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Commuting Regular Graphs for Non-Commutative Semigroups

Research Article

Azam Firuzkuhy and Hossein Doostie *

Department of Mathematics, Science and Research Branch, Islamic Azad University, P.O. Box 14515-1775, Tehran, Iran
Corresponding author: doostih@gmail.com

Abstract. To study the commuting regularity of a semigroup, we use a graph. Indeed, we define a multi-graph for a semigroup and identify this graph for the semidirect product of two monogenic semigroups. For a non-group semigroup S, the ordered pair (x, y) of the elements of S is called a commuting regular pair if for some $z \in S$, $xy = yxzyx$ holds, and S is called a commuting regular semigroup if every ordered pair of S is commuting regular. As a result of Abueida in 2013 concerning the heterogenous decomposition of uniform complete multi-graphs into the spanning edge-disjoint trees, we show that for a semigroup of order n, the commuting regular graph of S, $\Gamma(S)$ has at most n spanning edge-disjoint trees.

Keywords. Commuting regular graphs; Commuting regular semigroups; Multi-graphs

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1. Introduction

For a finite non-group semigroup S, an ordered pair (x, y) of the elements of S is called a commuting regular pair if for some $z \in S$, $xy = yxzyx$, and a semigroup is commuting regular if every pair of its elements is commuting regular. This notion studied by certain authors during the years for its interesting properties (one may see [10], [5] and [8]). This property is indeed the generalization of the commutativity in groups as studied in [4].

We assume that the reader is familiar with the notions of regular semigroup, inverse semigroup and rectangular band. For more information one may consult [7] and [3]. We refer to the recently obtained result of Pourfaraj [8] which states that the commutativity and commuting regularity of a semigroup are equivalent just for the rectangular bands. Also we may recall the result of Sadeghieh [5] which proves that a commutative, regular semigroup is commuting regular. Evidently, every group is a commuting regular semigroup, so all semigroups here are non-group finite semigroups.

Considering this property, we define the commuting regular graph as follows:

Definition 1.1. The commuting regular multi-graph of a semigroup S, is an undirected multi-graph $\Gamma(S)$, where every elements of S is a vertex and for two different vertices x and y, there are k number of edges between x and y where,

$$k = \begin{cases} 0, & \text{non of the pairs } (x,y) \text{ and } (y,x) \text{ are commuting regular,} \\ 1, & \text{exactly one of the pairs } (x,y) \text{ or } (y,x) \text{ is commuting regular,} \\ 2, & \text{both of the pairs } (x,y) \text{ and } (y,x) \text{ are commuting regular.} \end{cases}$$

Following [1], we recall the definition of uniform complete multi-graph K_n^r where, r is its edge multiplicity.

Obviously, if S is a group of order n, $\Gamma(S)$ is indeed the coinciding of two complete graphs, i.e.; $\Gamma(S) = K_n^2$. The natural question may be posed here that "is this multi-graph a complete multi-graph for non-group semigroups?". Investigating this question leads us to consider two types of non-group semigroups, commutative and non-commutative.

Here, our notation are merely standard and we follow [2, 9] to use $Sg(\pi)$ and $Gp(\pi)$ to distinguish between the semigroup and the group presented by the formal presentation $\pi = \langle X \mid R \rangle$ where, X is the generating set and R is the set of relators. We need the definition of semidirect product of two semigroups. As usual, for two semigroups S_1 and S_2 and for a homomorphism $\phi : S_2 \to End(S_1)$ where, $End(S_1)$ is the semigroup of endomorphisms, the set $S_1 \times S_2$ is a semigroup where the multiplication defined as:

$$(a,b)(c,d) = (a\varphi_b(c), bd)$$

such that $\varphi_b \in End(S_1)$ is the image of $b \in S_2$.

Our main results on non-commutative semigroups involving the semidirect product of monogenic semigroups are Propositions A and B. Proposition C is to establish the graph of commutative rectangular bands.

Proposition A. *For every integers $m,n \geq 3$, where m is even, the semigroup $S = Sg(\pi)$ where $\pi = \langle a,b \mid a^{n+1} = a, b^{m+1} = b, ba = a^{n-1}b, a^n = b^m \rangle$ is a non- commutative and commuting regular semigroup.*

Proposition B. *If S is the non-group semigroup of the semidirect product of two monogenic semigroups of orders m and n, then $\Gamma(S) = K_{mn}^2$.*

Proposition C. *For every commutative rectangular band S of order n, $\Gamma(S) = K_n^2$.*

2. The Proofs

Proof of Proposition A. Let $m,n \geq 3$ and m is even. For every integers $i,j \geq 1$ the relators $b^i a^j = a^{j(n-1)^i} b^i$, $a^n b^i = b^{m+i}$, $a^{n+i} = a^i$ and $b^{m+i} = b^i$ hold in S (one may use an inductive method to check them.) So, the elements of S are of three types: $\{a^i \mid 1 \le i \le n\}$, $\{b^i \mid 1 \le i \le m-1\}$ and $\{a^i b^j \mid 1 \le i \le n-1,\ 1 \le j \le m-1\}$. This shows that $|S| = nm$. This semigroup is not a rectangular band as well, because of the equations $aba = aa^{n-1}b = a^n b = b^{m+1} = b \ne a$.

According to these three types of the elements of S, any idempotent element e of S should be in one of the forms a^i, b^j or $a^i b^j$. Let e be the idempotent element. If $e = a^i$ then, $a^{2i} = a^i$ where, $1 \le i \le n$. The integer i is greater than or equal to $\frac{n}{2}$, otherwise, it results $2i < n$ and $a^{2i} \ne a^i$. If $i > \frac{n}{2}$, then i must be equal to n. So a^n is an idempotent element. The same discussion will be occurred for b^j where $1 \le j \le m-1$ which results that there is no value for j to make b^j as an idempotent element. Hence, there is no idempotent in the form of $a^i b^j$ as well, too.

We conclude that S is not a rectangular band. Because in a rectangular band, all elements are idempotent.

Obviously, S is a non-commutative semigroup. Now, we investigate the regularity and commuting regularity of S. To show its regularity, it is sufficient to consider the Table 1 which gives us a suitable y for every x such that $xyx = x$ holds.

Table 1. Regular elements of S

x	y	
a^i	a^j	$j = nk - i$
b^i	b^j	$j = mk - i$
$a^i b^j$	$b^{m-j} a^{n-i}$	

In Table 1, k is the smallest integer such that j is a positive integer. Finally, S is a commuting regular semigroup, Table 2 provides a suitable z of S satisfying the commuting regularity property, for every pair (x, y) where, $x, y \in S$. This completes the proof.

Table 2. Commuting regular pairs of S

x	y	z	
a^i	a^j	a^t	$t \equiv 2n - (i+j) \pmod n$
b^i	b^j	b^t	$t \equiv 2m - (i+j) \pmod m$
a^i	b^j	$b^{m-j} a^{n-t+1}$	$t \equiv 2(n-1)^j \pmod n$
b^j	a^i	$b^{m-j} a^{2n-2i+t}$	$t \equiv i(n-1)^j \pmod n$
a^i	$a^r b^s$	$b^{m-s} a^t$	$2r + 2i(n-1)^s + t \equiv i + r \pmod n$
$a^r b^s$	a^i	$b^{m-s} a^t$	$r + i(n-1)^s \equiv 2(r+i) + t \pmod n$
b^i	$a^r b^s$	$b^{2m-(s+i)} a^t$	$2r + n + t \equiv r(n-1)^i \pmod n$
$a^r b^s$	b^i	$b^{2m-(s+i)} a^t$	$t + 2r(n-1)^i \equiv r \pmod n$
$a^i b^j$	$a^r b^s$	$b^{2m-(s+j)} a^t$	$t + 2r + 2i(n-1)^s \equiv i + r(n-1)^j \pmod n$

$\square$

Proof of Proposition B. For every integers $m,n \geq 3$ the semidirect product of two monogenic semigroups $\langle a|a^{n+1}=a\rangle$ and $\langle b|b^{m+1}=b\rangle$ may be presented by:

$$\pi = \langle A,B|A^{n+1}=A,B^{m+1}=B,BA=A^{n-1}B,A^n=B^m\rangle$$

where, $A=(a,b^m)$ and $B=(a^n,b)$ (for a proof one may consider the main definition of the semidirect product of two semigroups and [6].)

So, according to Proposition A it is obvious that the semidirect product of two monogenic semigroups is a non-commutative and commuting regular non-group semigroup with mn elements.

Now, by Definition 1.1, in its commuting regular graph, all vertices connected to each other by two edges. So, $\Gamma(S)=K^2_{mn}$. □

Proof of Proposition C. Every commutative rectangular band S of order n is a commuting regular semigroup (one may consider [8]). So, the corresponding commuting regular multi-graph is a graph by n vertices such that there are exactly two edges between each pair of vertices. This shows $\Gamma(S)=K^2_n$. □

3. Conclusion

For a non-commutative commuting regular semigroup of order n, we get $\Gamma(S)=K^2_n$. Investigating and enumerating all spanning trees of this graph is of interest when these trees are edge-disjoint (one may see the literature for engineering applications of this enumeration in airline scheduling.) For this investigation, we consider Abueida [1]. In this paper, it is shown that for K^2_n, where $n \geq 8$, there are exactly n heterogeneous spanning trees as a decomposition of graph. Indeed, these n trees are $T_0,T_1,\dots,T_{n-1}$. By labelling the vertices of K^2_n by the elements of Z_n, all these trees are characterized completely. The tree $T_0(n)$ is a path on n vertices. At the vertex 1 the tree $T_1(n)$ has the maximum degree 3. Also, the tree $T_2(n)$ has the maximum degree 3 at the vertices 2 and 3. Finally, the degrees of the vertices 1, 2 and 3 of the tree $T_3(n)$ is 3. For every i, $(4 \leq i \leq n)$, there is a unique vertex of the maximum degree i in the tree $T_i(n)$ which is labelled i. This investigation yields that for a finite semigroup of order n, $\Gamma(S)$ has at most n edge-disjoint spanning trees.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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