T^m\omega)^2 = \lim_{n \to \infty} \sup d(T^m_{n\to \infty} x_n, T^m\omega)^2 \]
\[ \leq \lim_{n \to \infty} \sup(k_m d(x_n, \omega)^2 + k(d(x_n, T^mx_n) + d(\omega, T^m\omega))^2). \]

Since $\lim_{n \to \infty} k_n = 1$ and $\limsup_{m \to \infty} \limsup_{n \to \infty} d(x_n, T^mx_n) = 0$. Taking $\limsup$ of both sides the above inequality, we have

\[ \limsup_{m \to \infty} \Phi(T^m\omega)^2 \leq \limsup_{m \to \infty} k_m \limsup_{n \to \infty} d(x_n, \omega)^2 + k \limsup_{m \to \infty} \limsup_{n \to \infty} d(\omega, T^m\omega)^2 \]
\[ = \Phi(\omega)^2 + k \limsup_{m \to \infty} d(\omega, T^m\omega)^2. \quad (3.2) \]

By the (CN) inequality implies that

\[ d\left( x_n, \frac{\omega \oplus T^m\omega}{2} \right)^2 \leq \frac{1}{2} d(x_n, \omega)^2 + \frac{1}{2} d(x_n, T^m\omega)^2 - \frac{1}{4} d(\omega, T^m\omega)^2. \]
for any \( n, m \geq 1 \). Letting \( n \to \infty \) and taking limit supremum on the both sides, we get
\[
\Phi\left(\frac{\omega + T\omega}{2}\right)^2 \leq \frac{1}{2} \Phi(\omega)^2 + \frac{1}{2} \Phi(T^m \omega)^2 - \frac{1}{4} d(\omega, T^m \omega)^2
\]
for any \( m \geq 1 \). Since \( A((x_n)) = \{\omega\} \), we get
\[
\Phi(\omega)^2 \leq \Phi\left(\frac{\omega + T\omega}{2}\right)^2 
\leq \frac{1}{2} \Phi(\omega)^2 + \frac{1}{2} \Phi(T^m \omega)^2 - \frac{1}{4} d(\omega, T^m \omega)^2
\]
which implies that
\[
d(\omega, T^m \omega)^2 \leq 2\Phi(T^m \omega)^2 - 2\Phi(\omega)^2.
\]
Taking limit supremum on the both sides, we obtain
\[
\limsup_{m \to \infty} d(\omega, T^m \omega)^2 \leq 2\limsup_{m \to \infty} \Phi(T^m \omega)^2 - 2\Phi(\omega)^2. \tag{3.3}
\]
From inequalities (3.2) and (3.3), we get
\[
\limsup_{m \to \infty} d(\omega, T^m \omega)^2 \leq 2(\Phi(\omega)^2 + k\limsup_{m \to \infty} d(\omega, T^m \omega)^2) - 2\Phi(\omega)^2 
\leq 2k\limsup_{m \to \infty} d(\omega, T^m \omega).
\]
So, we obtain
\[
(1 - 2k)\limsup_{m \to \infty} d(\omega, T^m \omega)^2 \leq 0. \tag{3.4}
\]
Since \( k \in [0, \frac{1}{2}) \), we get \( \limsup_{m \to \infty} d(\omega, T^m \omega) = 0 \), which implies \( \lim_{n \to \infty} T^m(\omega) = \omega \) therefore \( T\omega = \omega \). The proof is completed.

**Theorem 3.3.** Let \( C \) be a nonempty closed convex subset of a complete CAT(0) space \( X \) and let \( T: C \to C \) be an asymptotically \( k \)-strictly pseudo-contractive mapping with \( k \in [0, \frac{1}{2}) \) and a sequence \( \{k_n\} \) in \([1, \infty)\) such that \( \lim_{n \to \infty} k_n = 1 \). Let \( \{x_n\} \) be a sequence in \( C \) defined by (1.3) and \( \{\alpha_n\} \) is a sequence in \((0, 1)\). Then \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) and \( \lim_{n \to \infty} d(x_n, p) \) exist for all \( p \in F(T) \).

**Proof.** First, we will prove that \( \lim_{n \to \infty} d(x_n, p) \) exist. It follow from Theorem 3.1 such that \( F(T) \neq \emptyset \) and \( p \in F(T) \). Since \( T \) is asymptotically \( k \)-strictly pseudo-contractive mapping and using Lemma 2.1 and Mann iteration (1.3), we have
\[
d(x_{n+1}, p)^2 = d(ax_n + (1 - \alpha_n)T^n x_n, p)^2 
\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n)d(T^n x_n, p)^2 - \alpha_n(1 - \alpha_n)d(x_n, T^n x_n)^2 
\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n)(k_n d(x_n, p)^2 + k d(x_n, T^n x_n)^2) - \alpha_n(1 - \alpha_n)d(x_n, T^n x_n)^2 
= \alpha_n d(x_n, p)^2 + k_n d(x_n, p)^2 - k_n \alpha_n d(x_n, p)^2 + (1 - \alpha_n)k d(x_n, T^n x_n)^2 
- \alpha_n(1 - \alpha_n)d(x_n, T^n x_n)^2.
\]
From \( \lim_{n \to \infty} k_n = 1 \), taking limit as \( n \to \infty \), we obtain
\[
d(n_{n+1}, p)^2 \leq d(x_n, p)^2 - [(1 - \alpha_n)(\alpha_n - k)d(x_n, T^n x_n)^2] \tag{3.5}
\leq d(x_n, p)^2. \tag{3.6}
\]
It follow that the sequence \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(x_n, p) \) exist for all \( p \in F(T) \).
Assume that \( \lim_{n \to \infty} d(x_n, p) = r \). From (3.6), we get

\[
\lim_{n \to \infty} d(x_{n+1}, p) = r. \tag{3.7}
\]

By (3.5), we also have

\[
d(x_n, T^n x_n)^2 \leq \frac{1}{(1 - \alpha_n)(\alpha_n - k)} [d(x_n, p)^2 - d(n_{n+1}, p)^2]. \tag{3.8}
\]

Since \( \lim_{n \to \infty} d(x_n, p) \) exist, we obtain

\[
\lim_{n \to \infty} d(x_n, T^n x_n) = 0. \tag{3.9}
\]

Next, we prove that \( \lim_{n \to \infty} d(x_n, T x_n) = 0 \). It follow from (1.3) and (3.9), we obtain

\[
d(x_{n+1}, x_n) = d(\alpha x_n \oplus (1 - \alpha_n) T^n x_n, x_n)
\leq (1 - \alpha_n) d(x_n, T^n x_n) \to 0 \quad \text{as } n \to \infty. \tag{3.10}
\]

Since \( T \) is an uniformly \( L \)-lipschitzian mapping, from (3.9) and (3.10) for any \( n \geq 1 \), we have

\[
d(x_n, T x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^n x_{n+1}) + d(T^{n+1} x_{n+1}, T x_n)
\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^n x_{n+1}) + Ld(x_{n+1}, x_n) + Ld(T^n x_n, x_n)
= (1 + L)d(x_n, x_{n+1}) + d(x_{n+1}, T^n x_{n+1}) + Ld(x_n, T^n x_n)
\to 0 \quad \text{as } n \to \infty.
\]

The proof is completed. \( \square \)

Next, we are ready to prove our results of an \( \Delta \)-convergence theorem.

**Theorem 3.4.** Let \( C \) be a nonempty closed convex subset of a complete \( \text{CAT}(0) \) space \( X \) and let \( T: C \to C \) be an asymptotically \( k \)-strictly pseudo-contractive mapping with \( k \in [0, \frac{1}{2}) \) and a sequence \( \{k_n\} \) in \([1, \infty)\) such that \( \lim k_n = 1 \). Let \( \{x_n\} \) be a sequence in \( C \) defined by (1.3) and \( \{\alpha_n\} \) is a sequence in \((0, 1)\). Then the sequence \( \{x_n\} \) is \( \Delta \)-convergent to a fixed point of \( T \).

**Proof.** The first, we prove that

\[
W_{\Delta}(x_n) = \bigcup_{\{u_n\} \subseteq \{x_n\}} A((u_n)) \subseteq F(T).
\]

Let \( u \in W_{\Delta}(x_n) \). Then, there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A((u_n)) = \{u\} \). By Lemma 2.5, there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_n v_n = v \in C \). By Theorem 3.3 and Theorem 3.2, we have \( v \in F(T) \). Since \( \lim d(x_n, v) \) exists, so \( u = v \) by Lemma 2.3. This show that \( W_{\Delta}(x_n) \subseteq F(T) \).

Next, we prove that \( \Delta \)-converges to a point in \( F(T) \), it is sufficient to show that \( W_{\Delta}(x_n) \) consists of exactly one point. Let \( \{u_n\} \) be a subsequence of \( \{x_n\} \) with \( A((u_n)) = \{u\} \) and let \( A((x_n)) = \{x\} \). We have already seen that \( u = v \) and \( v \in F(T) \). Since \( u \in W_{\Delta}(x_n) \subseteq F(T) \), by Theorem 3.3, \( \lim_{n \to \infty} d(x_n, u) \) exists. Hence, we obtain \( x = u \) by Lemma 2.3. This shows \( W_{\Delta}(x_n) = \{x\} \). This completes the proof. \( \square \)

**Theorem 3.5.** Let \( C \) be a nonempty closed convex subset of a complete \( \text{CAT}(0) \) space \( X \) and let \( T: C \to C \) be an uniformly continuous asymptotically \( k \)-strictly pseudo-contractive mapping with \( k \in [0, \frac{1}{2}) \) and a sequence \( \{k_n\} \) in \([1, \infty)\) such that \( \lim k_n = 1 \). Let \( \{x_n\} \) be a sequence in \( C \) defined
by (1.3) and \(\{a_n\}\) is a sequence in \((0, 1)\). Assume that \(T^s\) is semi-compact for some \(s \in \mathbb{N}\). Then the sequence \(\{x_n\}\) is converges strongly to a fixed point of \(T\).

**Proof.** By Theorem 3.3, we have \(\lim\limits_{n \to \infty} d(x_n, Tx_n) = 0\). Since \(T\) is an uniformly continuous, then

\[
d(x_n, T^s(x_n)) = d(x_n, T(x_n)) + d(T(x_n), T^2(x_n)) + \ldots + d(T^{s-1}(x_n), T^s(x_n)) \to 0 \text{ as } n \to \infty.
\]

That is, \(\{x_n\}\) is an approximate fixed point sequence for \(T^s\). By Definition 2.6, there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) and \(p \in C\) such that \(\lim_{k \to \infty} x_{n_k} = p\). Again, by the uniform continuity of \(T\), we obtain

\[
d(T(p), p) \leq d(T(p), T(x_{n_k})) + d(T(x_{n_k}), x_{n_k}) + d(x_{n_k}, p) \to 0 \text{ as } k \to \infty.
\]

That is, \(p \in F(T)\). From again Theorem 3.3, we get \(\lim\limits_{n \to \infty} d(x_n, p)\) exist, therefore \(p\) is the strong limit of the sequence \(\{x_n\}\) itself. The proof is completed. \(\square\)

### 4. Conclusion

In this work, we studied and proved strong and \(\Delta\)-convergence theorems by using modified Mann iteration process for an asymptotically \(k\)-strictly pseudo-contractive mapping in a \(\text{CAT}(0)\) spaces.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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