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Research Article

Independent Perfect Secure Domination in Graphs

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Abstract. For a graph $G = (V,E)$, a subset $Y \subseteq V$ will be a dominating set if each vertex $y \in V \setminus Y$ possesses a neighbour in Y. A perfect secure dominating set of G is a dominating set in which every vertex $y \in V \setminus Y$, has a unique vertex $x \in Y$ such that $xy \in E$ and $(Y \setminus \{x\}) \cup \{y\}$ is a dominating set. In addition, if Y is an independent set, then Y is an independent perfect secure dominating set of G. We have introduced the concept of independent perfect secure domination, presented the fundamental properties of this new parameter and investigated the independent perfect secure domination in certain classes of graphs such as the connected split graphs and the spiders in this paper. The complexity of the parameter is also discussed.

Keywords. Secure dominating set, Perfect secure dominating set, Independent perfect secure dominating set, Computational complexity

Mathematics Subject Classification (2020). 05C69, 68Q25, 05C05

1. Introduction

Let $G = (V,E)$ be an undirected simple graph. A vertex x is said to be a *neighbour* of vertex z if $xz \in E$. The vertices in the *open neighbourhood* of a vertex $x \in V$ are the vertices defined in the set $N(x) = \{z \in V : xz \in E\}$. The *closed neighbourhood* of x consists of the vertices of $\{x\} \cup N(x)$. For a subset $Y \subseteq V$ and $y \in Y$, the *private neighbours* of y with respect to Y are the vertices in the set $\mathrm{pn}(y,Y) = N[y] \setminus N[Y \setminus \{y\}]$. This neighbourhood is called as the *private neighbourhood* of y with respect to Y. The *external private neighbourhood* of y with respect to Y is the set $\mathrm{epn}(y,Y) = \mathrm{pn}(y,Y) \setminus \{y\} = N(y) \setminus N[Y \setminus \{y\}]$. A *clique* of G is a subset X of V such that $G[X]$ is complete. A vertex z is a *leaf* if $\deg(z) = 1$. The neighbour of z is called as the *support* of z. A *dominating set* Y of a graph G is a subset of V for which the vertices in $V \setminus Y$ have a neighbour in Y. The developments in the study of some prominent domination parameters can be referred in Haynes *et al*. [12–14].

The introduction of Roman domination gave a conceptualization of using domination parameters for the allocation of defense units (Cockayne *et al*. [8]). Later, the concepts of weak Roman domination (Henning and Hedetniemi [15]) and secure domination (Cockayne *et al*. [9]) were introduced. These variants are some of the strategies used for the allocation of defense units. In a secure dominating set, each vertex $y \in V \setminus Y$ has a neighbour $x \in Y$ such that $(Y \setminus \{x\}) \cup \{y\}$ is a dominating set (Cockayne *et al*. [9]). Then y is said to be *defended* by x or it can be said that x *defends* y. For a detailed study of the parameter, we refer Araki and Yumoto [1], Burger *et al*. [3, 4], Cockayne [7], Grobler and Mynhardt [11], Klostermeyer and Mynhardt [17], Li and Xu [18], Merouane and Chellali [20], and Wang *et al*. [23]. The study on perfect secure domination was initiated by Rashmi *et al*. [21]. A secure dominating set Y can be a perfect secure dominating set if each vertex $y \in V \setminus Y$ has a unique vertex $x \in Y$ such that x is a neighbour of y and $(Y \setminus \{x\}) \cup \{y\}$ is a dominating set (Rashmi *et al*. [21]). The results on the complexity of perfect secure domination were studied by Chakradhar and Reddy [6].

A set of vertices is independent if no two elements of the set are adjacent. A set is an independent dominating set if it is both a dominating set and an independent set. A survey on the important results of independent domination can be referred in Goddard and Henning [10]. There are numerous studies on efficient domination, independent domination and perfect domination. The studies were primarily driven by the diverse applications of these domination parameters, especially in the domain of computer science (Biggs [2], Hurink and Nieberg [16]).

Some of the applications related to secure domination are facility location problems and optimal placement of monitoring devices. Suppose that multiple drones are employed for surveillance and threat detection in a fixed large area. Each drone monitors the surroundings within its range for any possible threat. When an unusual activity is detected at a point of time within the drone's perimeter, the drone is expected to move to that spot and collect detailed information about the threat or unusual activity. In this example, when the locations of the drones are allocated based on secure domination, it would ensure the continuity of surveillance in the entire area even when a drone is displaced from its allocated location to the spot of unusual activity. An allocation of the locations of drones based on independent perfect secure domination would additionally render minimal interference between the drones. It would also ensure that each spot in the given area is accessed by a unique drone. Such an application could be employed in monitoring terrorist activities, monitoring traffic, maritime surveillance, environmental management and monitoring, crowd management, monitoring zones prone to volcanic activity etc.

For a *split graph*, the vertex set can be partitioned into sets I and K where I and K correspond to an independent set and a clique respectively. A tree is known to be a *spider* if it has at most one vertex whose degree is greater than or equal to three. When $\Delta \geq 3$, a spider can be obtained by subdividing the edges of $K_{1,n}$, $n = \Delta$. The vertex v with $\deg(v) \geq 3$ is referred to as the *head* of the spider. Each vertex of $N(v)$ is said to be a *hip* of the spider (Varghese *et al*. [22]). The *leg* of the spider is a path induced with a hip and a leaf as the end vertices such that the head will not be its internal vertex. Let the number of vertices in a leg be referred to as the *length* of the leg.

Certain fundamental results of secure domination are as follows:

Proposition 1.1 ([9]). *Let D be a dominating set. A vertex $x \in D$ defends a vertex $y \in V \setminus D$ if and only if $G[\{y,x\} \cup epn(x,D)]$ is complete.*

Proposition 1.2 ([9]). $\gamma_s(K_n) = 1$.

Theorem 1.1 ([9]). $\gamma_s(P_n) = \gamma_s(C_n) = \lceil \frac{3n}{7} \rceil$.

We introduce the concept of independent perfect secure domination in this paper and initiate a study on the novel parameter.

2. Independent Perfect Secure Domination

A formal definition for the concept of independent perfect secure domination is given in this section. We also present the basic properties of the new parameter.

Definition 2.1. A dominating set S of a graph G is an independent perfect secure dominating set of G if S is an independent set and for every vertex $w \in V \setminus S$, there is a unique vertex $u \in S$ such that $u \in N(w)$ and $(S \setminus \{u\}) \cup \{w\}$ is a dominating set for G.
The independent perfect secure domination number of G, denoted by $\gamma_{ips}(G)$ gives the minimum cardinality of an independent perfect secure dominating set of G. An independent perfect secure dominating set whose cardinality is $\gamma_{ips}(G)$ is called as a γ_{ips}-set of G.

Remark 2.1. Independent perfect secure dominating set does not exist for all the graphs.

Remark 2.2. $\gamma_s(G) \le \gamma_{ips}(G)$.

In the following results, we denote S to be an independent perfect secure dominating set.

Proposition 2.1. $\gamma_{ips}(G) = 1$ *if and only if G is complete.*

Proposition 2.2. *If G has a vertex of degree $|V|-1$, then G can have an independent perfect secure dominating set only when G is complete.*

Proof. Let us denote W to be the set of vertices whose degrees are $|V|-1$. Assume that G satisfies independent perfect secure domination. If G is not complete, then $|S| > 1$. Consider a vertex $w \in W$. If $w \notin S$, then $|N(w) \cap S| > 1$ and w will be adjacent to every external private neighbour of all the vertices of S. By Proposition 1.1, w can be defended by the vertices of $N(w) \cap S$. Hence w cannot be defended by a unique vertex of S. Thus, G should be complete when $w \notin S$.
Suppose that $w \in S$, then $|S| \ngtr 1$ since S is an independent set. By Proposition 2.1, G should be complete if any vertex of W is in S. □

Proposition 2.3. *A complete multipartite graph $K_{n_1,n_2,\dots,n_r}$, $r > 1$ does not satisfy independent perfect secure domination if any $n_i > 1$ for $1 \le i \le r$.*

Proof. The vertices of S cannot be in different partite sets as S is independent. Let X be a partite set and consider $S \subseteq X$. Then, we should have $S = X$ otherwise the vertices of $X \setminus S$ cannot be defended by an independent set. When $|X| \ge 2$, we have $N(v_i) = N(v_j)$ for any two vertices of $X \cap S$ and hence the vertices of $V \setminus X$ cannot be defended by a unique vertex.

Therefore, $|X| = 1$ is the only possibility. By Proposition 2.1, $|S| = 1$ only when $n_1 = \cdots = n_r = 1$. Thus $K_{n_1,n_2,\ldots,n_r}$ does not satisfy independent perfect secure domination when an $n_i > 1$ for $1 \le i \le r$. □

Remark 2.3. If two or more leaves are attached to the same support vertex, then such a graph G will not have an independent perfect secure dominating set.

Remark 2.4. Let us denote a leaf by l and its support by s. If $s \in S$, then $l \in \text{epn}(s,S)$ and it is the only vertex defended by s. If $s \notin S$, then $l \in S$ and s is defended by l.
On the other hand, if a vertex $b \in (N(s)\backslash l)$ is an element of S, then $l \in S$ and $s \notin S$. Also, $|\text{epn}(b,S)| > 0$. Since l defends s, s will not be a neighbour to all the vertices of $\text{epn}(b,S)$ by Proposition 1.1.

Remark 2.5. For a vertex $u \in S$, if $\text{epn}(u,S) = \emptyset$, then u defends all its neighbours.

Remark 2.6. For a vertex $u \in S$, if $\text{epn}(u,S) = \emptyset$, then each vertex of $(N(N(u))\backslash u) \cap S$ will have external private neighbours in $V\backslash N[u]$. Let $v \in N(u)$, then v will not have a neighbour $w \in S\backslash\{u\}$ such that v is a neighbour to all the vertices of $\text{epn}(w,S)$. Otherwise, v cannot be defended by a unique vertex.

Theorem 2.1. *An independent perfect secure dominating set exists for a path P_n when $n \equiv 0,1,2,4,6 \pmod 7$ and $\gamma_{ips}(P_n) = \lceil \frac{3n}{7} \rceil$.*

Proof. Let the vertices of a path P_n be labelled as $v_1, v_2, \ldots, v_n$. The absence of cycles implies that $|\text{epn}(v,S)| \le 1$. If a non-leaf vertex $v_i \in S$ and has $|\text{epn}(v_i,S)| = 1$, then $v_{i+1} \in \text{epn}(v_i,S)$ or $v_{i-1} \in \text{epn}(v_i,S)$. Without loss of generality, we consider $v_{i-1} \in \text{epn}(v_i,S)$ and let $i+5 \le n$. Since $v_{i+1} \notin N(v_{i-1})$, v_{i+1} cannot be defended by v_i by Proposition 1.1. Hence, $v_{i+2} \in S$ and v_{i+1} is defended by v_{i+2}. Since S is independent, $v_{i+3} \notin S$. If $v_{i+4} \notin S$, we have $v_{i+3} \in \text{epn}(v_{i+2},S)$. Then by Proposition 1.1, v_{i+1} cannot be defended by v_{i+2}, a contradiction. Thus, $v_{i+4} \in S$ and $|\text{epn}(v_{i+2},S)| = 0$. Hence, v_{i+1} and v_{i+3} are defended by v_{i+2}. From Remark 2.6, we have $|\text{epn}(v_{i+4},S)| = 1$ and $v_{i+5} \in \text{epn}(v_{i+4},S)$. Thus, the subgraph induced by $\{v_{i-1}, v_i, \ldots, v_{i+5}\}$ is a P_7 and S contains the vertices $\{v_i, v_{i+2}, v_{i+4}\}$. When $v_{i+1} \in \text{epn}(v_i,S)$ with $i-5 \ge 1$, the vertices $\{v_{i-5}, v_{i-4}, \ldots, v_{i+1}\}$ induce a path P_7 and are defended by $\{v_{i-4}, v_{i-2}, v_i\}$.
From Remark 2.4 and the above reasoning, only P_n which can be packed with $k.P_7 \cup i.P_2 \cup j.P_4$ can have independent perfect secure dominating sets when $i+j \le 2$ and $7k+2i+4j = n$. Among the vertices which are packed with a P_2, at least one vertex should be a leaf. The same holds for the vertices which are packed with a P_4.
Consider $n \equiv r \pmod 7$, $r = 0,1,2,4,6$. When $n \ge 9$, $V(P_n)\backslash R$ can be packed with P_7 where

$$R = \begin{cases} \{v_1, v_2, v_3, v_4, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}, & \text{when } r = 1, \\ \{v_{n-1}, v_n\}, & \text{when } r = 2, \\ \{v_1, v_2, v_{n-1}, v_n\}, & \text{when } r = 4, \\ \{v_1, v_2, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}, & \text{when } r = 6 \\ \emptyset, & \text{when } r = 0. \end{cases}$$

Since $\gamma_{ips}(G) \geq \gamma_s(G)$,

$$S = \begin{cases} \{v_1, v_3, v_{n-2}, v_n\} \cup \left(\bigcup_{\beta=1}^{\lfloor \frac{n}{7} \rfloor - 1} \{v_{6+7(\beta-1)}, v_{8+7(\beta-1)}, v_{10+7(\beta-1)}\} \right), & \text{when } r = 1,\ n > 8, \\ \{v_n\} \cup \left(\bigcup_{\beta=1}^{\lfloor \frac{n}{7} \rfloor} \{v_{2+7(\beta-1)}, v_{4+7(\beta-1)}, v_{6+7(\beta-1)}\} \right), & \text{when } r = 2,\ n > 2, \\ \{v_1, v_n\} \cup \left(\bigcup_{\beta=1}^{\lfloor \frac{n}{7} \rfloor} \{v_{4+7(\beta-1)}, v_{6+7(\beta-1)}, v_{8+7(\beta-1)}\} \right), & \text{when } r = 4,\ n > 4, \\ \{v_1, v_{n-2}, v_n\} \cup \left(\bigcup_{\beta=1}^{\lfloor \frac{n}{7} \rfloor} \{v_{4+7(\beta-1)}, v_{6+7(\beta-1)}, v_{8+7(\beta-1)}\} \right), & \text{when } r = 6 \text{ and } n > 6, \\ \bigcup_{\beta=1}^{\lfloor \frac{n}{7} \rfloor} \{v_{2+7(\beta-1)}, v_{4+7(\beta-1)}, v_{6+7(\beta-1)}\}, & \text{when } r = 0 \end{cases}$$

is a minimum independent perfect secure dominating set with $\gamma_{ips}(P_n) = \left\lceil \frac{3n}{7} \right\rceil$.
When $n = 1, 2, 4, 6$ and 8, we have

$$S = \begin{cases} \{v_1\}, & \text{when } n = 1, \\ \{v_2\}, & \text{when } n = 2, \\ \{v_1, v_4\}, & \text{when } n = 4, \\ \{v_1, v_4, v_6\}, & \text{when } n = 6, \\ \{v_1, v_3, v_6, v_8\}, & \text{when } n = 8 \end{cases}$$

to be a minimum independent perfect secure dominating set with $\gamma_{ips}(P_n) = \left\lceil \frac{3n}{7} \right\rceil$. □

Theorem 2.2. *An independent perfect secure dominating set exists for a cycle C_n, $n > 3$ when $n \equiv 0 \pmod 7$. $\gamma_{ips}(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

3. Complexity Results

In this section, we show that the decision problem pertaining to the existence of an independent perfect secure dominating set is NP-complete even when restricted to bipartite graphs.

Independent Perfect Secure Dominating Set Existence Problem (IPSDE)

Instance: A graph $G^* = (V^*, E^*)$.

Question: Does G^* have an independent perfect secure dominating set?

The proof of the result is by reduction from the efficient domination problem. McRae [19] had proved the NP-completeness of efficient domination problem for bipartite and chordal graphs and we refer [13] for the proof.

Efficient Dominating Set Problem (ED)

Instance: A graph $G = (V, E)$.

Question: Does G have an efficient dominating set?

Theorem 3.1. *Problem IPSDE is NP-complete for bipartite graphs.*

Proof. It is obvious that IPSDE is in NP. For a bipartite graph $G = (V,E)$, let $G' = (V',E')$ be a copy of G such that V' and E' are copies of V and E respectively. We denote the vertices of G by v_i and the vertices of G' by v_i'. Let G'' be a graph with vertex set $V'' = V \cup V'$ and edge set $E'' = E \cup E' \cup E_{cross}$ where $E_{cross} = \{uv', u'v : uv \in E\}$. Corresponding to each $v_i \in V$, we consider a subgraph H_i. We assume each H_i to be a P_{19} whose vertices are denoted by $h_{i,j}$ along with an edge linking vertices $h_{i,2}$ and $h_{i,15}$. Then, G^* can be obtained from G'' and $\{H_i\}_{i=1}^{|V|}$ by linking each $h_{i,1}$ to its corresponding v_i for every i. Thus, G^* is a bipartite graph with $21|V|$ vertices and $4|E| + 20|V|$ edges. Therefore, G^* can be constructed from G in polynomial time.

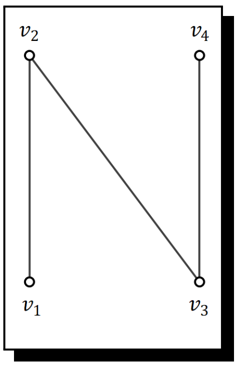

Figure 1. G

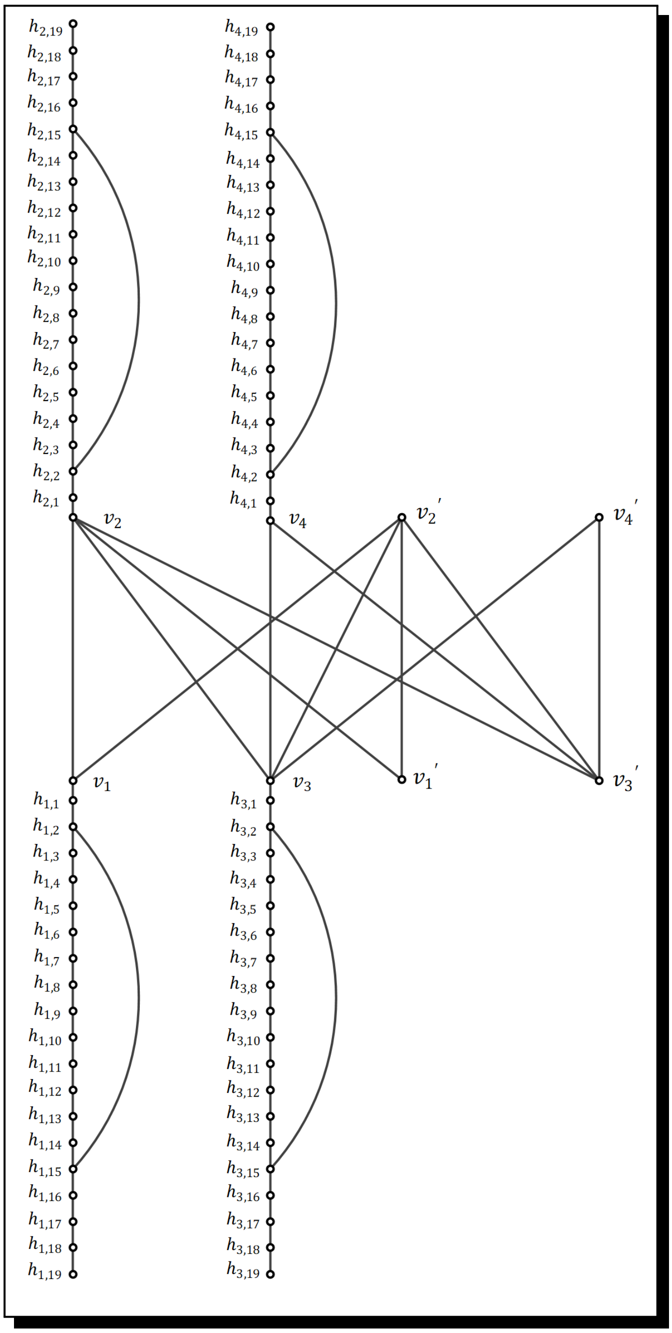

Figure 2. G^*

Let S and S^* be an efficient dominating set and an independent perfect secure dominating set of G and G^* respectively. Then, we prove that S exists when S^* exists and conversely. We first consider that G has an efficient dominating set S. When $v_i \in S$, let $v_i \cup v_i' \cup \{h_{i,3}, h_{i,5}, h_{i,7}, h_{i,10}, h_{i,12}, h_{i,14}, h_{i,17}, h_{i,19}\} \subset S^*$. Suppose that $v_q \notin S$, then let $\{h_{q,2}, h_{q,4}, h_{q,6}, h_{q,9}, h_{q,11}, h_{q,13}, h_{q,16}, h_{q,18}\} \subset S^*$. Then, S^* is an independent perfect secure dominating set of G^*.

Conversely, let S^* be an independent perfect secure dominating set of G^*, then we need to prove that there will be an efficient dominating set S for G.

Observation 1. *$h_{i,1} \notin S^*$ for any i.*

Proof of the claim. Suppose that $h_{i,1} \in S^*$ has an external private neighbour which is external to the vertices of H_i, then $h_{i,2}$ must be defended by $h_{i,3}$ or $h_{i,15}$. Then at least one vertex of $H_i \backslash \{h_{i,1}, h_{i,2}, h_{i,3}, h_{i,15}\}$ will not be defended by a unique vertex. Suppose that $h_{i,2}$ is the external private neighbour of $h_{i,1}$, then $h_{i,3}, h_{i,15} \notin S^*$. Then, there cannot be a configuration in which the vertices of $\{h_{i,3}, \dots, h_{i,15}\}$ are defended by unique vertices of an independent set. Thus when $h_{i,1} \in S^*$, it cannot have an external private neighbour.
Suppose that $h_{i,1} \in S^*$ does not have an external private neighbour. By Remark 2.6, every vertex of $N(h_{i,1})$ should have a neighbour $z \neq h_{i,1}$ in S^* such that z has external private neighbours. Let $h_{i,3} \in S^*$ and if the vertices $h_{i,3}, \dots, h_{i,15}$ are defended by unique vertices of an independent set, then $h_{i,2}$ cannot be defended by a unique vertex of S^*. When $h_{i,3} \notin S^*$, then $h_{i,15} \in S^*$. If we consider $\{h_{i,3}, \dots, h_{i,16}\}$ to be defended by unique vertices of an independent set, then $\{h_{i,17}, h_{i,18}, h_{i,19}\}$ cannot be defended by unique vertices of an independent set.
Thus, we conclude that $h_{i,1} \notin S^*$ for any i.

Observation 2. *$v_i \in S^*$ if and only if $v_i' \in S^*$.*

Proof of the claim. If we consider $v_i \in S^*$ and $v_i' \notin S^*$, then v_i' should be defended by a vertex of $N(v_i') \cap S^*$. But as $N(v_i') \subset N(v_i)$, it leads to the contradiction that S^* is an independent set. Thus, if $v_i \in S^*$, then $v_i' \in S^*$.
Conversely, consider $v_i' \in S^*$ and $v_i \notin S^*$. Since $N(v_i) \backslash N(v_i') = \{h_{i,1}\}$, v_i should be defended by $h_{i,1}$. But by Observation 1, $h_{i,1} \notin S^*$. Therefore $v_i \in S^*$ whenever $v_i' \in S^*$. Hence the claim.
Since S^* is independent, $S^* \cap V$ is also independent. Further, if we prove that $S^* \cap V$ is a dominating set for G with $N(v_\alpha) \cap N(v_\beta) = \emptyset$ for any $v_\alpha \in S^* \cap V$ and $v_\beta \in S^* \cap V$, the proof is complete.
Suppose that $S^* \cap V$ is not a dominating set for G. Then, let us consider a vertex $v_q \in V$ for which $N[v_q] \cap (S^* \cap V) = \emptyset$. Since S^* is a dominating set, v_q should have a neighbour in $S^* \backslash V$. If v_q is dominated by a vertex $v_k{}' \in S^* \cap V'$, then by Observation 2 there should exist a vertex $v_k \in S^* \cap V$ which dominates v_q, a contradiction. Also from Observation 1, $h_{q,1} \notin S^*$. Thus, v_q does not have a neighbour of S^* in $N[v_q] \backslash V$. Hence if S^* is an independent perfect secure dominating set, then the vertices of V should have a vertex of $S^* \cap V$ in their closed neighbourhood. Therefore $S^* \cap V$ is a dominating set for G.
Next, let us consider $N(v_i) \cap N(v_j) \neq \emptyset$ for some $v_i \in S^* \cap V$ and $v_j \in S^* \cap V$. From Observation 2, v_i' and v_j' should be elements of S^*. Then, $N(v_i) \cap N(v_j)$ cannot be defended by unique vertices of S^* as they can be defended by both v_i' and v_j'. Hence, $N(v_i) \cap N(v_j) = \emptyset$ for any $v_i \in S^* \cap V$ and $v_j \in S^* \cap V$. Therefore, $S = S^* \cap V$ is an efficient dominating set.
Hence the problem IPSDE is NP-complete for bipartite graphs. □

4. Further Results on Independent Perfect Secure Domination in Graphs

In this section, we provide upper bounds for $|S|$ and $|E|$ of a non-trivial, simple connected graph where S is taken to be an arbitrary independent perfect secure dominating set in the result. The independent perfect secure domination of connected split graphs is also investigated. The existence of a graph which satisfies independent perfect secure domination for arbitrary values of $|V|$ and $\gamma_{ips}(G)$ is also proved.

Theorem 4.1. *Let G be a non-trivial simple connected graph which satisfies independent perfect secure domination, then $|S| \le \lfloor \frac{n}{2} \rfloor$.*

Proof. If $|S| > \lfloor \frac{n}{2} \rfloor$, then $|V \setminus S| < |S|$. Let $W \subset S$ contain all the vertices of S which have external private neighbours. Let $T = \bigcup_{v \in W} (v \cup \text{epn}(v,S))$ and let T' be a subset of $N(W) \setminus \bigcup_{v \in W} \text{epn}(v,S)$ which consists of vertices that can be defended by the vertices of W. Then, the vertices in $V \setminus (T \cup T')$ are the vertices of $(S \setminus W) \cup \left((V \setminus S) \setminus \left(\left(\bigcup_{v \in W} \text{epn}(v,S)\right) \cup T'\right)\right)$. Since $|W| \le \left|\left(\bigcup_{v \in W} \text{epn}(v,S)\right) \cup T'\right|$, we have $|S \setminus W| > \left|(V \setminus S) \setminus \left(\left(\bigcup_{v \in W} \text{epn}(v,S)\right) \cup T'\right)\right|$. As S is independent and also as G is connected, we have each vertex of $S \setminus W$ to be adjacent to a vertex of $(V \setminus S) \setminus \left(\left(\bigcup_{v \in W} \text{epn}(v,S)\right) \cup T'\right)$. By pigeonhole principle, there will exist a vertex $u \in (V \setminus S) \setminus \left(\left(\bigcup_{v \in W} \text{epn}(v,S)\right) \cup T'\right)$ which is adjacent to more than one vertex of $S \setminus W$.
Therefore, u cannot be defended by a unique vertex of S. This leads to a contradiction that G satisfies independent perfect secure domination. Thus, $|S| \le \lfloor \frac{n}{2} \rfloor$.
If $T \cup T' = \emptyset$ for a non-trivial simple connected graph G, then the vertices of $V \setminus S$ cannot be defended by unique vertices of S.
We know that $|S| > |V \setminus S|$ when $|S| > \lfloor \frac{n}{2} \rfloor$. Consider G to be a non-trivial simple connected graph and assume $S = W$. By pigeonhole principle we infer that at least one vertex of S cannot have an external private neighbour, a contradiction to our assumption. Thus, $|S| \le \lfloor \frac{n}{2} \rfloor$. □

In the following theorem, we consider $S = \{v_1, v_2, \dots, v_k\}$ to be a γ_{ips}-set of a simple connected graph G. Also, let $p = \sum_{i=1}^{k} |\text{epn}(v_i, S)|$.

Theorem 4.2. *For a simple connected graph G with $\gamma_{ips}(G) = k$, $|E| \le \frac{(|V|-k)(|V|-k+1)}{2}$.*

Proof. There are no edges among the vertices of S since S is an independent set. Every vertex of $\bigcup_{i=1}^{k} \text{epn}(v_i,S)$ is adjacent to exactly one vertex of S. Thus there are p edges between the vertices of S and the vertices of $\bigcup_{i=1}^{k} \text{epn}(v_i,S)$. Also, there are at most $\frac{p(p-1)}{2}$ edges among the vertices of $\bigcup_{i=1}^{k} \text{epn}(v_i,S)$.
Consider a vertex $w \in V \setminus \left(S \cup \bigcup_{i=1}^{k} \text{epn}(v_i,S)\right)$ and let it be defended by a unique vertex $v_j \in S$. Then, w can be adjacent to at most $|\text{epn}(v_i,S)|$ vertices of $\{v_i\} \cup \text{epn}(v_i,S)$ for $1 \le i \le k$ and $i \ne j$.

Also, w should be adjacent to all the vertices of $\{v_j\} \cup \text{epn}(v_j,S)$. Thus, there can be at most $\left(\sum_{i=1}^{k} |\text{epn}(v_i,S)|\right) + 1 = p+1$ edges between each vertex of $V \setminus \left(S \cup \bigcup_{i=1}^{k} \text{epn}(v_i,S)\right)$ and the vertices of $S \cup \bigcup_{i=1}^{k} \text{epn}(v_i,S)$. Also, there can be at most $\frac{(|V|-k-p)(|V|-k-p-1)}{2}$ edges among the vertices of $V \setminus \left(S \cup \bigcup_{i=1}^{k} \text{epn}(v_i,S)\right)$. Therefore, $|E| \le \frac{(|V|-k)(|V|-k+1)}{2}$. □

In the following result, we denote G to be a connected split graph consisting of clique K and independent set I. Let S be a minimum independent perfect secure dominating set. We assume K to be a maximal clique and $I \neq \emptyset$.

Theorem 4.3. *A connected split graph G with a maximal clique K and $I \neq \emptyset$ satisfies independent perfect secure domination only if each vertex $v \in K$ has $|N(v) \cap I| = 1$. Then, $\gamma_{ips}(G) = |I|$.*

Proof. When a connected split graph satisfies independent perfect secure domination and has $I \neq \emptyset$, we make the following claims:

Claim 1. $|N(v) \cap I| \ge 1$ for every $v \in K$

Proof of the claim. Let $v \in K$ be a vertex with $N(v) \cap I = \emptyset$. For G to be connected, every vertex in the set I should have at least one neighbour in $K \setminus \{v\}$. In order to defend v, a vertex of K should be an element of S.

Suppose $v \in S$. By Proposition 2.1, we have $|S| > 1$. Since S is independent and $v \in S$, vertices of $K \setminus \{v\}$ cannot be in S. Thus, we have $I \subset S$. Since $I \neq \emptyset$, there exists vertices in $N(I) \cap K$ which will be defended by both v and their corresponding neighbour in $I \cap (S \setminus \{v\})$. In this case, we infer that the split graph considered cannot satisfy independent perfect secure domination.

Suppose that $v \notin S$ and v is defended by a vertex $z \in K$. As the vertices of $K \setminus \{z\}$ cannot be in S, we have $I \setminus N(z) \subset S$ and $v \in \text{epn}(z,S)$. By Proposition 1.1, the vertices which are defended by z must be adjacent to v. As v is not adjacent to any vertex of I and since the vertices of $N(z) \cap I$ cannot be adjacent to any other vertices of I, we have $N(z) \cap I = \emptyset$. Then as in the previous case, there exists vertices in $N(I) \cap K$ which will be defended by z and their corresponding neighbours in $I \cap (S \setminus \{z\})$. In this case, we infer that the split graph considered cannot satisfy independent perfect secure domination.

From the above cases, we conclude that every vertex of K should have at least one neighbour in I when G satisfies independent perfect secure domination.

Claim 2. $|N(v) \cap I| = 1$ for every $v \in K$.

Proof of the claim. Suppose that a vertex $p \in K$ is an element of S. By Proposition 2.1, we have $|S| > 1$. Since I is independent, $S \setminus \{p\} = I \setminus N(p)$. From Claim 1, $|N(p) \cap I| \ge 1$. If $|N(p) \cap I| > 1$, at most one vertex of $N(p) \cap I$ can be defended by p. Since $N(N(p) \cap I) \setminus \{p\} \subseteq K$, G cannot admit an independent perfect secure dominating set. Thus, $|N(p) \cap I| = 1$.

Let $x, y \in I \setminus N(p)$ and suppose that $N(x) \cap N(y) \neq \emptyset$. Then, $N(x) \cap N(y)$ can be defended by both x and y. Hence, for arbitrary vertices x and y of $S \cap I$, $N(x) \cap N(y) = \emptyset$. Also, it can be observed that a vertex $u \in K \setminus \{p\}$ which is adjacent to $N(p) \cap I$ and a vertex $x \in I \setminus N(p)$, will be defended by x and p. Then, u cannot be defended by a unique vertex of S. Therefore, every vertex of K is adjacent to exactly one vertex of I.

If $S \cap K = \emptyset$, then $S = I$. Let $x, y \in I$ and suppose that $N(x) \cap N(y) \neq \emptyset$. As $x, y \in S$ with $\text{epn}(x,S) \subset K$, $\text{epn}(y,S) \subset K$ and $N(x) \cap N(y) \subset K$, $N(x) \cap N(y)$ can be defended by both x and y. Then, $N(x) \cap N(y)$ cannot be defended by unique vertices. Thus, $N(x) \cap N(y) = \emptyset$ for arbitrary vertices x and y of $S \cap I$. Therefore, every vertex of K is adjacent to exactly one vertex of I. Hence the claim.

If $|S \cap K| = 1$ with $S \cap K = \{v\}$, then $\{v\} \cup (I \setminus N(v))$ is an independent perfect secure dominating set. Else if $|S \cap K| = 0$, then $S = I$ is an independent perfect secure dominating set. In both the cases, $\gamma_{ips}(G) = |I|$. □

Theorem 4.4. *For any integers $n \geq 2$ and $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, there exists a graph G with n vertices that satisfies independent perfect secure domination such that $\gamma_{ips}(G) = s$.*

Proof. Since $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, $n = 2s + t$. Consider a path P_s with vertices $\{u_1, \dots, u_s\}$. Let each u_i be adjacent to a leaf v_i. Then, let G be a graph consisting of P_s along with their leaves v_i and a clique K_t (if $t \geq 1$) such that each vertex of K_t is adjacent to u_1 and v_1. Thus, G is a graph with n vertices that satisfies independent perfect secure domination. Also, $S = \{v_1, \dots, v_s\}$ would be a γ_{ips}-set of G and $\gamma_{ips}(G) = s$. □

5. Some Results on Independent Perfect Secure Domination in Trees

In this section, we study the independent perfect secure domination of the spiders. Also, we demonstrate the construction of trees T for certain values of $|V|$ and $\gamma_{ips}(T)$.

Theorem 5.1. *For a given integer $k \geq 3$, there exists a tree T with n $(2k \leq n \leq 3k-2)$ vertices that satisfies independent perfect secure domination. Also, $\gamma_{ips}(T) = k$.*

Proof. When $2k+1 \leq n \leq 3k-2$, let $\{u_1, \dots, u_{k-1}\} \cup \{v_1, \dots, v_{k-1}\} \cup \{x\} \cup \{w_1, \dots, w_{n-(2k-1)}\}$ be the vertices of the tree T. In T, consider each u_i to be a neighbour of v_i for $1 \leq i \leq k-1$ and let x be adjacent to all w_j where $1 \leq j \leq n-(2k-1)$. Also, let w_i be adjacent to u_i for $1 \leq i \leq n-(2k-1)$. For T to be connected, let each vertex $v_i \in \{v_{n-(2k-1)+1}, \dots, v_{k-1}\}$ be adjacent to v_{i-1}. For the tree T thus considered, $\{u_1, \dots, u_{k-1}\} \cup \{x\}$ is a minimum independent perfect secure dominating set and $\gamma_{ips}(T) = k$.

When $n = 2k$, consider T to be a tree with path P_k and each vertex u_i of P_k adjacent to a leaf v_i, $1 \leq i \leq k$. Then, $\{v_1, \dots, v_k\}$ is a minimum independent perfect secure dominating set and $\gamma_{ips}(T) = k$. □

For a spider T, consider v to be the vertex with $\deg(v) > 2$. Consider l_i, $1 \leq i \leq n = \Delta$ to be the length of each leg of the spider. We denote x_j, $0 \leq j \leq 6$ to be the number of legs whose length $l_i \equiv j \pmod 7$.

Theorem 5.2. *If T satisfies independent perfect secure domination with $\Delta > 2$, then $|V| \equiv (ax_a + bx_b + cx_c + y) \pmod 7$ where one of the given combinations would follow:*

(i) $a = 3,\ b = 5,\ c = 0,\ y = 1$. $\gamma_{ips}(T) = n + 1 + \sum_{i=1}^{n} \left\lceil \frac{3(l_i - 3)}{7} \right\rceil$ *where each* $l_i \equiv 0, 3, 5 \pmod 7$.

(ii) $a = 2,\ b = 5,\ c = 0,\ y \in \{1, 2, 4\}$. $\gamma_{ips}(T) = 2(n-1) + 1 + \sum_{\substack{i=1 \\ i \neq \omega}}^{n} \left\lceil \frac{3(l_i - 5)}{7} \right\rceil + \left\lceil \frac{3(l_\omega - 1)}{7} \right\rceil$ *where ω is a distinct index used for the leg whose length is l_ω and $l_\omega \equiv 1, 3, 5 \pmod 7$. Each $l_i \equiv 0, 2, 5 \pmod 7$ for $i \neq \omega$. When $l_i \equiv 2 \pmod 7$, $l_i \geq 9$.*

(iii) $a=2,\ b=4,\ c=0,\ y\in\{0,2,4\}$. $\gamma_{ips}(T)=\sum_{\substack{i=1\\i\neq\alpha}}^{n}\left\lceil\frac{3l_i}{7}\right\rceil+\left\lceil\frac{3(l_\alpha+1)}{7}\right\rceil$ *where* α *is a distinct index used for the leg whose length is* l_α *and* $l_\alpha\equiv 1,3,6 \pmod 7$. *Each* $l_i\equiv 0,2,4 \pmod 7$ *for* $i\neq\alpha$.

(iv) $a=2,\ b=4,\ c=6,\ y\in\{1,2\}$ *and there are at least two legs for which* $l_q\not\equiv 0 \pmod 7$. $\gamma_{ips}(T)=2+\sum_{\substack{i=1\\i\neq\alpha}}^{n}\left\lceil\frac{3l_i}{7}\right\rceil+\left\lceil\frac{3(l_\alpha-4)}{7}\right\rceil$ *where* α *is a distinct index used for the leg whose length is* l_α *and* $l_\alpha\equiv 1,4,6 \pmod 7$. *Each* $l_i\equiv 0,2,4,6 \pmod 7$ *for* $i\neq\alpha$.

Proof. Let us denote the vertices at distance t from v by V_t. Assume S to be a minimum independent perfect secure dominating set.

Case 1: $v\in S$

Subcase 1.1: v does not have an external private neighbour

By Remark 2.6, $V_2\subset S$ and $\text{epn}(V_2,S)=V_3$. Since $V_3=\text{epn}(V_2,S)$, vertices of V_4 cannot be in S. Thus, $V_4=\text{epn}(V_5,S)$. $G[V\setminus(V_1\cup V_2\cup V_3\cup\{v\})]$ has disjoint paths and we infer from the proof of Theorem 2.1 that the vertices of $V\setminus(\{v\}\cup V_1\cup V_2\cup V_3)$ corresponding to a particular leg of length l_i should be packed with $p.P_7\cup q.P_2\cup r.P_4$ where $q+r\le 1$ and $7p+2q+4r=l_i-3$. Hence, each $l_i\equiv 3,5,0 \pmod 7$ and $|V|\equiv(3x_3+5x_5+1) \pmod 7$. Also, $\gamma_{ips}(T)=n+1+\sum_{i=1}^{n}\left\lceil\frac{3(l_i-3)}{7}\right\rceil$.

Subcase 1.2: v has an external private neighbour

The vertex v can at most have one external private neighbour and let it be $u\in V_1$. Thus the vertex $N(u)\cap V_2$ should not be an element of S and it should be an external private neighbour to its neighbour in $V_3\cap S$. Suppose that the leg containing u is denoted by L_ω and its length by l_ω. As L_ω is a path, we infer from the proof of Theorem 2.1 that $V(L_\omega)\setminus\{u\}$ should be packed with $p.P_7\cup q.P_2\cup r.P_4$ where $q+r\le 1$ and $7p+2q+4r=l_\omega-1$. Thus, $l_\omega\equiv 1,3,5 \pmod 7$.

The vertices of $V_1\setminus\{u\}$ are defended by their corresponding neighbours in $V_2\setminus N(u)$. Since S is independent, vertices of $V_3\setminus V(L_\omega)$ cannot be in S. If the vertices of $V_4\setminus V(L_\omega)$ are not elements of S, we have the vertices of $V_3\setminus V(L_\omega)$ to be in $\text{epn}(V_2\setminus V(L_\omega),S)$. Then by Proposition 1.1, it leads to a contradiction and vertices of $V_1\setminus\{u\}$ cannot be defended by vertices of $V_2\setminus N(u)$. Thus, $V_4\setminus V(L_\omega)\subset S$ and $|\text{epn}(V_2\setminus V(L_\omega),S)|=0$. From Remark 2.6, every vertex $x\in V_4\setminus V(L_\omega)$ will have $|\text{epn}(x,S)|=1$ and $V_5\setminus V(L_\omega)\subseteq\text{epn}(V_4\setminus V(L_\omega),S)$. As $G[V\setminus(V_1\cup V_2\cup V_3\cup V_4\cup V_5\cup\{v\}\cup V(L_\omega))]$ has disjoint paths, we infer from the proof of Theorem 2.1 that vertices of $V\setminus(V_1\cup V_2\cup V_3\cup V_4\cup V_5\cup\{v\}\cup V(L_\omega))$ corresponding to a particular leg should be packed with $p.P_7\cup q.P_2\cup r.P_4$ where $q+r\le 1$ and $7p+2q+4r=l_i-5$ for $1\le i\le n,\ i\neq\omega$. Also, $l_i\ge 5$. Hence, each $l_i\equiv 0,2,5 \pmod 7$ for $1\le i\le n,\ i\neq\omega$ and $|V|\equiv(2x_2+5x_5+y) \pmod 7$ where $y\in\{1,2,4\}$ when $l_\omega\equiv 5,1,3 \pmod 7$, respectively. Also, $\gamma_{ips}(T)=2(n-1)+1+\sum_{\substack{i=1\\i\neq\omega}}^{n}\left\lceil\frac{3(l_i-5)}{7}\right\rceil+\left\lceil\frac{3(l_\omega-1)}{7}\right\rceil$.

Case 2: $v\notin S$

Let us consider $u\in S$ to be a unique vertex that defends v. Let the leg containing u be denoted by L_α and its length by l_α.

Subcase 2.1: $v \in \text{epn}(u,S)$

Since $v \in \text{epn}(u,S)$, $V_1 \setminus \{u\}$ cannot be in S and they are the external private neighbours of their corresponding neighbours in $V_2 \setminus V(L_\alpha) \subset S$. Since $G[V \setminus (\{v\} \cup V(L_\alpha))]$ has disjoint paths, we infer from the proof of Theorem 2.1 that vertices of $V \setminus (\{v\} \cup V(L_\alpha))$ corresponding to a particular leg should be packed with $p.P_7 \cup q.P_2 \cup r.P_4$ where $q + r \le 1$ and $7p + 2q + 4r = l_i$ for $1 \le i \le n$ and $i \ne \alpha$. Thus, each leg other than L_α should have $l_i \equiv 0,2,4 \pmod 7$.

The subgraph with vertices of $V(L_\alpha) \cup \{v\}$ can be considered as a path $P_{l_\alpha+1}$ and since $v \in \text{epn}(u,S)$, $l_\alpha + 1 \equiv 0,2,4 \pmod 7$. Then, $l_\alpha \equiv 6,1,3 \pmod 7$. Therefore, $|V| \equiv (2x_2 + 4x_4 + y) \pmod 7$ where $y \in \{0,2,4\}$ when $l_\alpha + 1 \equiv 0,2,4 \pmod 7$, respectively. Hence, $\gamma_{ips}(T) = \sum\limits_{\substack{i=1\\ i\ne\alpha}}^{n} \left\lceil \frac{3l_i}{7} \right\rceil + \left\lceil \frac{3(l_\alpha+1)}{7} \right\rceil$.

Subcase 2.2: $v \notin \text{epn}(u,S)$

When v is not an external private neighbour of u, there will be some vertices of $N(v) \setminus \{u\}$ in S. Each vertex of $(V_1 \setminus \{u\}) \cap S$ will have an external private neighbour in V_2. Let us denote the set of such legs by L_A. Since $G[V \setminus (\{v\} \cup V(L_\alpha) \cup V(L_A))]$ has distinct paths, we infer from the proof of Theorem 2.1 that each leg other than L_α and those in L_A would have their length $l_i \equiv 0,2,4 \pmod 7$.

For those legs in L_A, vertices of $V_3 \cap V(L_A)$ should be external private neighbours of $V_4 \cap V(L_A) \subset S$. As $G[V(L_A)]$ has disjoint paths, we infer from the proof of Theorem 2.1 that any leg in L_A whose length is l_j will have $l_j - 2 \equiv 0,2,4 \pmod 7$. Thus, $l_j \equiv 2,4,6 \pmod 7$ when l_j represents the length of the legs in L_A.

Consider the leg denoted by L_α. Since u cannot have external private neighbours, $V(L_\alpha) \cap V_4 \subseteq \text{epn}(V(L_\alpha) \cap V_3, S)$ by Remark 2.6. Since $V(L_\alpha) \setminus (\{u\} \cup V_2 \cup V_3 \cup V_4)$ is a path, we have from the inferences of the proof of Theorem 2.1 that $l_\alpha - 4 \equiv 0,2,4 \pmod 7$. Then, $l_\alpha \equiv 4,6,1 \pmod 7$. Therefore, $|V| \equiv (2x_2 + 4x_4 + 6x_6 + y) \pmod 7$ where $y \in \{1,2\}$ when $l_\alpha \not\equiv 1 \pmod 7$ and $l_\alpha \equiv 1 \pmod 7$, respectively. Also, $\gamma_{ips}(T) = 2 + \sum\limits_{\substack{i=1\\ i\ne\alpha}}^{n} \left\lceil \frac{3l_i}{7} \right\rceil + \left\lceil \frac{3(l_\alpha-4)}{7} \right\rceil$. □

Remark 5.1. When $\Delta(T) = 2$ for a spider T, the resultant is a path and hence $|V| \equiv 0,1,2,4,6 \pmod 7$. $\gamma_{ips}(T) = \left\lceil \frac{3|V|}{7} \right\rceil$ by Theorem 2.1.

6. Conclusion

The concept of independent perfect secure domination was introduced in this paper and a study into the properties of the new parameter was taken up. Further, the following problems can be taken up for investigation:

Problem 6.1. *Characterize the classes of graphs which satisfy independent perfect secure domination.*

Problem 6.2. *Characterize the graphs for which $\gamma_{ips}(G) = \gamma_s(G)$ and the graphs for which $\gamma_{ips}(G) > \gamma_s(G)$.*

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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